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Decimation-invariant sequences and their automaticity

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Abstract

This paper deals with one-dimensional bidirectional sequences $\underline{a}:\mathbb{Z}\to V$, V a finite set, such that any p-decimation ($|p|\geqslant 2$) of the sequence reproduces the sequence (modulo a certain shift). We develop a procedure for solving the underlying decimation-invariance (DI) equations and find that the number of solutions is always finite. Conditions for equivalency among solutions of differently parametrized DI-problems, and for possible periodicity and symmetry of solutions, are derived. It is shown that the set of all possible p-based decimations of a such a DI sequence (the so-called full kernel of the sequence) is finite. This implies finiteness of the kernel for the separate right and left parts of the sequence, and also $|p|$-automaticity of these parts. An algorithm is presented that constructs the kernel and associated $|p|$-automaton of a DI-sequence explicitly. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A *p-decimation* of a (one-dimensional) sequence is the operation that produces a new sequence by taking every pth element in the original sequence (and so there are $|p|$ different decimations). Such decimations are central in the study of so-called *p-automatic sequences* (or uniform tag sequences): these are sequences whose nth entry can be obtained as the output of a *finite automaton* that is fed with the p-adic representation of n (using standard representation, see [10, 9, 1]; for two-dimensional sequences, see [13]; using more exotic numeration systems, see [2]; for further gen-

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eralizations see [11]). Automaticity of a sequence is equivalent with *finiteness of its associated kernel*, this being the set of all possible compositions of p-decimations of the sequence.

Decimation is also a particular way of *coarse-graining* an object which is defined on a regular lattice, by considering the object only at the sites of a regular sublattice, deleting all the rest, and then performing a proper rescaling of this sublattice. Coarse-graining, and so decimation, is a *standard rescaling operation* in what is known as *renormalization*, a common procedure in statistical physics for the analysis of critical phenomena (percolation, ferromagnetism,…[15; 14 Chapters 2, 5]). At some critical parameter value, the geometrical structure of the object on the lattice typically lacks a characteristic scale: it is an object (mostly fractal-like) which is *invariant under coarse-graining* (mostly in a statistical sense). A certain decimation invariance is also exhibited by the so-called *fractal matrices*, deterministic fractal objects defined by an arithmetically based substitution system [3]. For other examples of particular coarse-graining and decimation-invariant objects defined on the $\mathbb{N} \times \mathbb{Z}$- and on the $\mathbb{N} \times \mathbb{Z}^2$-lattices, consider the *cellular automaton related orbits* presented in [4–6, 8].

In this paper, we study *one-dimensional bidirectionally infinite sequences* $\underline{a} : \mathbb{Z} \to V$, where V is a *finite* set, such that these sequences are *invariant under any p-decimation.*

Here follows a brief overview of the paper. The definition of p-decimation invariance (DI) is introduced in Section 2, where DI-equations are derived whose solutions are DI-sequences. Section 3 deals with the procedure for solving the DI-equations. Possible periodicity of solutions is discussed in Section 4. p-Automaticity of DI-sequences is the subject of Section 5: here we present a way for the explicit determination of the kernel and associated automaton.

2. Complete p-decimation invariance

Let $\underline{a}$ denote the bidirectionally infinite sequence $(a(k))_{k\in\mathbb{Z}}$ which maps the integer set $\mathbb{Z}$ into a finite symbol set V. For convenience, the kth element of $\underline{a}$ will usually be denoted by $a(k)$ instead of by $\underline{a}(k)$.

Definition 1. Let $p\in\mathbb{Z}$, $|p|\geqslant 2$ and $\mu\in[|p|]\stackrel{\text{def}}{=}\{0,1,\ldots,|p|-1\}$. The $(p;\mu)$-*decimation* of $\underline{a}$, symbolized by $D_p^\mu\underline{a}$, is the subsequence $(a(kp+\mu))_{k\in\mathbb{Z}}$. That is, with $D_p^\mu\underline{a}(k)$ denoting the kth element of the sequence $D_p^\mu\underline{a}$, we have

$$\forall k \in \mathbb{Z}:\ D_p^\mu\underline{a}(k) = a(kp+\mu). \tag{1}$$

Definition 2. Let $\kappa\in\mathbb{Z}$. The κ-*shift* of sequence $\underline{a}$, denoted by ${}^\kappa\underline{a}$, is the sequence $(a(k+\kappa))_{k\in\mathbb{Z}}$, that is

$$\forall k \in \mathbb{Z}:\ {}^\kappa\underline{a}(k) = a(k+\kappa). \tag{2}$$

Definition 3. Let $p \in \mathbb{Z}$, $|p| \geqslant 2$ and let $\boldsymbol{\kappa} \stackrel{\text{def}}{=} [\kappa_0, \kappa_1, \ldots, \kappa_{|p|-1}]$ be a set of $|p|$ shift-parameters, then $\underline{a}$ is (*complete*) $[p, \boldsymbol{\kappa}]$-*decimation invariant* (DI) iff

$$\forall \mu \in [|p|]: \quad D_p^{\mu} \underline{a} = {}^{\kappa_\mu} \underline{a} \tag{3}$$

or, using (1), (2)

$$\forall \mu \in [|p|], \quad \forall k \in \mathbb{Z}: \quad a(kp + \mu) = a(k + \kappa_\mu). \tag{4}$$

Eq. (4) are called the (*complete*) *decimation-invariance equations* (DI-equations).

Remark 1. $[p, \boldsymbol{\kappa}]$-DI for $p \geqslant 2$ means that *all* decimated sequences $D_p^{\mu} \underline{a}$ (meaning over *all* $\mu \in \{0, 1, \ldots, p-1\}$, hence the adjective *complete*) are identical to the original sequence $\underline{a}$ *modulo a shift* κ_μ. This is also the case for $p \leqslant -2$, but only after a (left–right) *reversal* of the decimated sequences. In order to emphasize this difference, the $[p, \boldsymbol{\kappa}]$-DI with *negative* p will be called *reverse decimation invariance*, while $[p, \boldsymbol{\kappa}]$-DI with positive p will sometimes be called *direct* DI.

3. Solving the $[p, \boldsymbol{\kappa}]$-decimation invariance equations

First observe that any constant sequence [i.e. $(a(k))_{k \in \mathbb{Z}} \equiv \text{constant} \in V$] is a *trivial solution* of the DI-equations. In this section, it will be shown how the DI-equations can be solved for nontrivial solutions, starting from a finite set of initial $a(k)$-values, the so-called *seed*. This seed will have to be determined first. It will become clear that the solution procedure will differ for direct and reverse decimations.

Definition 4. The (equivalent) *standard form* of the DI-equations (4) is

$$\forall k \in \mathbb{Z}: \; a(k) = a\left(\frac{k - \mu}{p} + \kappa_\mu\right), \quad \text{for } k \equiv \mu \bmod |p|, \; \mu \in [|p|]. \tag{5}$$

The induced integer mapping

$$k \mapsto \operatorname{Res}(k) \stackrel{\text{def}}{=} \frac{k - \mu}{p} + \kappa_\mu, \quad k \equiv \mu \bmod |p|, \; \mu \in [|p|] \tag{6}$$

will be called the *rescaling map*.

We will further need the affine contractions

$$\mathbb{R}\mathrm{es}_\mu(x) = \frac{x - \mu}{p} + \kappa_\mu, \quad x \in \mathbb{R}, \; \mu \in [|p|]. \tag{7}$$

The way to solve the DI-equations (5) is suggested by considering the graphs associated with the rescaling maps. We present two generic examples of such graphs, one for the direct case and one for the reverse case.

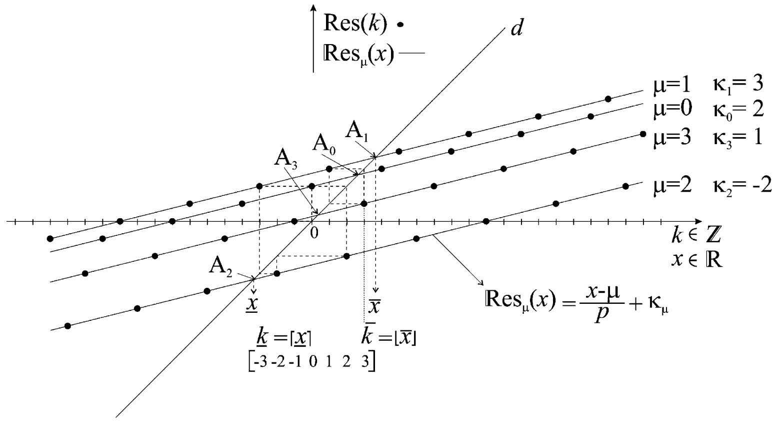

Fig. 1. Graphs of the mappings $k \mapsto \mathrm{Res}(k)$ and $x \mapsto \mathbb{R}\mathrm{es}_\mu(x)$ for the *direct* $[p, \boldsymbol{\kappa}]$-decimation invariance problem ($p \geqslant 2$), with $p = 4$, $\boldsymbol{\kappa} = [2, 3, -2, 1]$. The core C is the interval $I[-3, 3]$. There are two classes of loop-equivalent points in the core. The points inside each class are connected by dashed lines [connecting k to $\mathrm{Res}(k)$], thus producing the following chains:
1st chain: $-1 \mapsto 0 \mapsto 2 \mapsto -2 \mapsto -3$ ($\mapsto 2$).
2nd chain: $1 \mapsto 3$ ($\mapsto 1$).

3.1. Solving direct $[p, \boldsymbol{\kappa}]$-DI equations (case $p \geqslant 2$)

Example 1. Take $p = 4$, $\boldsymbol{\kappa} = [\kappa_0, \kappa_1, \kappa_2, \kappa_3] = [2, 3, -2, 1]$.

Fig. 1 shows the graphs of $\mathrm{Res}(k)$ versus k (the big dots), and $\mathbb{R}\mathrm{es}_\mu(x)$ versus x (the straight lines connecting the dots associated with a fixed μ).

The fixed point of the map $\mathbb{R}\mathrm{es}_\mu$ is the point

$$x_\mu = \frac{p\kappa_\mu - \mu}{p - 1}. \tag{8}$$

(This is the x-coordinate of the intersection point A_μ of the line $\mathbb{R}\mathrm{es}_\mu(x)$ with the diagonal line d.) Let $\underline{x} = \min_\mu x_\mu$, $\bar{x} = \max_\mu x_\mu$ (see Fig. 1), and define $\underline{k} = \lceil \underline{x} \rceil$, $\bar{k} = \lfloor \bar{x} \rfloor$, with $\lfloor\ \rfloor$ and $\lceil\ \rceil$ the floor and ceiling functions, respectively. For calculations, one can use the equivalent expressions

$$\underline{k} = \min_\mu \lceil x_\mu \rceil, \quad \bar{k} = \max_\mu \lfloor x_\mu \rfloor. \tag{9}$$

Since the slope of the $\mathbb{R}\mathrm{es}_\mu$-lines is $1/p$, the next lemma follows.

Lemma 1. *With* $\underline{k}$ *and* $\bar{k}$ *as defined in* (9) *above, and* $I[\underline{k}, \bar{k}]$ *denoting the integer interval* $\{\underline{k}, \underline{k} + 1, \cdots, \bar{k}\}$, *the following properties hold*:

$$\begin{array}{ll} \text{a.} & \underline{k} \leqslant \bar{k}, \\ \text{b.} & k < \underline{k} \Rightarrow k < \mathrm{Res}(k) \leqslant \bar{k}, \\ \text{c.} & k > \bar{k} \Rightarrow k > \mathrm{Res}(k) \geqslant \underline{k}, \\ \text{d.} & k \in I[\underline{k}, \bar{k}] \Rightarrow \mathrm{Res}(k) \in I[\underline{k}, \bar{k}]. \end{array} \tag{10}$$

Remark 2. When $\kappa_\mu = \kappa =$ constant and $p-1$ does not divide κ, then $\underline{k} = \bar{k}$. When κ_μ depends on μ, then it is always true that $\underline{k} < \bar{k}$.

Definition 5. A (finite) sequence $\underline{s}: I[\underline{k}, \bar{k}] \to V$ is $[p, \boldsymbol{\kappa}]$-DI on $I[\underline{k}, \bar{k}]$ if $s(kp+\mu) = s(k+\kappa_\mu)$ for all $\mu \in [p]$ and for all k such that both sides of this equation are defined; or equivalently, if

$$s(k) = s(\mathrm{Res}(k)), \tag{11}$$

where both k and $\mathrm{Res}(k) \in I[\underline{k}, \bar{k}]$.

We will now describe the set of all $[p, \boldsymbol{\kappa}]$-DI sequences $\underline{s}$ on $I[\underline{k}, \bar{k}]$. Consider the map Res (formula (6)) restricted to $I[\underline{k}, \bar{k}]$. According to Lemma 1d: $\mathrm{Res}(I[\underline{k}, \bar{k}]) \subseteq I[\underline{k}, \bar{k}]$, and since $I[\underline{k}, \bar{k}]$ is finite, any sequence k, $\mathrm{Res}(k)$, $\mathrm{Res}^2(k)$, $\mathrm{Res}^3(k), \ldots$ that starts with $k \in I[\underline{k}, \bar{k}]$ must necessarily enter a loop (Res^n means the nth iterate of Res).

Definition 6. • $k', k'' \in I[\underline{k}, \bar{k}]$ are *loop-equivalent* if there are $n, m \in \mathbb{N}$ such that $\mathrm{Res}^n(k') = \mathrm{Res}^m(k'')$, meaning that the Res-iterations of k' and k'' ultimately enter the *same* loop. (Or: k' and k'' are connected to the same loop). We write $k' \cong k''$.

Let N_l denote the number of equivalence classes of $I[\underline{k}, \bar{k}]$ with respect to this loop-equivalence, that is: N_l is the number of different loops.

• Let $\underline{b}$ and $\underline{b}'$ be two sequences defined on some set $Z \subseteq \mathbb{Z}$, having values in V, that is $\underline{b}: Z \to V$ and $\underline{b}': Z \to V$. Then $\underline{b}$ and $\underline{b}'$ are called *symbolically equivalent* (SE) when there exists a bijection $F: V \to V$ such that $\underline{b} = F\underline{b}'$ (or: if both sequences are equivalent under a 1–1 replacement of symbols).

Definitions 5 and 6 lead to the following lemma.

Lemma 2. (a) *The sequence* $\underline{s}: I[\underline{k}, \bar{k}] \to V$ *is a solution of a* $[p, \boldsymbol{\kappa}]$*-DI problem on* $I[\underline{k}, \bar{k}]$ *if and only if* $s(k') = s(k'')$ *for* $k' \cong k''$.

(b) *The number of different* $[p, \boldsymbol{\kappa}]$*-DI sequences on* $I[\underline{k}, \bar{k}]$ *is* $|V|^{N_l}$. *When* $|V| \geqslant N_l$, *the number of different* (*modulo SE*) $[p, \boldsymbol{\kappa}]$*-DI sequences equals* $B(N_l)$, *where* $B(N_l)$ *is the so-called Bell-number of* N_l, *which is the number of partitions of a set of* N_l *elements. The number of different* (*mod SE*) $[p, \boldsymbol{\kappa}]$*-DI sequences with* $N (\leqslant N_l)$ *different symbols equals* $S(N_l, N)$, *a Stirling number of the second kind* [12].

Obviously, when $\underline{a}: \mathbb{Z} \to V$ is a $[p, \boldsymbol{\kappa}]$-DI sequence then $\underline{a} | I[\underline{k}, \bar{k}]$ is $[p, \boldsymbol{\kappa}]$-DI on $I[\underline{k}, \bar{k}]$.

Lemma 3. *Let* $\underline{s}: I[\underline{k}, \bar{k}] \to V$ *be* $[p, \boldsymbol{\kappa}]$*-DI on* $I[\underline{k}, \bar{k}]$. *Then there is a unique* $[p, \boldsymbol{\kappa}]$*-DI sequence* $\underline{a}: \mathbb{Z} \to V$ *such that* $a(k) = s(k)$ *for* $k \in I[\underline{k}, \bar{k}]$.

Proof. We have to prove that there is a unique extension $\underline{a}: \mathbb{Z} \to V$ of the sequence $\underline{s}: I[\underline{k}, \bar{k}] \to V$ and that this extension is a $[p, \boldsymbol{\kappa}]$-DI sequence.

Let $k \in \mathbb{Z}$, $k < \underline{k}$. Lemma 1b implies that there is a number $n_0 \in \mathbb{N}$ such that $\mathrm{Res}^{n_0}(k) \in I[\underline{k}, \bar{k}]$. Then define

$$a(k) = s(\mathrm{Res}^n(k)), \quad \text{where } n \geqslant n_0.$$

This definition is correct: $s(\mathrm{Res}^n(k))$ does not depend on the number n (when $n \geqslant n_0$), because of Lemma 1d and because $\underline{s}$ is $[p, \boldsymbol{\kappa}]$-DI on $I[\underline{k}, \bar{k}]$. This can be interpreted in the sense that if $\mathrm{Res}^n(k) \in I[\underline{k}, \bar{k}]$, then $a(k) = s(\mathrm{Res}^n(k))$, which in its turn implies that $a(k) = s(\mathrm{Res}^{n-1}(\mathrm{Res}(k)) = a(\mathrm{Res}(k))$.

For $k > \bar{k}$, we use Lemma 1c and define $a(k)$ in the same way. It then follows that the sequence $\underline{a} : \mathbb{Z} \to V$ satisfies $a(k) = a(\mathrm{Res}(k))$ for all $k \in \mathbb{Z}$, that is, it is $[p, \boldsymbol{\kappa}]$-DI. Moreover, it follows from the construction that such an extension is unique. □

Remark 3. Observe that the definitions of $a(k)$ for $k < \underline{k}$ and $k > \bar{k}$ are independent. As a consequence of Lemma 1b, the practical way to extend the restricted $[p, \boldsymbol{\kappa}]$-DI solution $\underline{s}$ to the full solution $\underline{a}$, is to propagate the solution recursively to the left using the DI-equation (5) for $k < \underline{k}$. In the same way, the solution can be propagated independently to the right for $k > \bar{k}$, as a consequence of Lemma 1c. Later on, we will see that this left–right independency no longer holds for reverse DI-sequences.

Definition 7. The interval $C \stackrel{\text{def}}{=} I[\underline{k}, \bar{k}]$ will be called the *core* (*-interval*) for the underlying DI-equations, and any particular DI-solution $\underline{s}$ on it will be called a *seed* (from which the whole solution can be grown). Eqs. (11), which define the seed, will be called the *seed-internal* equations.

Remark 4. Because a seed extends in a unique way to a full DI-sequence, Lemma 2b concerning the number of seeds also applies to the full DI-sequences.

Example 1 (*Continued*)**.** For the $[p = 4, \boldsymbol{\kappa} = [2, 3, -2, 1]]$-DI problem, $N_l = 2$. With $V = \{0, 1, 2, 3\}$, there are $4^2 = 16$ solutions, four of which are trivial. The number of different symbolically equivalent solutions is $B(2) = 2$, one of which is trivial. Part of the nontrivial solution is graphically represented in Fig. 2.

3.2. Solving reverse $[p, \boldsymbol{\kappa}]$-DI equations (case $p \leqslant -2$)

The next example will show that reverse decimation-invariant equations cannot be propagated *separately* to the left and to the right from a seed as was the case for the direct DI-equations. Another procedure is proposed.

Example 2. Take $p = -4$ and $\boldsymbol{\kappa} = [-3, 1, 2, 4]$. Fig. 3 shows the graphs of $\mathrm{Res}(k)$ and $\mathbb{R}\mathrm{es}_\mu(x)$ associated with the mappings (6), (7).

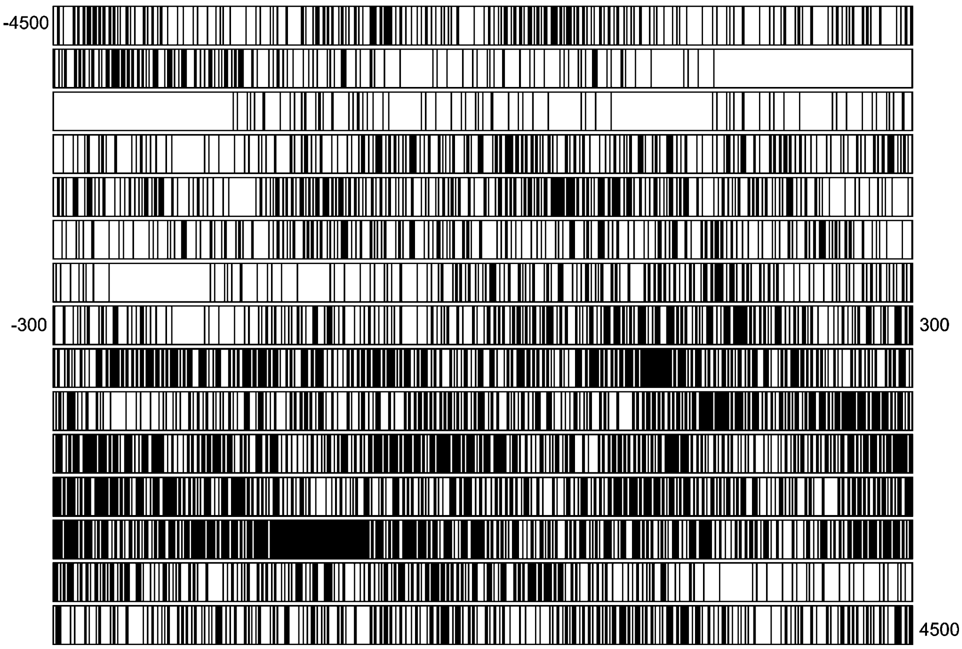

Fig. 2. Graphical representation of the two-valued nontrivial solution of the $[p=4, \boldsymbol{\kappa}=[2,3,-2,1]]$-DI problem introduced in Example 1 (display range from $k=-4500$ to 4500).

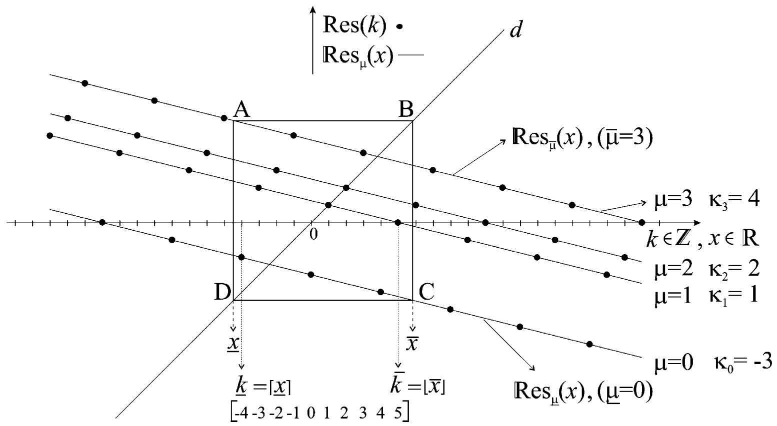

Fig. 3. Graphs of the mappings $k \mapsto \mathrm{Res}(k)$; $x \mapsto \mathbb{R}\mathrm{es}_{\mu}(x)$ for the inverse decimation invariance problem with $p=-4$, $\boldsymbol{\kappa}=[-3,1,2,4]$. The core C is the interval $I[4,5]$. The three loop-equivalent classes in the core correspond to the following chains:

1st chain: $-1 \mapsto 5 \mapsto 0 \mapsto -3 \mapsto 2$ $(\mapsto 2)$

2nd chain: 1 $(\mapsto 1)$

3d chain: $-4 \mapsto -2 \mapsto 3 \mapsto 4$ $(\mapsto -4)$.

This DI-problem has $B(3)=5$ different symbolically equivalent solutions, one of which is the trivial solution.

Let $\underline{\mu}$ and $\bar{\mu}$ be the μ-values corresponding respectively to the lowest and highest $\mathbb{R}\mathrm{es}_\mu(x)$-line, i.e.

$$\bar{\mu} = \arg\max_{\mu}\{\mu - p\kappa_\mu\}, \qquad \underline{\mu} = \arg\min_{\mu}\{\mu - p\kappa_\mu\}.$$

Then, it is always possible to find a unique square ABCD (see Fig. 3), with vertices A and C on the $\mathbb{R}\mathrm{es}_{\bar{\mu}}(x)$ and $\mathbb{R}\mathrm{es}_{\underline{\mu}}(x)$-lines respectively, and with B and D on the *diagonal* line d. (The existence of such a square follows from the following geometrical argument: this square is symmetric with respect to the diagonal d. So C is the unique intersection point of the diagonal-symmetric image of the $\mathbb{R}\mathrm{es}_{\bar{\mu}}(x)$-line with the $\mathbb{R}\mathrm{es}_{\underline{\mu}}(x)$-line. C completely defines the square.) With $\bar{x}$ and $\underline{x}$ the x-coordinates of B, C and A, D, respectively, these satisfy $\bar{x} = \mathbb{R}\mathrm{es}_{\bar{\mu}}(\underline{x})$, $\underline{x} = \mathbb{R}\mathrm{es}_{\underline{\mu}}(\bar{x})$, or

$$\bar{x} = \mathbb{R}\mathrm{es}_{\bar{\mu}}(\mathbb{R}\mathrm{es}_{\underline{\mu}}(\bar{x})), \qquad \underline{x} = \mathbb{R}\mathrm{es}_{\underline{\mu}}(\mathbb{R}\mathrm{es}_{\bar{\mu}}(\underline{x})).$$

Using (7), this produces the explicit solutions

$$\underline{x} = \frac{p^2\kappa_{\underline{\mu}} + p\kappa_{\bar{\mu}} - p\underline{\mu} - \bar{\mu}}{p^2 - 1} \quad \text{and} \quad \bar{x} = \frac{p^2\kappa_{\bar{\mu}} + p\kappa_{\underline{\mu}} - p\bar{\mu} - \underline{\mu}}{p^2 - 1}. \tag{12}$$

From this construction, it is clear that $x \in [\underline{x}, \bar{x}] \Rightarrow \mathbb{R}\mathrm{es}_\mu(x) \in [\underline{x}, \bar{x}]$ for all $\mu \in [|p|]$. Restricting this property to $k \in \mathbb{Z}$ and $\mathrm{Res}(k)$, and defining

$$\underline{k} = \lceil \underline{x} \rceil \quad \text{and} \quad \bar{k} = \lfloor \bar{x} \rfloor, \tag{13}$$

this leads to point a. in the following lemma.

Lemma 4 (Reverse decimation). *Let* $\underline{k}$ *and* $\bar{k}$ *be defined by Eqs.* (12) *and* (13). *Then*

a. $k \in I[\underline{k}, \bar{k}] \Rightarrow \mathrm{Res}(k) \in I[\underline{k}, \bar{k}]$.
b. *For* $k < \underline{k}$: $\mathrm{Res}^n(k) \in I[\underline{k}, \bar{k}]$ *after* $n \leqslant (\underline{k} - k)$ *steps.*
c. *For* $k > \bar{k}$: $\mathrm{Res}^n(k) \in I[\underline{k}, \bar{k}]$ *after* $n \leqslant (k - \bar{k})$ *steps.*

Proof (Of points b and c)**.** Start from a k-value to the left of $\underline{k}$, that is $\underline{k} - k > 0$. When $\mathrm{Res}(k) \notin I[\underline{k}, \bar{k}]$, it follows from the contractivity of the $\mathbb{R}\mathrm{es}_\mu$-mappings (see Fig. 3), that $\underline{x} - k > \mathrm{Res}(k) - \bar{x} > 0$, implying that

$$\underline{k} - k > \mathrm{Res}(k) - \bar{k} > 0 \quad \text{when } \neg(\underline{k} - k = 1 \;\&\; k \equiv \bar{\mu} \bmod [|p|]). \tag{14}$$

When $\underline{k} - k = 1$ and $k \equiv \bar{\mu} \bmod [|p|]$, then $\mathrm{Res}(k) = \mathbb{R}\mathrm{es}_{\bar{\mu}}(k) = \bar{x} + 1/p > \bar{k}$, and as $\bar{x} + 1/p \leqslant \bar{k} + 1$, it necessarily follows that $\mathrm{Res}(k) = \bar{k} + 1$. (This is the case for the example shown.) Anyhow, $\mathrm{Res}(k)$ is located to the right of $I[\underline{k}, \bar{k}]$ when $k < \underline{k}$.

With a starting point $k' > \bar{k}$ and $\mathrm{Res}(k') \notin I[\underline{k}, \bar{k}]$, it holds that

$$k' - \bar{k} > \underline{k} - \mathrm{Res}(k') > 0 \quad \text{when } \neg(k - \bar{k} = 1 \;\&\; k \equiv \underline{\mu} \bmod [|p|]). \tag{15}$$

($k - \bar{k} = 1$ & $k \equiv \underline{\mu} \bmod [|p|]$ implies that $\mathrm{Res}(k) = \underline{k} - 1$.)

Eqs. (14) and (15) imply that, when starting from $k < \underline{k}$ or from $k > \bar{k}$, there is an $n \in \mathbb{N}$ such that $\mathrm{Res}^n(k)$ enters $I[\underline{k}, \bar{k}]$ after at most $\underline{k} - k$, respectively $k - \bar{k}$ steps. □

Like in the previous section (dealing with the case $p \geqslant 2$), the map $\mathrm{Res} : I[\underline{k}, \bar{k}] \to I[\underline{k}, \bar{k}]$ defines here also a *loop-equivalence* relation on the set $I[\underline{k}, \bar{k}]$. Let N_l also here denote the number of such equivalence classes (loops). Then Lemma 2 concerning DI-solutions $\underline{s}$ restricted to $I[\underline{k}, \bar{k}]$ also holds for $p \leqslant -2$.

So does Lemma 3 for the extension of $\underline{s}$ to $\underline{a}$. Its proof for $p \leqslant -2$, which goes along the same lines as the one for $p \geqslant 2$ (define $a(k) = s(\mathrm{Res}^n(k))$, n such that $\mathrm{Res}^n(k) \in I[\underline{k}, \bar{k}]$), will now invoke Lemma 4b and c, instead of Lemma 1b and c. This implies that the extension of $\underline{s}$ to $\underline{a}$ can no longer be done independently for $k < \underline{k}$ and for $k > \bar{k}$.

Also here it is appropriate to call $C = I[\underline{k}, \bar{k}]$ the *core* and any $[p, \boldsymbol{\kappa}]$-DI solution $\underline{s}$ on C a *seed* for the underlying $[p, \boldsymbol{\kappa}]$-DI equations.

Remark 5. In itself, generating the address-sequence $(\mathrm{Res}^n(k))_{k \in \mathbb{N}}$ until $\mathrm{Res}^{n_0}(k) \in C$, memorizing these addresses and putting all elements at these addresses equal to $a(\mathrm{Res}^{n_0}(k))$, is a workable but inefficient procedure to generate the sequence $\underline{a}$. A more efficient "inverse" procedure grows the solution starting from the seed in a way that will be explained now. It uses the mappings

$$U_\mu : \mathbb{Z} \to \mathbb{Z} : k \to p(k - \kappa_\mu) + \mu,$$

which are somehow the inverse of the $\mathrm{Res}(k)$-mapping defined in (6), and goes back to the original DI-equation (4), which can now be written as

$$\forall k \in \mathbb{Z}, \forall \mu \in [|p|]:\ a(U_\mu(k)) = a(k).$$

Let $U(k) \stackrel{\mathrm{def}}{=} \{U_\mu(k) \mid \mu \in [|p|]\}$. Then $a(k)$ can be propagated stepwise from the seed by generating the address-set $E_1 = U(C) \backslash C$ and attributing to each element of E_1 the value associated with its preimage. Then produce $E_2 = U(E_1) \backslash E_1$ and attribute to each element of E_2 the value associated with its preimage. The whole sequence $\underline{a}$ is then generated by continuing this procedure, whereby at step $n+1$ the address-set $E_{n+1} = U(E_n) \backslash E_n$ is determined, while attributing to each element of E_{n+1} the value associated with its preimage.(This procedure actually also works for solving direct DI-equations.)

Remark 4 concerning the number of solutions also holds for negative p. This allows us to end this section on solving the DI-equations by concluding that we have proved the following:

Theorem 1. *Let* $p \in \mathbb{Z}$, $|p| \geqslant 2$, $\boldsymbol{\kappa} = [\kappa_0, \kappa_1, \ldots, \kappa_{|p|-1}] \subset \mathbb{Z}$. *The number of* $[p, \boldsymbol{\kappa}]$-*DI solutions* (*modulo symbolic equivalence*) *is* $B(N_l)$, *the Bell-number of* N_l, *where* N_l *is the number of different loops* (*equivalence classes*) *with respect to the map* $\mathrm{Res} : I[\underline{k}, \bar{k}] \to I[\underline{k}, \bar{k}]$ *defined by* (6), *where* $\underline{k}$ *and* $\bar{k}$ *are defined by* (9) *for* $p \geqslant 2$, *and by* (12), (13) *for* $p \leqslant -2$. *The number of different* $(\mathrm{mod}\, SE)$ $[p, \boldsymbol{\kappa}]$-*DI sequences with* $N (\leqslant N_l)$ *symbols is* $S(N_l, N)$ (*Stirling-number of the second kind*).

Definition 8. A solution to a $[p, \boldsymbol{\kappa}]$-DI problem is a *solution of maximal diversity* when it is obtained by attributing different values to the different loop-equivalence classes in the core.

4. Periodicity of DI-sequences

If $\underline{a}$ is a $[p, \boldsymbol{\kappa}]$-DI sequence,what about the reverse sequence? What are the conditions for a DI-sequence to be periodic? These are the questions dealt with in this section. The answers will be used in the next section on automaticity of DI-sequences.

Definition 9.

- Let $a, p \in \mathbb{Z}$. The *integer quotient* $\imath\frac{a}{p}$ is defined through

$$a = \imath\frac{a}{p} \times p + a \bmod |p|, \quad \text{with } a \bmod |p| \in [|p|]. \tag{16}$$

- Let $\underline{a} = (a(k))_{k\in\mathbb{Z}}$ be a sequence, then the *reverse* sequence of $\underline{a}$ is $\underline{a}^- \stackrel{\text{def}}{=} (a(-k))_{k\in\mathbb{Z}}$.

Theorem 2. *Let $\underline{a} = (a(k))_{k\in\mathbb{Z}}$ be a $[p, \boldsymbol{\kappa}]$-DI sequence $(|p| \geqslant 2)$. Then the reverse sequence $\underline{a}^-$ is $[p, \boldsymbol{\kappa}^-]$-DI, where*

$$\kappa_0^- = -\kappa_0 \quad \textit{and} \quad \kappa_\mu^- = \text{sign}(p) - \kappa_{|p|-\mu} \quad \textit{for } \mu \in [|p|]\backslash\{0\} \tag{17}$$

with $\text{sign}(p) = 1$ *when* $p > 0$, $\text{sign}(p) = -1$ *when* $p < 0$.

Proof.

$$\begin{aligned} a^-(kp + \mu) &= a(-kp - \mu) \\ &\stackrel{(4)}{=} a\left(-k + \imath\frac{-\mu}{p} + \kappa_{-\mu \bmod |p|}\right) \\ &= a^-\left(k - \imath\frac{-\mu}{p} - \kappa_{-\mu \bmod |p|}\right). \end{aligned}$$

Identifying $-\imath\frac{-\mu}{p} - \kappa_{-\mu \bmod |p|}$ with κ_μ^-, which ultimately leads to expressions (17), and invoking Definition 3, establishes the theorem. □

Before formulating a theorem concerning the possible periodicity of DI-sequences, we need a few lemmas first. The development that follows goes along the same lines as the ones leading to periodicity conditions for solutions of the so-called more general coarse-graining invariant equations that appear in cellular automata defined over a finite field (see [5]). Lemma 5 follows directly from the definition of $\text{Res}(k)$ [Eq. (6)].

Lemma 5.

(a)

$$\forall k, s \in \mathbb{Z}: \ \text{Res}(k + sp) = \text{Res}(k) + s. \tag{18}$$

(b)

$$\{\text{Res}(k_0 + sp) | s \in \mathbb{Z}\} = \mathbb{Z}, \{\text{Res}(k) \,|\, k \in \mathbb{Z}\} = \mathbb{Z}. \tag{19}$$

(c) $\forall k \in \mathbb{Z}$:

(i) *when* μ ($\stackrel{\text{def}}{=} k \bmod |p|) \in \{0, 1, \ldots, |p| - 2\}$:

$$\mathrm{Res}(k + 1) - \mathrm{Res}(k) = \kappa_{\mu+1} - \kappa_\mu, \tag{20}$$

(ii) *when* μ ($\stackrel{\text{def}}{=} k \bmod |p|) = |p| - 1$:

$$\mathrm{Res}(k + 1) - \mathrm{Res}(k) = \kappa_0 - \kappa_{|p|-1} + \mathrm{sign}(p). \tag{21}$$

Lemma 6. *Suppose that* $\underline{a}$ *is* $[p, \boldsymbol{\kappa}]$*-DI with minimal period* P, *then*

(a) $\forall k \in \mathbb{Z}$: $a(\mathrm{Res}(k)) = a(\mathrm{Res}(k + P))$

(b) *when* $\forall k \in \mathbb{Z}$: $a(\mathrm{Res}(k)) = a(\mathrm{Res}(k + \gamma))$ *for some* $\gamma \in \mathbb{N}$, *then* P *divides* γ.

Proof. (a) $a(k + P) - a(k) \stackrel{(5,6)}{=} a(\mathrm{Res}(k + P)) - a(\mathrm{Res}(k)) = 0$.

(b) From the premises, it follows that $\forall k, s, l, \in \mathbb{Z}$:

$$a(\mathrm{Res}(k)) = a(\mathrm{Res}(k + sp\gamma)) \stackrel{(18)}{=} a(\mathrm{Res}(k) + s\gamma) = a(\mathrm{Res}(k) + lP).$$

As Lemma 5(b) says that $\{\mathrm{Res}(k) \mid k \in \mathbb{Z}\} = \mathbb{Z}$, it follows by letting $\mathrm{Res}(k) = m$, that

$$\forall m, s, l \in \mathbb{Z}: \ a(m) = a(m + s\gamma) = a(m + lP).$$

This means that the sequence $\underline{a}$ has two periods: γ and P. As P is the minimal period, P must divide γ. $\square$

Lemma 7. *A necessary condition for* P *to be a possible minimal period of a* $[p, \kappa]$*-DI solution is that*

$$\forall k \in \mathbb{Z}: \quad P \text{ divides } p[\mathrm{Res}(k + 1) - \mathrm{Res}(k)] - 1. \tag{22}$$

Proof. From (18), it follows that

$$\forall k \in \mathbb{Z}: \quad \mathrm{Res}(k + p[\mathrm{Res}(k + 1) - \mathrm{Res}(k)]) = \mathrm{Res}(k + 1),$$

and, as a consequence:

$$\forall k \in \mathbb{Z}: \ a(\mathrm{Res}(k + 1 + p[\mathrm{Res}(k + 1) - \mathrm{Res}(k)] - 1)) = a(\mathrm{Res}(k + 1)).$$

Apply Lemma 6b to conclude that (22) holds. $\square$

Theorem 3. *A necessary condition for a solution* $\underline{a}$ *of a* $[p, \boldsymbol{\kappa}]$*-DI problem to be periodic with minimal period* P *is that*

$$\forall \mu \in \{0, 1, \ldots, |p| - 2\}: \ P \text{ divides } p(\kappa_{\mu+1} - \kappa_\mu) - 1. \tag{23}$$

Proof. We will show that (23) is equivalent to (22).

(a) Combine (22) and (20) to conclude that
$\forall k \in \mathbb{Z}$ such that $k \equiv \mu \bmod |p|$, with $\mu \in \{0, 1, \ldots, |p| - 2\}$:

$$P \text{ divides } p[\mathrm{Res}(k + 1) - \mathrm{Res}(k)] - 1$$

is equivalent to

$$\forall\mu\in\{0,1,\ldots,|p|-2\}:\quad P \text{ divides } p(\kappa_{\mu+1}-\kappa_\mu)-1. \tag{24}$$

(b) From (24) it follows that P divides

$$\begin{aligned}-\sum_{\mu=0}^{|p|-2} p(\kappa_{\mu+1}-\kappa_\mu)-1 &= -p(\kappa_{p-1}-\kappa_0)+(|p|-1)\\ &= p[\kappa_0-\kappa_{p-1}+sgn(p)]-1\\ &\overset{(21)}{=} p[\mathrm{Res}(k+1)-\mathrm{Res}(k)]-1 \text{ for all } k\\ &\quad \text{for which } \mu=k \bmod |p|=|p|-1.\end{aligned}$$

So, when (24) is satisfied, (22) is also automatically satisfied for those k for which $k \bmod |p|=|p|-1$.

(c) From (a) and (b) together,it follows that (23) is equivalent to (22). □

Corollary 1. *A periodic solution to a* $[p,\boldsymbol{\kappa}]$*-DI problem cannot have a minimal period* P *which has a factor in common with* p.

Proof. Assume that $P=p_1 l$ and $p=p_1 m$, with $p_1\neq 1$. According to Theorem 3, $p_1 l$ must divide $p_1 m[\kappa_{\mu+1}-\kappa_\mu]-1$, for all $\mu\in[|p|-1]$., i.e. $p_1 m[\kappa_{\mu-1}-\kappa_\mu]-1=p_1 lz$, $l\in\mathbb{Z}$, or $p_1[m(\kappa_{\mu-1}-\kappa_\mu)-lz]=1$. This implies that the product of two integers, one being different from 1, should equal 1. As this is impossible, the starting assumption cannot be true, and so, P and p cannot have a common factor. □

5. Automaticity of DI-sequences

A one-dimensional unidirectional sequence $(a(k))_{k\in\mathbb{N}}$ is said to be p-automatic, when $a(k)$ can be obtained from a finite automaton whose input is the p-adic representation of k. It is known [10, 9] that p-automaticity is equivalent to the sequence being generated by a p-substitution system, and to finiteness of the set of all possible p^n-decimations ($n\in\mathbb{N}$) of $(a(k))_{k\in\mathbb{N}}$ (the so-called *kernel*).

In this section, we will prove automaticity of $[p,\boldsymbol{\kappa}]$-DI sequences by explicit construction of the kernel.

Definition 10. Let $\underline{a}=(a(k))_{k\in\mathbb{Z}}$ be a bidirectionally infinite sequence, then
$\mathrm{R}\underline{a}\overset{\text{def}}{=}(a(k))_{k\in\mathbb{N}}$ is the righ-half infinite part of $\underline{a}$,
$\mathrm{L}\underline{a}\overset{\text{def}}{=}(a(-k))_{k\in\mathbb{N}}=\mathrm{R}\underline{a}^-$ is the left-half infinite part of $\underline{a}$.

Let $\underline{a}$ be a $[p,\boldsymbol{\kappa}]$-DI-sequence. This section considers the $|p|$-automaticity of the sequences $\mathrm{R}\underline{a}$ and $\mathrm{L}\underline{a}$. For an explicit determination of the $|p|$-kernel, we will have to distinguish between direct and reverse DI-sequences, for reasons that are similar to those appearing in the solution procedure of the DI-equations.

Applying (1) consecutively, one obtains

Lemma 8.

$$D_p^{\mu_{n-1}} D_p^{\mu_{n-2}} \dots D_p^{\mu_1} D_p^{\mu_0} \underline{a}(k) = a(kp^n + \mu_{n-1}p^{n-1} + \mu_{n-2}p^{n-2} + \dots + \mu_1 p + \mu_0). \tag{25}$$

Remark 6. When $p \geqslant 2$, the right-hand side of (25) also equals $D_{p^n}^m$, with $m = \mu_{n-1}p^{n-1} + \mu_{n-2}p^{n-2} + \dots + \mu_0 \in [p^n]$ (the string $\mu_{n-1} \dots \mu_0$ being the n-digit p-adic representation of m). It will sometimes be convenient to represent $D_{p^n}^m$ as $D_{p^n}^{\mu_{n-1}\dots\mu_0}$.

5.1. *p-automaticity of direct* $[p, \boldsymbol{\kappa}]$*-DI sequences* $(p \geqslant 2)$

Definition 11. Let $\underline{a}$ be a bidirectional sequence, then its *full* (*direct*) *p-kernel*, denoted by $\mathrm{FK}_p(\underline{a})$, is the set consisting of $\underline{a}$ itself and of all its p^n-decimations (see [2, Section 2.2]):

$$\mathrm{FK}_p(\underline{a}) = \{\underline{a}, D_{p^n}^m \underline{a} \mid n \in \mathbb{N}_0, m \in [p^n]\}. \tag{26}$$

Its *full reverse kernel*, denoted by $\mathrm{FK}_p^-(\underline{a})$, is given by

$$\mathrm{FK}_p^-(\underline{a}) = \mathrm{FK}_p(\underline{a}^-) = \{\underline{a}^-, D_{p^n}^m \underline{a}^- \mid n \in \mathbb{N}_0, m \in [p^n]\}. \tag{27}$$

Remark 7. Recall Theorem 2, stating that when $\underline{a}$ is a $[p, \boldsymbol{\kappa}]$-DI solution, then $\underline{a}^-$ is a solution of a related $[p, \boldsymbol{\kappa}^-]$-DI problem. Because of this and because $\mathrm{FK}_p^-(\underline{a}) = \mathrm{FK}_p(\underline{a}^-)$, considering the reverse kernel in the $[p, \boldsymbol{\kappa}]$-DI problem is equivalent to considering the direct kernel in the related $[p, \boldsymbol{\kappa}^-]$-DI problem. The implication of Theorem 2 is that it suffices to study the properties concerning direct kernels (and associated automata).

5.1.1. *Finding* $\mathrm{FK}_p(\underline{a})$:

Let $s \in \{0, 1, \dots, p-1\}^* = [p]^*$ be a string of p-adic digits and $\mu \in [p]$, then μs represents the string obtained from s by extending it to the left with μ. ϕ represents the empty string, so $\mu\phi = \mu$.

Theorem 4 $(p \geqslant 2)$. *Let* $\underline{a}$ *be a* $[p, \boldsymbol{\kappa}]$*-DI sequence. Then*

$$D_{p^n}^m \underline{a} = (a(k + a_{\mu_{n-1}\mu_{n-2}\dots\mu_0}))_{k \in \mathbb{Z}}, \tag{28}$$

where $\mu_{n-1}\mu_{n-2}\dots\mu_0$ *is the n-digit p-adic representation of m, and where* $a_{\mu_{n-1}\mu_{n-2}\dots\mu_0} \in \mathbb{Z}$ *is obtained from the recursion*

$$a_{\mu_j s} = F_{\mu_j}(a_s) \stackrel{\text{def}}{=} \imath\frac{a_s + \mu_j}{p} + \kappa_{(a_s + \mu_j) \bmod p}, \tag{29}$$

starting with $a_\phi = 0$.

Proof (By induction)**.** (A) Let $s = \mu_{j-1}\mu_{j-2}\dots\mu_0$ and suppose that (28) is satisfied for $n = j$, that is $D^s_{p^j}\underline{a} = (a(k + a_s))_{k\in\mathbb{Z}} \overset{\text{def}}{=} (a'(k))_{k\in\mathbb{Z}}$. Then (28) is also satisfied for $n = j + 1$, where $a_{\mu_j s}$ is related to a_s through (29).

Indeed,

$$
\begin{aligned}
D^{\mu_j s}_{p^{j+1}}\underline{a} = D^{\mu_j}_{p}\underline{a}' &\overset{(1)}{=} (a'(kp + \mu_j))_{k\in\mathbb{Z}} \\
&= (a(kp + \mu_j + a_s))_{k\in\mathbb{Z}} \\
&= \left(a\left(\left[k + \imath\frac{a_s + \mu_j}{p} \right] p + (a_s + \mu_j) \bmod p \right) \right)_{k\in\mathbb{Z}} \\
&\overset{(4)}{=} \left(a\left(k + \imath\frac{a_s + \mu_j}{p} + \kappa_{(a_s+\mu_j) \bmod p} \right) \right)_{k\in\mathbb{Z}}.
\end{aligned}
$$

Using (29) transforms this into

$$D^{\mu_j s}_{p^{j+1}}\underline{a} = (a(k + a_{\mu_j s}))_{k\in\mathbb{Z}}. \tag{30}$$

(B) Eq. (28) is obtained by starting the induction from $s = 0$, with $a_s = a_\phi = 0$. Indeed, this yields

$$D^{\mu_0}_{p}\underline{a} = (a(k + a_{\mu_0}))_{k\in\mathbb{Z}}, \quad \text{with } a_{\mu_0} = \kappa_{\mu_0}$$

which is seen to be correct from comparison with the DI-equations (3, 4). □

Corollary 2. *The kernel* $\mathrm{FK}_p(\underline{a})$ *of a* $[p, \boldsymbol{\kappa}]$*-DI sequence* $\underline{a}$ *consists of the following shifted versions of* $\underline{a}$:

$$\mathrm{FK}_p(\underline{a}) = \{(a(k + a_s))_{k\in\mathbb{Z}} \mid s \in [p]^*\}, \tag{31}$$

where $a_s \in \mathbb{Z}$ *is generated through recursion* (29), *and this for all* p*-adic strings* s (*including the empty string* ϕ).

Remark 8. By induction with respect to the length $|s|$ of the word s, it can be proven that $|a_s| \leqslant 1 + (p/(p-1))\max_\mu |\kappa_\mu|$. This implies that the full kernel is *finite*. We shall now derive the exact information concerning the a_s-values appearing in (31).

In Theorem 4, the following discrete mappings from $\mathbb{Z} \times [p]$ into $\mathbb{Z}$ were defined

$$\forall \mu \in [p]: \quad F_\mu : a_s \mapsto a_{\mu s} = \imath\frac{a_s + \mu}{p} + \kappa_{(a_s+\mu) \bmod p}. \tag{32}$$

These mappings can be graphically represented in an $(a_s, a_{\mu s})$-diagram. From the structure of this diagram, and from the fact that the initial condition $a_\phi = 0$, it will follow that the finite set $\{a_s \mid s \in [p]^*\}$ appearing in (31) can be constructed easily. This will be illustrated now, using a generic example that leads to the statement of some general results.

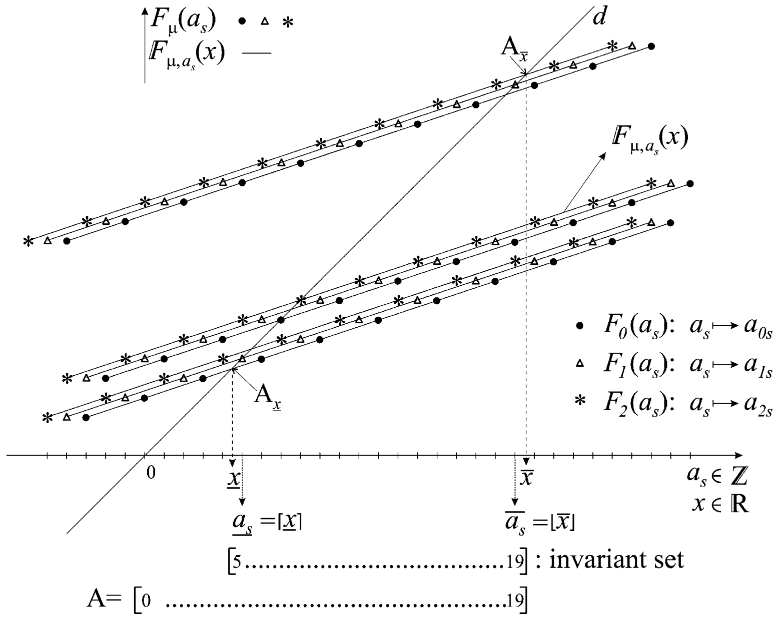

Fig. 4. The $F_\mu(a_s)$- and $\mathbb{F}_{\mu,a_s}(x)$-diagrams for the DI-problem with $p=3$, $\boldsymbol{\kappa}=[3,5,13]$.

Example 3. Take $p=3$, $\boldsymbol{\kappa}=[3,5,13]$. The corresponding $F_\mu(a_s)$-diagrams are shown in Fig. 4.

It is easily derived from (50), when a_s is restricted to $[p]$ (μ always being restricted to $[p]$), that

$$F_\mu(a_s) = \begin{cases} \kappa_{\mu+a_s} & \text{when } \mu + a_s < p, \\ 1 + \kappa_{\mu+a_s-p} & \text{when } \mu + a_s \geqslant p. \end{cases} \tag{33}$$

It also follows from (32) that

$$\forall l \in \mathbb{Z}: \quad F_\mu(a_s + pl) = F_\mu(a_s) + l. \tag{34}$$

Fix $a_s \in [p]$, $\mu \in [p]$, then the last equality implies that all points $\{(a_s + pl, F_\mu(a_s + pl) \mid l \in \mathbb{Z}\}$ in the $(a_s, F_\mu(a_s))$-diagram are *collinear*. The line connecting these points is the graph of the function

$$\mathbb{F}_{\mu,a_s}(x) \stackrel{\text{def}}{=} \frac{F_\mu(a_s + p) - F_\mu(a_s)}{p}(x - a_s) + F_\mu(a_s) \quad (x, \mathbb{F}_{\mu,a_s}(x) \in \mathbb{R}),$$

or, using (34)

$$\mathbb{F}_{\mu,a_s}(x) = \frac{x - a_s}{p} + F_\mu(a_s). \tag{35}$$

There are p^2 such lines: one for each couple $(a_s, \mu) \in [p]^2$. Some of these lines may coincide.

Referring to Fig. 4 again, let $A_{\bar{x}}$ and $A_{\underline{x}}$ be the intersection points of the highest and lowest of these lines with the diagonal line d. $\underline{x}$ and $\bar{x}$ are the respective abscissas. Then it is easily found that

$$\bar{x} = \max_{(\mu, a_s) \in [p]^2} \frac{pF_\mu(a_s) - a_s}{p-1},$$

$$\underline{x} = \min_{(\mu, a_s) \in [p]^2} \frac{pF_\mu(a_s) - a_s}{p-1}.$$

Let

$$\bar{a}_s = \lfloor \bar{x} \rfloor \quad \text{and} \quad \underline{a}_s = \lceil \underline{x} \rceil, \tag{36}$$

then it is clear from the graph that when $a_s \in I[\underline{a}_s, \bar{a}_s]$, then $F_\mu(a_s) \in I[\underline{a}_s, \bar{a}_s]$ (i.e. the integer set $\{\underline{a}_s, \underline{a}_s + 1, \dots, \bar{a}_s\}$). Or $I[\underline{a}_s, \bar{a}_s]$ is *invariant* under the mappings F_μ. According to the previous development, we have to apply these mappings recursively over all possible μ-strings, starting from $a_s = 0$. Three different possibilities have to be distinguished:

1. $0 < \underline{a}_s$ (like in the example). As $F_\mu(a_s) > a_s$ when $a_s < \underline{a}_s$, a_s must eventually enter the invariant set $I[\underline{a}_s, \bar{a}_s]$ and stay there
2. $0 \in I[\underline{a}_s, \bar{a}_s] : a_s$ never leaves the invariant set
3. $0 > \bar{a}_s : a_s$ decreases until it enters the invariant set and stays there.

As an intermediate conclusion, we are now able to formulate the following lemma.

Lemma 9. *With* $\underline{a}_s$ *and* $\bar{a}_s$ *defined in* (36), *the range of* a_s *in the kernel* $\mathrm{FK}_p(\underline{a})$ *of the DI-sequence* $\underline{a}$ *as given by* (31) *is restricted to the finite set*

$$\begin{aligned} \mathsf{A} &= I[0, \bar{a}_s] \quad \text{when } 0 < \underline{a}_s \\ &= I[\underline{a}_s, 0] \quad \text{when } 0 > \bar{a}_s \\ &= I[\underline{a}_s, \bar{a}_s] \quad \text{when } 0 \in I[\underline{a}_s, \bar{a}_s]. \end{aligned} \tag{37}$$

For Example 3, it is easily seen from the diagram in Fig. 4 that $\underline{a}_s = 5$, $\bar{a}_s = 19$, and so $\mathsf{A} = I[0, 19]$. The transition table $a_s \overset{\mu}{\mapsto} a_{\mu s} = F_\mu(a_s)$ is

	$a_s \in \mathsf{A}$																			
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
μ 0	3	5	13	4	6	14	5	7	15	6	8	16	7	9	17	8	10	18	9	11
1	5	13	4	6	14	5	7	15	6	8	16	7	9	17	8	10	18	9	11	19
2	13	4	6	14	5	7	15	6	8	16	7	9	17	8	10	18	9	11	19	10

Observe that the elements 1, 2, 12 do not appear inside the transition table: $a_{\mu s}$ never takes these values when starting from $a_s = 0$. So, not all elements in A are *reachable* from $a_s = 0$. Which elements are reachable or not can be derived from inspection of the table.

Definition 12. An *unreachable* element in A is any element that is not reached under the mappings $F_\mu(a_s)$ when starting from $a_s = 0$. Call this set of unreachable elements $\mathsf{A}_{\text{nreach}}$. The *reachable* set (from $a_s = 0$) is $\mathsf{A}_{\text{reach}} = \mathsf{A} \backslash \mathsf{A}_{\text{nreach}}$.

As a consequence of Lemma 9, and using the concept of the set of reachable states, we formulate

Proposition 1. *The* $\mathrm{FK}_p(\underline{a})$*-kernel of a* $[p, \boldsymbol{\kappa}]$*-DI sequence* $\underline{a}$ *is the following finite set of shifted versions of* $\underline{a}$:

$$\mathrm{FK}_p(\underline{a}) = \{(a(k + a_s))_{k \in \mathbb{Z}} \mid a_s \in \mathsf{A}_{\text{reach}}\}. \tag{38}$$

Definition 13. $\mathsf{A}_{\text{reach}}$ will be called the *formal kernel-defining shift-set*.

Remark 9. $I[\underline{a}_s, \bar{a}_s]$, A, $\mathsf{A}_{\text{reach}}$, $\mathsf{A}_{\text{nreach}}$ depend on p and $\boldsymbol{\kappa}$ in the DI-problem, and *not* on the particular solution $\underline{a}$ itself (hence the adjective "formal" in the previous definition). Dependent on the solution however, it might happen, for $a_s \neq a_{s'}$, that $(a(k + a_s))_{k \in \mathbb{Z}} \equiv (a(k + a_{s'}))_{k \in \mathbb{Z}}$.

Definition 14. The equivalence relation R over $\mathsf{A}_{\text{reach}}$ which makes

$$a_s \overset{R}{\cong} a_{s'} \text{ iff } (a(k + a_s))_{k \in \mathbb{Z}} \equiv (a(k + a_{s'}))_{k \in \mathbb{Z}}, \tag{39}$$

defines the set of equivalence classes $\mathsf{A}_{\text{reach}}/R$, and depends on the particular solution considered. Define $\mathsf{A}_{\text{red}} \subseteq \mathsf{A}_{\text{reach}}$ as a set of representatives of these equivalence classes, and call it the *reduced kernel-defining shift-set*. Then

$$\mathrm{FK}_p(\underline{a}) = \{(a(k + a_s))_{k \in \mathbb{Z}} \mid a_s \in \mathsf{A}_{\text{red}}\}. \tag{40}$$

Remark 10. The original transition table $a_s \mapsto F_\mu(a_s)$, with $a_s \in \mathsf{A}_{\text{reach}}$, reduces in an obvious way to a smaller table for a_s restricted to A_{red} under this equivalence relation.

5.1.2. Deriving the p*-automaton for* $\mathrm{R}\underline{a}$:

We recall first that a sequence $\underline{b} : \mathbb{N} \to V$ is generated by a *reverse-reading* p-automaton $\mathscr{A} = (\mathsf{A}, s_0, T, V, \tau)$ with states $s_i \in \mathsf{A}$, initial state s_0, state transition function $T : \mathsf{A} \times V \to \mathsf{A}$, output function τ: $\mathsf{A} \to V$; if $\underline{b}(m)$, $m \in \mathbb{N}$ is found as output of the automaton starting in state s_0 and state transitions determined by the sequence $\mu_0, \mu_1, \ldots, \mu_{n-1}$ where $\mu_{n-1}\mu_{n-2}\ldots\mu_0$ is the p-adic representation of m with $\mu_{n-1} \neq 0$ (see, for example, [1]).

Theorem 5. (A) *The formal automaton* $\mathscr{A}_{\text{reach}} = (\mathsf{A}_{\text{reach}}, 0, F_\mu, V, \tau)$ *with states* $a_s \in \mathsf{A}_{\text{reach}}$, *initial state* 0, *state transition function* F_μ: $a_s \mapsto a_{\mu s}$ *for all* $\mu \in [p]$, *output map* $\tau : \mathsf{A}_{\text{reach}} \to V : a_s \mapsto \underline{a}(a_s)$ *is a reverse-reading* p-*automaton for* R$\underline{a}$.

(B) *The reduced automaton* $\mathscr{A}_{\text{red}} = (\mathsf{A}_{\text{red}}, 0, F_\mu, V, \tau)$ *with reduced state set* A_{red} (*which depends on the specific DI-solution considered*), *is also a* p-*automaton for* R$\underline{a}$.

Proof. (A) (a). Observe that $\underline{a}(0) = a(a_\phi)$, and so the automaton starts in the right state.

(b). Let $m > 0$ and $m = \mu_{n-1} p^{n-1} + \mu_{n-2} p^{n-2} + \cdots + \mu_0$, $(\mu_{n-1} \neq 0)$. Then

$$a(m) = D^m_{p^n} \underline{a}(0) \overset{(28)}{=} a(a_{\mu_{n-1}\mu_{n-2}\cdots\mu_0}) \overset{(29)}{=} a(F_{\mu_{n-1}}(\ldots F_{\mu_1}(F_{\mu_0}(a_\phi))\ldots))$$

and so $a(m)$ is indeed generated from the given automaton with input sequence $\mu_0, \mu_1, \ldots, \mu_{n-1}$.

(B) Is a direct consequence of (A) and of the equivalency definition (39). □

Proposition 2. *For all nonperiodic solutions to a* $[p, \boldsymbol{\kappa}]$-*DI problem, the formal and reduced kernel-defining shift sets and the corresponding automata are equal* (*or*: *the formal kernel and the formal automaton are irreducible*):

$$\mathsf{A}_{\text{reach}} = \mathsf{A}_{\text{red}}, \quad \mathscr{A}_{\text{reach}} = \mathscr{A}_{\text{red}}.$$

Proof. The condition for effective reducibility of $\mathsf{A}_{\text{reach}}$ and A_{red} is that there are different a_s ans $a_{s'}$ such that $(a(k + a_s))_{k \in \mathbb{Z}} \equiv (a(k + a_{s'}))_{k \in \mathbb{Z}}$ (see Remark 8). This is clearly a periodicity condition for $\underline{a}$, implying a period length $(a_s - a_{s'})$. So $\mathsf{A}_{\text{reach}}$ is not reducible to a smaller A_{red}-set for nonperiodic solutions. □

As a consequence of the periodicity Theorem 3 and the previous Proposition 2, it is possible to formulate

Corollary 3. *When* $gcd\{p(\kappa_{j+1} - \kappa_j) - 1 \mid j \in [p-1]\} = 1$, *then* $\mathsf{A}_{\text{reach}} = \mathsf{A}_{\text{red}}$ *and* $\mathscr{A}_{\text{reach}} = \mathscr{A}_{\text{red}}$ *for any nontrivial* (*i.e. nonconstant*) *solution. For the trivial solutions*, $\mathsf{A}_{\text{red}} = \{0\}$.

Remark 11. Because of the isomorphism between $a_s \overset{\mu}{\mapsto} a_{\mu s} = F_\mu(a_s)$ and $(a(k + a_s))_{k \in \mathbb{Z}} \overset{\mu}{\mapsto} (a(k + a_{\mu s}))_{k \in \mathbb{Z}} \overset{(28)}{=} D^\mu_p (a(k + a_s))_{k \in \mathbb{Z}}$, one could as well consider $(a(k + a_s))_{k \in \mathbb{Z}}$, $a_s \in \mathsf{A}_{\text{reach}}$ (or $a_s \in \mathsf{A}_{\text{red}}$) as states of the formal or reduced automaton of R$\underline{a}$, while the state transition under μ is defined as the D^μ_p-decimation.

Until now, we have related the automaticity of the unidirectional sequence R$\underline{a}$, in which $\underline{a}$ is a bidirectional $[p, k]$-DI sequence, to the so-called full p-kernel of $\underline{a}$. In doing so, we obtained the related automaton $\mathscr{A}_{\text{red}}$. Is this the automaton with a minimal number of states? Note that, traditionally (see e.g., [1]), automaticity of R$\underline{a}$ is related to a kernel of decimated sequences involving only R$\underline{a}$ itself (to be called standard kernel in the present context). Therefore, it is important to investigate the relationship between the automata based on both the full and the standard kernel. This will answer the minimality question.

Definition 15.

- Let $\mathrm{SK}_p(\underline{a})$ denote the *standard* p-kernel of $\mathrm{R}[\mathrm{FK}_p(\underline{a})]$, that is: $\mathrm{SK}_p(\underline{a}) = \{\mathrm{R}\underline{a}, D^m_{p^n}(\mathrm{R}\underline{a}) \mid n \in \mathbb{N}_0,\ m \in [p^n]\}$. (see [1]).
- Let $\mathrm{R}[\mathrm{FK}_p(\underline{a})] \overset{\mathrm{def}}{=} \mathrm{R}\{\underline{c} \mid \underline{c} \in \mathrm{FK}_p(\underline{a})\}$ (i.e. the set of right infinite parts of all sequences in $\mathrm{FK}_p(\underline{a})$, starting from $k = 0$).

Proposition 3. $\mathrm{SK}_p(\mathrm{R}\underline{a}) = \mathrm{R}[\mathrm{FK}_p(\underline{a})] = \{(a(k + a_s))_{k\in\mathbb{N}} \mid a_s \in \mathsf{A}_{\mathrm{red}}\}$.

Proof. Observe that, because $m \geqslant 0$, $D^m_{p^n}(\mathrm{R}\underline{a}) = \mathrm{R}[D^m_{p^n}(\underline{a})]$. $\square$

Remark 12. $\mathsf{A}_{\mathrm{red}}$ need not be the minimal standard kernel-defining shift-set, as $(a(k + a_s))_{k\in\mathbb{N}}$ might be identical to $(a(k + a_{s'}))_{k\in\mathbb{N}}$, even when $(a(k + a_s))_{k\in\mathbb{Z}} \not\equiv (a(k + a_{s'}))_{k\in\mathbb{Z}}$. But again, by constructing the set of equivalence classes over $\mathsf{A}_{\mathrm{red}}$, such that $a_s \cong a_{s'}$ whenever $(a(k + a_s))_{k\in\mathbb{N}} \equiv (a(k + a_{s'}))_{k\in\mathbb{N}}$, $(a_s, a_{s'} \in \mathsf{A}_{\mathrm{red}})$, $\mathsf{A}_{\mathrm{red}}$ reduces to $\mathsf{A}_{\mathrm{min}}$, the *minimal* kernel-defining shift-set for the standard kernel $\mathrm{SK}_p(\mathrm{R}\underline{a})$. The transition table for $a_s \overset{\mu}{\mapsto} a_{\mu s} = F_\mu(a_s)$ becomes a minimal one under this equivalence relation, and the corresponding p-automaton $\mathscr{A}_{\mathrm{min}} = \{\mathsf{A}_{\mathrm{min}}, 0, F_\mu, V, \tau\}$ is the *minimal* p-automaton for R$\underline{a}$.

Proposition 4. *For any solution of a* $[p, \boldsymbol{\kappa}]$*-DI problem which is not periodic to the right, it holds that* $\mathsf{A}_{\mathrm{min}} = \mathsf{A}_{\mathrm{red}} = \mathsf{A}_{\mathrm{reach}}$, and $\mathscr{A}_{\mathrm{min}} = \mathscr{A}_{\mathrm{red}} = \mathscr{A}_{\mathrm{reach}}$.

Proof. The condition for reducibility of $\mathsf{A}_{\mathrm{red}}$ is that there are different a_s and $a_{s'} \in \mathsf{A}_{\mathrm{red}}$ such that $(a(k + a_s))_{k\in\mathbb{N}} = a(k + a_{s'}))_{k\in\mathbb{N}}$. This implies periodicity to the right of $\underline{a}$, a condition which is not satisfied. (The equality of $\mathsf{A}_{\mathrm{reach}}$ and $\mathsf{A}_{\mathrm{red}}$ was already established in Proposition 2). $\square$

As a consequence of the periodicity Theorem 3, which can be shown to hold also for ultimate periodicity to the right (or to the left), it is possible to formulate

Corollary 4. *When* $\gcd\{p(\kappa_{j+1} - \kappa_j) - 1 \mid j \in [p-1]\} = 1$, *the minimal kernel-defining shift set and the minimal* p*-automaton for nontrivial* R$\underline{a}$ *coincide with the formal kernel-defining shift set and with the formal* p*-automaton*:

$$\mathsf{A}_{\mathrm{min}} = \mathsf{A}_{\mathrm{reach}}, \quad \text{and} \quad \mathscr{A}_{\mathrm{min}} = \mathscr{A}_{\mathrm{reach}}.$$

5.2. $|p|$-automaticity of reverse $[-|p|,\boldsymbol{\kappa}]$-DI sequences ($p=-|p|\leqslant -2$)

The following theorem is the counterpart of Theorem 4 for direct DI-sequences (its proof proceeds along the same lines as the one for Theorem 4).

Theorem 6 ($p\leqslant -2$). *Let $\underline{a}$ be a $[-|p|,\boldsymbol{\kappa}]$-DI sequence. Then*

$$D^m_{|p|^n}\underline{a}=(a((-1)^n k+a_{\mu_{n-1}\mu_{n-2}\cdots\mu_0}))_{k\in\mathbb{Z}}, \tag{41}$$

where $\mu_{n-1}\mu_{n-2}\cdots\mu_0$ is the n-digit $|p|$-adic representation of m, and where $a_{\mu_{n-1}\mu_{n-2}\cdots\mu_0}\in\mathbb{Z}$ is obtained from the recursion

$$a_{\mu_j s}=G^j_{\mu_j}(a_s)\stackrel{\text{def}}{=}\imath\frac{a_s+(-1)^j\mu_j}{p}+\kappa_{(a_s+(-1)^j\mu_j)\bmod |p|}, \tag{42}$$

starting with $a_\phi=0$.

Corollary 5. *The kernel $\mathrm{FK}_{|p|}(\underline{a})$ of a $[-|p|,\boldsymbol{\kappa}]$-DI sequence $\underline{a}$ consists of the following shifted versions of $\underline{a}$:*

$$\mathrm{FK}_{|p|}(\underline{a})=\{(a((-1)^{|s|}k+a_s))_{k\in\mathbb{Z}}\mid s\in[|p|]^*\}, \tag{43}$$

where a_s is generated through recursion (42), and this for all $|p|$-adic strings s (including the empty string). $|s|$ represents the length of string s).

Remark 13. By induction with respect to the length $|s|$ of the string s, it can be proven that $|a_s|\leqslant 1+[|p|/(|p|-1)]\max_\mu|\kappa_\mu|$, which again implies that the full-kernel is finite (cf. Remark 9).

Just like in the case of direct DI-sequences (Section 5.1), we will derive the kernel-defining shift-sets $\mathsf{A}_{\text{reach}}$, A_{red}, A_{min} and their associated $|p|$-automata for reverse $[-|p|,\boldsymbol{\kappa}]$-DI sequences from considerations concerning the $(a_s, G^j_{\mu_j}(a_s))$-diagrams defined in (42). Because j affects $G^j_{\mu_j}(a_s)$ only through $(-1)^j$, there is only need to distinguish between the two cases in which j is either even or odd. When the value a_s is obtained from a string with length $|s|=$ even, we denote this value by a_s^+, otherwise by a_s^-. The following $|p|\times 2$ mappings then replace the $G^j_{\mu_j}(a_s)$-mappings in (42):

$$G^+_\mu:\; a_s^+\mapsto a_{\mu s}^-=\imath\frac{a_s^++\mu_j}{p}+\kappa_{(a_s^++\mu_j)\bmod |p|} \tag{44}$$

and

$$G^-_\mu:\; a_s^-\mapsto a_{\mu s}^+=\imath\frac{a_s^--\mu_j}{p}+\kappa_{(a_s^--\mu_j)\bmod |p|}. \tag{45}$$

These mappings have to be applied *recursively and alternatively*, starting with $a_\phi^+=0^+$ ($|\phi|=0=$ even, so we start with a $+$-sign).

Again, we proceed with a generic example.

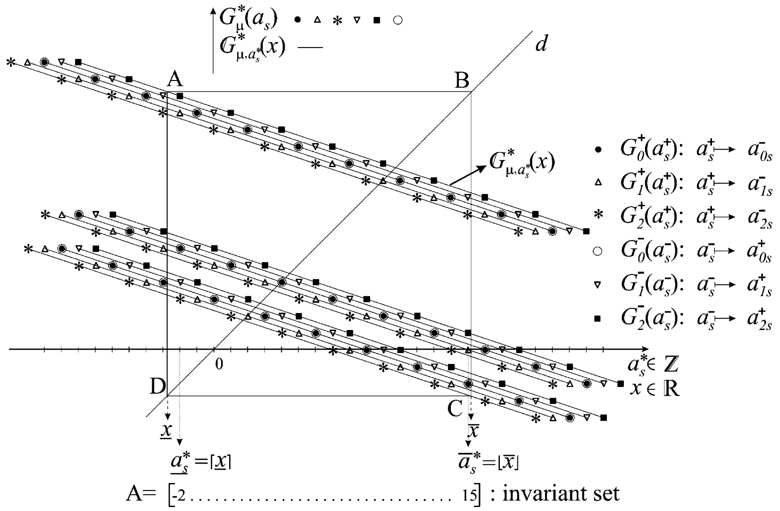

Fig. 5. The $G^*_\mu(a_s)$- and $\mathbb{G}^*_{\mu,a_s^*}(x)$-diagrams for the reverse DI-problem with $p = -3$, $\boldsymbol{\kappa} = [3,5,13]$.

Example 4. Take the counterpart of the problem introduced in Example 3: $p = -3$, $\boldsymbol{\kappa} = [3,5,13]$. Fig. 5 shows the discrete $G^*_\mu(a_s^*)$-mappings, and their $\mathbb{R}$-extensions $\mathbb{G}^*_{\mu,a_s^*}(x)$ $(* \in \{-,+\}, \mu \in [|p|], a_s^* \in [|p|])$.

For a_s^* restricted to $[|p|]$, it follows from (44,45) that

$$G^+_\mu(a_s^+) = \begin{cases} \kappa_{a_s^+ + \mu} & \text{when } a_s^+ + \mu < |p|, \\ -1 + \kappa_{a_s^+ + \mu - |p|} & \text{when } a_s^+ + \mu \geqslant |p| \end{cases} \tag{46}$$

and

$$G^-_\mu(a_s^-) = \begin{cases} \kappa_{a_s^- - \mu} & \text{when } a_s^- - \mu \geqslant 0, \\ 1 + \kappa_{a_s^- - \mu + |p|} & \text{when } a_s^- - \mu < 0. \end{cases} \tag{47}$$

As it also follows from (44,45) that

$$\forall l \in \mathbb{Z},\ \forall * \in \{-,+\}:\ G^*_\mu(a_s^* + |p| l) = G^*_\mu(a_s^*) - l, \tag{48}$$

one finds that the lines connecting all points $\{(a_s^*, G^*_\mu(a_s^*)) \,|\, (\mu, *) \text{ fixed}\}$ satisfy

$$\mathbb{G}^*_{\mu,a_s^*}(x) = \frac{x - a_s^*}{-|p|} + G^*_\mu(a_s^*). \tag{49}$$

There are $2p^2$ such lines: one for each triple $(*, a_s^*, \mu) \in \{-,+\} \times [|p|]^2$.

Let $(\bar{*}, \overline{a_s^*}, \bar{\mu})$ and $(\underline{*}, \underline{a_s^*}, \underline{\mu})$ be the $(*, a_s^*, \mu)$-triples corresponding to the highest and lowest of these lines, i.e.

$$(\bar{*}, \overline{a_s^*}, \bar{\mu}) = \arg\max \left[\frac{a_s^*}{|p|} + G^*_\mu(a_s^*) \right], \tag{50}$$

$$(\underline{*}, \underline{a_s^*}, \underline{\mu}) = \arg\min \left[\frac{a_s^*}{|p|} + G_\mu^*(a_s^*) \right]. \tag{51}$$

Just like in Section 3.2 where the procedure for finding the core of the solution of reverse DI-sequences was discussed, it is possible to find a square ABCD (see Fig. 5), with vertices A and C on the $\mathbb{G}^{\bar{*}}_{\bar{\mu},\overline{a_s^*}}(x)$ and $\mathbb{G}^{\underline{*}}_{\underline{\mu},\underline{a_s^*}}(x)$-lines respectively, and with B and D on the diagonal line d.

With $\underline{x}$ and $\bar{x}$ the x-coordinates of A, D and B, C respectively, $\underline{x}$ and $\bar{x}$ satisfy

$$\underline{x} = \mathbb{G}^{\underline{*}}_{\underline{\mu},\underline{a_s^*}}(\bar{x}), \quad \bar{x} = \mathbb{G}^{\bar{*}}_{\bar{\mu},\overline{a_s^*}}(\underline{x}). \tag{52}$$

Let

$$\underline{a}_s^* = \lceil \underline{x} \rceil \quad \text{and} \quad \bar{a}_s^* = \lfloor \bar{x} \rfloor, \tag{53}$$

then $I[\underline{a}_s^*, \bar{a}_s^*]$ is an *attracting invariant set* under the alternating recursion (44), (45) (same arguments as developed in Section 3.2).

As the recursion has to start with $a_\phi^+ = 0^+$, this leads again to a lemma, which is actually similar to Lemma 9:

Lemma 10. *With $\underline{a}_s^*$ and $\bar{a}_s^*$ defined in* (53), (52), *the range of a_s in the kernel* $\mathrm{FK}_p(\underline{a})$ *of a* $[-|p|, \boldsymbol{\kappa}]$*-DI sequence $\underline{a}$ as given by* (43) *is restricted to the finite set*

$$\begin{aligned} \mathsf{A} &= I[0, \bar{a}_s^*] \quad \text{when } 0 < \underline{a}_s^* \\ &= I[\underline{a}_s^*, 0] \quad \text{when } 0 > \bar{a}_s^* \\ &= I[\underline{a}_s^*, \bar{a}_s^*] \quad \text{when } 0 \in I[\underline{a}_s^*, \bar{a}_s^*]. \end{aligned} \tag{54}$$

Example 4 (*Continued*)**.** Continuing the previous example ($p = -3, \boldsymbol{\kappa} = [3, 5, 13]$), it is readily seen from Fig. 5 that the top line corresponds to

$$\mathbb{G}^{\bar{*}}_{\bar{\mu},\overline{a_s^*}}(x) = \mathbb{G}^{-}_{\bar{2},\bar{1}}(x) = \frac{x-1}{-3} + 14,$$

$$\mathbb{G}^{\underline{*}}_{\underline{\mu},\underline{a_s^*}}(x) = \mathbb{G}^{-}_{\underline{2},\underline{1}}(x) = \frac{x-1}{-3} + 2.$$

Solving $\bar{x}$ and $\underline{x}$ from (52) gives $\bar{x} = 15.25$, $\underline{x} = -2.75$, and so the invariant set is $I[-2, 15]$, implying that $\mathsf{A} = I[-2, 15]$ too.

The transition table for $G_\mu^+(a_s^+): a_s^+ \mapsto a_{\mu s}^-$ and $G_\mu^-(a_s^-): a_s^- \mapsto a_{\mu s}^+$ restricted to $a_s \in \mathsf{A}$ is

		$a_s^* \mid a_s \in \mathsf{A}$										
	-2^-	-2^+	-1^-	-1^+	0^-	0^+	1^-	1^+	2^-	2^+	3^-	3^+
0	6^+	6^-	14^+	14^-	3^+	3^-	5^+	5^-	13^+	13^-	2^+	2^-
μ 1	4^+	14^-	6^+	3^-	14^+	5^-	3^+	13^-	5^+	2^-	13^+	4^-
2	15^+	3^-	4^+	5^-	6^+	13^-	14^+	2^-	3^+	4^-	5^+	12^-

	$a_s^* \mid a_s \in \mathsf{A}$											
	4^-	4^+	5^-	5^+	6^-	6^+	7^-	7^+	8^-	8^+	9^-	9^+
0	4^+	4^-	12^+	12^-	1^+	1^-	3^+	3^-	11^+	11^-	0^+	0^-
μ 1	2^+	12^-	4^+	1^-	12^+	3^-	1^+	11^-	3^+	0^-	11^+	2^-
2	13^+	1^-	2^+	3^-	4^+	11^-	12^+	0^-	1^+	2^-	3^+	10^-

	$a_s^* \mid a_s \in \mathsf{A}$											
	10^-	10^+	11^-	11^+	12^-	12^+	13^-	13^+	14^-	14^+	15^-	15^+
0	2^+	2^-	10^+	10^-	-1^+	-1^-	1^+	1^-	9^+	9^-	-2^+	-2^-
μ 1	0^+	10^-	2^+	-1^-	10^+	1^-	-1^+	9^-	1^+	-2^-	9^+	0^-
2	11^+	-1^-	0^+	1^-	2^+	9^-	10^+	-2^-	-1^+	0^-	1^+	8^-

Actually, this is a transition table from $\mathsf{A}^* \stackrel{\mathsf{def}}{=} \{a_s^+, a_s^- \mid a_s \in \mathsf{A}\}$ into A^* itself. Inspection of this table shows that here also, there is a set of elements that cannot be reached from $0^+ : \mathsf{A}^*_{\mathrm{nreach}} = \{-2^+, 6^-, 7^-, 7^+, 8^+, 15^-\}$. The set of reachable elements is $\mathsf{A}^*_{\mathrm{reach}} = \mathsf{A}^* \backslash \mathsf{A}^*_{\mathrm{nreach}}$. This set can be partitioned into $\mathsf{A}^+_{\mathrm{reach}}$ and $\mathsf{A}^-_{\mathrm{reach}}$, containing the reachable a_s^+ and a_s^-, respectively.

Here follows the reverse-DI equivalent of Proposition 1:

Proposition 5. *The* $\mathrm{FK}_p(\underline{a})$*-kernel of a reverse* $[-|p|, \boldsymbol{\kappa}]$*-DI sequence* $\underline{a}$ *is the following finite set of shifted versions of* $\underline{a}$:

$$\mathrm{FK}_p(\underline{a}) = \{(a(k + a_s^+))_{k \in \mathbb{Z}}, (a(-k + a_s^-))_{k \in \mathbb{Z}} \mid a_s^+ \in \mathsf{A}^+_{\mathrm{reach}}, a_s^- \in \mathsf{A}^-_{\mathrm{reach}}\}. \tag{55}$$

Again, the formal kernel-defining shift-set $\mathsf{A}^*_{\mathrm{reach}}$ need not be minimal: a reduced kernel-defining shift set $\mathsf{A}^*_{\mathrm{red}}$ and standard kernel-defining shift set $\mathsf{A}^*_{\mathrm{min}}$ can be derived from $\mathsf{A}^*_{\mathrm{reach}}$, by considering the equivalence classes resulting from putting $a_s^* \cong a_{s'}^{*'}$ whenever $(a(*k + a_s^*))_{k \in \mathbb{Z}} \equiv (a(*'k + a_{s'}^{*'}))_{k \in \mathbb{Z}}$ (for the reduction of $\mathsf{A}^*_{\mathrm{reach}}$ to $\mathsf{A}^*_{\mathrm{red}}$), or whenever $(a(*k + a_s^*))_{k \in \mathbb{N}} \equiv (a(*'k + a_{s'}^{*'}))_{k \in \mathbb{N}}$ (for the reduction of $\mathsf{A}^*_{\mathrm{reach}}$ or $\mathsf{A}^*_{\mathrm{red}}$ to $\mathsf{A}^*_{\mathrm{min}}$). Observe that, when $* = *'$, reduction under this equivalence relationship is only possible when $\underline{a}$ is periodic, and thus the earlier conditions for nonperiodicity can be invoked to establish irreducibility. When $* \neq *'$, reducibility implies the presence of so-called reverse symmetry in $\underline{a}$, which has been conjectured to imply periodicity [7].

Associated with these kernel-defining shift-sets are the formal, the reduced and standard $|p|$-automata $\mathscr{A}^*_{\mathrm{reach}}$, $\mathscr{A}^*_{\mathrm{red}}$, $\mathscr{A}^*_{\mathrm{min}}$ for $\mathrm{R}\underline{a}$.

We state here the reverse DI-sequence counterparts of some direct DI-sequence properties.

Theorem 7 (Counterpart of Theorem 5). (A) *The formal automaton* $\mathscr{A}^*_{\mathrm{reach}} = (\mathsf{A}^*_{\mathrm{reach}}, 0^+, G^*_\mu, V, \tau)$ *with states* $a_s^* \in \mathsf{A}_{\mathrm{reach}}$, *initial state* 0^+, *state transition function* $G^*_\mu : a_s^\pm \mapsto a_{\mu s}^\mp$ *for all* $\mu \in [|p|]$, *output map* $\tau : \mathsf{A}^*_{\mathrm{reach}} \to V : a_s^* \mapsto \underline{a}(a_s)$ *is a reverse-reading* $|p|$*-automaton for* $\mathrm{R}\underline{a}$.

(B) *The* reduced *automaton* $\mathscr{A}^*_{\text{red}} = (\mathsf{A}^*_{\text{red}}, 0^+, G^*_\mu, V, \tau)$ *with reduced state set* $\mathsf{A}^*_{\text{red}}$ (*which depends on the specific DI-solution considered*), *is also a* $|p|$*-automaton for* R$\underline{a}$. *So is the minimal automaton* $\mathscr{A}^*_{\min}$ (*with state set* $\mathsf{A}^*_{\min}$).

Proof. (A) (a). Observe that $\underline{a}(0^+) = a(a^+_\phi)$, and so the automaton starts in the right state.

(b) Let $m > 0$ and $m = \mu_{n-1}|p|^{n-1} + \mu_{n-2}|p|^{n-2} + \cdots + \mu_0$, $(\mu_{n-1} \neq 0)$. Then

$$a(m) = D^m_{|p|^n}\underline{a}(0) \stackrel{(41)}{=} a(a_{\mu^*_{n-1}\mu_{n-2}\dots\mu_0}) \stackrel{(44,45)}{=} a(G^*_{\mu_{n-1}}(\dots G^-_{\mu_1}(G^+_{\mu_0}(a^+_\phi))\dots),$$

where $* = +$ when $n =$ even, $* = -$ when $n =$ odd. So $a(m)$ is indeed generated from the given automaton with input sequence $\mu_0, \mu_1, \dots, \mu_{n-1}$.

(B) Is a direct consequence of (A) and of the reduction of $\mathsf{A}_{\text{reach}}$ to $\mathsf{A}^*_{\text{red}}$ (or $\mathsf{A}^*_{\min}$) under the proper equivalence relation. $\square$

6. Conclusion

In this paper, we have defined, through the basic Eq. (3), the concept of decimation-invariant sequences. We have presented an algorithm for solving these decimation invariance equations and for determining the number of solutions. Then we have investigated the automaticity of these sequences, and derived algorithms for finding the automaton generating the right- and left-infinite parts of decimation invariant sequences. It is also possible to construct the automaton which generates the whole bidirectional DI-sequence: it just extends the algorithms presented above by using numbering systems which allow the unique representation of all integers, negative as well as positive [2]. More details can be found in [7].

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