Fixed point properties for nonexpansive representations of topological semigroups

E. Nazari *, A.H. Riazi

Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Ave, Tehran, Iran

Received 8 September 2003
Available online 11 May 2006
Submitted by R. Triggiani

Abstract

In this paper, we study a fixed point and a nonlinear ergodic properties for an amenable semigroup of nonexpansive mappings on a nonempty subset of a Hilbert space.

Keywords: Nonexpansive mapping; Fixed point; Semitopological semigroup; Invariant means; Hilbert space

1. Introduction

Let $S$ be a semitopological semigroup with identity, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \rightarrow as$ and $s \rightarrow sa$ from $S$ into $S$ are continuous. Let $D$ be a nonempty subset of a Hilbert space $H$ and $\mathcal{S} = \{T_t: t \in S\}$ be a continuous representation of $S$ as mappings from $D$ into $D$, i.e., (i) $T_e = I$; (ii) $T_{st} = T_sT_t$ for all $s, t \in S$; (iii) the mapping $S \times C \rightarrow C$ defined by $(s, t) \rightarrow T_sx$, $s \in S$, $x \in C$, is continuous when $S \times C$ has the product topology. Let $\mathcal{S}$ be a continuous representation of the semigroup $S$ and each $T_s$, $s \in S$, be a nonexpansive self-mapping of $D$, i.e., $\|T_sx - T_sy\| \leq \|x - y\|$ for all $x, y \in D$ and $s \in S$, then $\mathcal{S}$ is called a continuous nonexpansive semigroup on $D$. Goebel and Schöneberg in [2] proved that if $T$ be a nonexpansive self-mapping of a nonempty subset $D$ of a Hilbert space $H$, then $T$ has a fixed point in $D$ if and only if $\{T^n x\}$ is bounded for some $x \in D$ and for $y \in \text{clco}\{T^n x: n > 0\}$, there is a unique $p \in D$ such that $\|y - p\| = \inf_{z \in D} \|y - z\|$. In this paper

* Corresponding author.
E-mail addresses: n7713914@aut.ac.ir (E. Nazari), riazi@aut.ac.ir (A.H. Riazi).

0022-247X/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2006.03.093
we study this theorem for nonexpansive continuous representation of an amenable semitopological semigroup on a nonempty subset \( D \) of a Hilbert space \( H \).

2. Some preliminaries

All topologies in this paper are assumed to be Hausdorff. Given a nonempty set \( S \), we denote by \( \ell_\infty(S) \) the Banach space of all bounded real-valued functions on \( S \) with supremum norm. Let \( S \) be a semigroup. Then a subspace \( X \) of \( \ell_\infty(S) \) is left (respectively right) translation invariant if \( \ell_a(X) \subseteq X \) (respectively \( r_a(X) \subseteq X \)) for all \( a \in S \), where \( \ell_a(f)(s) = f(as) \) and \( r_a(f)(s) = f(sa) \), \( s \in S \). If \( S \) is a semitopological semigroup, we denote by \( CB(S) \) the closed subalgebra of \( \ell_\infty(S) \) consisting of continuous functions. Let \( LUC(S) \) (respectively \( RUC(S) \)) be the space of left (respectively right) uniformly continuous functions on \( S \), i.e., all \( f \in CB(S) \) such that the mapping from \( S \) into \( CB(S) \) defined by \( s \mapsto \ell_s(f) \) (respectively \( s \mapsto r_s(f) \)) is continuous when \( CB(S) \) has the sup norm topology. It is well known that \( LUC(S) \) and \( RUC(S) \) are left and right translation invariant closed subalgebras of \( CB(S) \), respectively, containing constants [1]. Note that when \( S \) is a topological group, then \( LUC(S) \) is precisely the space of left uniformly continuous functions on \( S \) defined in [3]. Now suppose \( X \) be a subspace of \( \ell_\infty(S) \) containing constants. Then \( \mu \in X^* \) is called a mean on \( X \) if \( \|\mu\| = \mu(1) = 1 \). Moreover, let \( X \) be \( r_s \)-invariant then, a mean \( \mu \) on \( X \) is right invariant if \( \mu(r_s f) = \mu(f) \) for all \( s \in S \) and \( f \in X \). Similarly we can define left invariant mean. \( \mu \) is called an invariant mean if it is left and right invariant. The value of a mean \( \mu \) at \( f \in X \) will also be denoted by \( \mu(f) \), \( \langle \mu, f \rangle \) or \( \mu_1(f) \). A net \( \{\mu_\alpha\} \) on \( RUC(S) \) is called asymptotically invariant [9] if for each \( f \in RUC(S) \) and \( a \in S \),

\[
\mu_\alpha(r_a f) - \mu_\alpha(f) \to 0 \quad \text{and} \quad \mu_\alpha(l_a f) - \mu_\alpha(f) \to 0.
\]

Let \( \mu \) be a mean on \( X \), \( E \) be a Banach space, \( \phi : S \to E \) be a bounded function, and \( K \) be a closed convex subset of \( E \). Suppose for each \( x \in K \), the real-valued function \( f \) on \( S \) defined by

\[
f_t(x) = \|\phi(t) - x\|^2 \quad \text{for all} \ t \in S,
\]

belongs to \( X \). Then set

\[
r(x) = \langle \mu, f_x \rangle \quad \text{for all} \ x \in K,
\]

and define \( r = \inf_{x \in K} r(x) \) and \( M_\mu = \{y \in K : r(y) = r\} \).

Lemma 2.1. The nonnegative real-valued function \( r \) on \( K \) defined as above is continuous, convex and \( r(x_n) \to \infty \) as \( \|x_n\| \to \infty \). If \( E \) is reflexive or \( K \) is weakly compact, then \( M_\mu \) is a nonempty closed convex subset of \( K \). Furthermore, if \( E \) is a Hilbert space, then \( M_\mu \) contains a unique element \( y \) and \( r + \|y - x\| \leq r(x) \) for all \( x \in K \) [7].

Remark 2.2. Let \( D \) be a subset of the real Hilbert space \( H \), \( x \in H \) and \( \mathcal{S} = \{T_t : t \in S\} \) be a nonexpansive semigroup on \( D \). Suppose \( \{T_t z : t \in S\} \) is bounded for some \( z \in D \). Then the functions \( f_x(t) = \|T_t z - x\|^2 \) and \( g_x(t) = \langle T_t z, x \rangle \) are in \( RUC(S) \) [6], see also [5]. Now if \( \mu \) is a mean on \( RUC(S) \), by the Riesz representation theorem, there exists \( z_\mu \in H \) such that \( \mu(T_t z, x) = \langle z_\mu, x \rangle \) for each \( x \in H \) [8].

3. Fixed point theorems

Lemma 3.1. Let \( S \) be a semitopological semigroup, \( D \) be a nonempty subset of a real Hilbert space \( H \) and \( \mathcal{S} = \{T_t : t \in S\} \) be a nonexpansive continuous representation of a semitopological
semigroup $S$ on $D$. Suppose \{$T_t z$: $t \in S$\} is bounded for some $z \in D$ and RUC($S$) has a left invariant mean $\mu$. Then $z_\mu \in M_\mu$, and $\|T_t x - z_\mu\| \leq \|T_t x - z_\mu\|$ for all $x \in D$ and $t, s \in S$.

**Proof.** Let $\mu$ be a LIM on RUC($S$). By Remark 2.2 for $x \in H$, the function $f_t(x) = \|T_t z - x\|^2$, $t, s \in S$, is in RUC($S$). Let $M_\mu = \{y \in H: r(y) = r\}$ where $r = \inf \{r(x): x \in H\}$ and $r(x) = \langle \mu, f_x \rangle$. By Lemma 2.1, $M_\mu$ contains a unique element $y$ and $r + \|y - x\|^2 \leq r(x)$ for all $x \in H$.

Now let $x \in H$ and $t, s \in S$. Then,

$$
\|z_\mu - x\|^2 = \|T_t z - x\|^2 - \|T_t z - z_\mu\|^2 - 2\langle T_t z - z_\mu, z_\mu - x \rangle.
$$

So,

$$
0 \leq \|z_\mu - x\|^2 = \mu_t (\|T_t z - x\|^2 - \|T_t z - z_\mu\|^2 - 2\langle T_t z - z_\mu, z_\mu - x \rangle)
= \mu_t \|T_t z - x\|^2 - \mu_t \|T_t z - z_\mu\|^2 - 2\langle z_\mu - z_\mu, z_\mu - x \rangle
= \mu_t \|T_t z - x\|^2 - \mu_t \|T_t z - z_\mu\|^2.
$$

This implies that $M_\mu$ consists of a single point $z_\mu$.

To prove of the second part, let $x \in D$, then \{$T_t x$: $t \in S$\} is bounded because $\|T_t x - T_t z\| \leq \|x - z\|$ for $t \in S$. For $p = z_\mu \in M_\mu$ and $s, t, \theta \in S$, we have

$$
2\langle T_t x - T_{st} x, T_{\theta z} - p \rangle = \|T_t x - p\|^2 - \|T_{st} x - p\|^2 + \|T_{st} x - T_{\theta z}\|^2 - \|T_t x - T_{\theta z}\|^2.
$$

Now, applying $\mu$ to both sides of the above equality with respect to $\theta$ we have

$$
0 = 2\langle T_t x - T_{st} x, p - p \rangle
= \|T_t x - p\|^2 - \|T_{st} x - p\|^2 + \mu_{\theta} \|T_{st} x - T_{\theta z}\|^2 - \mu_{\theta} \|T_t x - T_{\theta z}\|^2.
$$

On the other hand, since

$$
\mu_{\theta} \|T_{st} x - T_{\theta z}\|^2 = \mu_{\theta} \|T_{st} x - T_{st \theta z}\|^2 \quad \text{(by left invariance $\mu$)}
\leq \mu_{\theta} \|T_t x - T_{\theta z}\|^2.
$$

Therefore,

$$
\|T_{st} x - p\|^2 \leq \|T_t x - p\|^2 + \mu_{\theta} \|T_t x - T_{\theta z}\|^2 - \mu_{\theta} \|T_t x - T_{\theta z}\|^2.
$$

So

$$
\|T_{st} x - p\| \leq \|T_t x - p\|.
$$

This completes the proof. \qed

We now state our main fixed point theorem extending Goebel–Schöneberg’s theorem [2] for nonexpansive continuous representation of an amenable semitopological semigroup on a subset $D$ of a Hilbert space.

**Theorem 3.2.** Let $S$ be a semitopological semigroup with identity, $D$ be a nonempty subset of a real Hilbert space $H$ and $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive continuous representation of $S$ on $D$. Suppose RUC($S$) has a left invariant mean $\mu$. Then $\mathcal{S}$ has common fixed point in $D$ if and only if \{$T_t x$: $t \in S$\} is bounded for some $x \in D$ and for any $y \in \text{clco}\{T_t x: t \in S\}$, there is a unique $p \in D$ such that $\|y - p\| = \inf_{z \in D} \|y - z\|$. 

Hence, we have \( \|T_p - c\| = \|T_{ce}p - c\| \leq \|T_{te}p - c\| = \|p - c\|. \)

Proof. Necessity is obvious. Let us prove the sufficiency. Assume \( \{T_x: x \in S\} \) is bounded for some \( x \in D \) and \( M_\mu \) be as in the proof of Lemma 3.1 and \( c = z_\mu \in M_\mu \). Then by [4] we have \( c \in clco \{T_x: x \in S\} \). So there exists a unique \( p \in D \) such that \( \|c - p\| \leq \|c - z\| \) for all \( z \in D \).

On the other hand, by Lemma 3.1 we know \( \|T_p - c\| = \|T_{te}p - c\| \leq \|T_{te}p - c\| = \|p - c\|. \)

Hence, we have \( \|c - p\| = \inf_{z \in D} \|c - z\| \leq \|c - T_p p\| = \|c - p\|, t \in S, \) and the uniqueness of \( p \) implies that \( T_p p = p \) for all \( t \in S \), i.e., \( p \) is a common fixed point for \( \mathcal{Z} \).

**Corollary 3.3.** (Goebel–Schöneberg [2]) Let \( T \) be a nonexpansive self-mapping of a nonempty subset \( D \) of a Hilbert space \( H \). Then \( T \) has a fixed point in \( D \) if and only if \( \{T^n x: n \geq 0\} \) is bounded for some \( x \in D \) and for any \( y \in clco \{T^n x: n \geq 0\} \) there is a unique \( p \in D \) such that \( \|y - p\| = \inf_{z \in D} \|y - z\|. \)

Proof. Let \( S = (N, +) \) and \( \mathcal{Z} = \{T^n: n \in N\} \), then since \( S \) is amenable by Theorem 3.2 the proof is compete.

4. Ergodic theorem

We are now ready to establish our nonlinear ergodic theorem by using Lemma 3.1 for an arbitrary subset \( D \) of a Hilbert space \( H \).

**Theorem 4.1.** Let \( H, D, S, \mathcal{Z} \) be as in Theorem 3.2. If \( \mu \) is an invariant mean on \( RUC(S) \), and \( \{T_z: z \in S\} \) is bounded for some \( z \in D \), then for any asymptotically invariant net \( \{\mu_s\} \) of means on \( RUC(S) \), the net \( Z_{\mu_s} \) converges weakly to \( Z_\mu \). In particular, if \( v \) is another invariant mean on \( RUC(S) \), then \( z_\mu = z_v \).

**Proof.** If \( \mu \) is invariant mean on \( RUC(S) \), then

\[
\mu_s \|T_z - x\|^2 \leq \inf_t \|T_{tuz} z - x\|^2 \quad \text{(by right invariance of } \mu)\]

for each \( x \in H \) [10]. On the other hand, let \( y \in M_\mu \) and \( s \in S \), by Lemma 3.1 we have

\[
\inf_u \sup_t \|y - T_{tu} T_s z\|^2 = \inf_u \sup_t \|y - T_s z\|^2 \leq 0.
\]

And hence

\[
\inf_u \sup_t \|T_{tuz} - y\|^2 \leq \inf_u \sup_t \|T_{tuz} - y\|^2 = \inf_u \sup_t \|T_{tuz} - y\|^2 \leq \|T_s z - y\|^2.
\]

So, we have

\[
\inf_s \sup_t \|T_{tuz} - y\|^2 \leq \mu_s \|T_s z - y\|^2.
\]

Therefore, for each \( y \in M_\mu \),

\[
\mu_s \|T_z - y\|^2 = \inf_s \sup_t \|T_{tuz} - y\|^2.
\]

Hence if \( v \) is another invariant mean on \( RUC(S) \), then by Lemma 3.1, \( z_v \in M_v \). Hence

\[
\mu_s \|T_z - z_v\|^2 = \inf_s \sup_t \|T_{tuz} - z_v\|^2 \leq \mu_s \|T_z - z_v\|^2
\]

\[
\leq \inf_s \sup_t \|T_{tuz} - z_v\|^2 = v \|T_z - z_v\|^2 \leq v \|T_z - z_v\|^2
\]

\[
\leq \inf_s \sup_t \|T_{tuz} - z_v\|^2 = \mu_s \|T_z - z_v\|^2.
\]
Hence \( \mu_t \| T_t z - z_\mu \|^2 = \mu_t \| T_t z - z_\nu \|^2 \). By uniqueness of the element in \( M_\mu \), we have \( z_\mu = z_\nu \).

Now, if \( \{ \mu_\alpha \} \) is an asymptotically invariant net and \( \mu \) is a cluster point of \( \{ \mu_\alpha \} \) in the weak*-topology, then \( \mu \) is invariant mean on \( RUC(S) \). Hence if \( \{ z_{\mu_\beta} \} \) is a subnet of the net \( \{ z_{\mu_\alpha} \} \) such that \( z_{\mu_\beta} \) converges weakly to some \( y \) in \( H \), then, since a cluster point \( \nu \) of \( \{ \mu_\beta \} \) is also a cluster point of \( \{ \mu_\alpha \} \), \( \nu \) is invariant mean. So, \( y = z_\mu = z_\nu \) by the above. This implies that \( z_{\mu_\alpha} \) converges weakly to \( z_\mu \). □

References