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Generating Functions for Column-Convex Polyominoes

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Using bijections and language theory, we give the generating function for the number of column-convex polyominoes with perimeter $2n$. We prove also a result about their number with area m and k columns. © 1988 Academic Press, Inc.

INTRODUCTION

Unit squares with vertices at integer points in the cartesian plane are called *cells*. A *polyomino* is a finite connected union of cells such that the interior is also connected (no cut point). The *area* of a polyomino is the number of cells, the *perimeter* is the length of the border.

Polyominoes are defined up to translation.

Polyominoes are very classical objects in combinatorics. Counting polyominoes according to their area or perimeter is a major unsolved problem in combinatorics. The first authors interested in this subject were Read [12] and Golomb [5]. Some authors Lunnon [9], Redelmeier [13] have given the first values for the number of polyominoes with a given area.

The physicists have given several asymptotic results. They call *animal* a set of points obtained by taking the centers of the cells of a polyomino. Giving a privileged direction for the growth of an animal allows them to obtain generating functions (see Viennot [16] and references therein).

A *column* (resp. *row*) of a polyomino is the intersection between the polyomino and any infinite vertical (resp. horizontal) strip of unit squares.

A polyomino is said to be *column-* (resp. *row-*) *convex* if all its columns (resp. rows) are connected. A *convex* polyomino is both row- and column-convex (see Fig. 1).

Asymptotic results on the number of convex polyominoes with area n have been given by Bender [1] and Klarner and Rivest [8].

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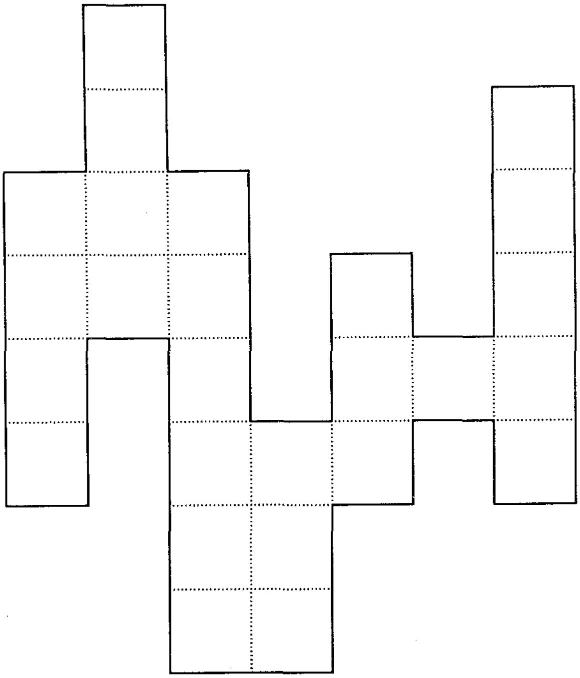


FIG. 1. A column-convex polyomino.

In [6], Klarner gives the expression for the generating function enumerating row-convex polyominoes according to area. This function is rational and is obtained by a combinatorial interpretation of a Fredholm integral [7].

Recently Delest and Viennot [3] found an exact formula for the number p_{2n} of convex polyominoes with perimeter $2n$. The method used, due to ideas of Schützenberger, is in three steps:

(i) A bijection is established between polyominoes and words of an algebraic language.

(ii) Solve the corresponding algebraic system gives the generating function

$$p(t) = \sum_{n \geqslant 0} p_{2n} t^{2n}.$$

(iii) Expanding $p(t)$ gives the number

$$p_{2n+8} = (2n+11)4^n - 4(2n+1)\binom{2n}{n}.$$

We use here the same method to enumerate column-convex polyominoes.

We give the generating function for the number $g_{n,m}$ of column-convex polyominoes with area n and m columns. This refines a result of Klarner [6] who gave the generating function of $g_n = \sum_{m \geqslant 1} g_{n,m}$.

Furthermore, making an extension of the bijection for the so-called *parallellogram* polyominoes given in [3], we found the generating function for the number c_n of column-convex polyominoes with perimeter $2n+2$. This last computation has been possible by means of the M.I.T. [10] symbolic manipulation system MACSYMA. The complexity of this function does not permit us to give a formula for c_n.

NOTATIONS

Let X be an alphabet, we denote by X^* the free monoid generated by X that is the set of the words written with letters from X, the product is defined as the *concatenation* of two words: for $u = u_1 \cdots u_p$ and $v = v_1 \cdots v_q$, then $uv = u_1 \cdots u_p v_1 \cdots v_q$. The *empty word* is denoted by ε. The number of occurrences of the letter x in the word w is denoted by $|w|_x$, the length of w by

$$|w| = \sum_{x \in X} |w|_x.$$

A *language* is a subset of X^*. For any languages A, B of X^*, we denote by AB the set of the words $f = uv$ with $u \in A$ and $v \in B$. The submonoid generated by A is denoted by A^*. The language A is a code iff every words of A^* has a unique factorization $f = a_1 \cdots a_p$ with $p \geqslant 1$ and for $1 \leqslant i \leqslant p$, $a_i \in A$. Let $K \langle\langle X \rangle\rangle$ (resp. $K[[X]]$) be the algebra of non-commutative (resp. commutative) power series with variables from X and coefficients in the ring K. We denote by α the canonical morphism which make the variables commuting. For any language L in X^*, we denote by $\mathbf{L}$ the generating function

$$\mathbf{L} = \sum_{w \in L} w,$$

which is an element of $Z \langle\langle X \rangle\rangle$. Note that for any languages A, B, C, the equality $\mathbf{C} = \mathbf{A} + \mathbf{B}$ means that $C = A \cup B$ and $A \cap B = \emptyset$. The equality $\mathbf{C} = \mathbf{AB}$ means that any word w of C has a unique factorization $w = uv$ with $u \in A$ and $v \in B$. The equality $\mathbf{A}^* = (1 - \mathbf{A})^{-1}$ means that A is a code.

The Dyck language D is the set of words w in $\{x, \bar{x}\}$ verifying two conditions:

(i) for any factor f such that $w = fg$, $|f|_x \geqslant |f|_{\bar{x}}$,

(ii) $|w|_x = |w|_{\bar{x}}$.

A path ω is a sequence $\omega = (s_0, s_1, ..., s_n)$ of integer points in the cartesian plane. The point s_0 (resp. s_n) is the *starting* (resp. *final*) point. The length of ω is n. Each pair (s_i, s_{i+1}) is an *elementary step* of the path. The elementary step is called *north* (resp. *south*, resp. *east*, resp. *west*) iff $s_i = (x, y)$, $s_{i+1} = (x', y')$ with $x = x'$, $y' = y + 1$ (resp. $x = x'$, $y' = y - 1$, resp. $x' = x + 1$, $y = y'$, resp. $x' = x - 1$, $y = y'$).

1. Generating Function according to the Area

Let P be any column-convex polyomino. It can be defined by two sequences of integers

$$C(P) = (c_1, ..., c_k), \qquad A(P) = (a_1, ..., a_{k-1})$$

with the convention that if $k = 1$ then $A(P)$ is empty. These two sequences are defined by the following conditions:

for every $i \in [1, k]$, c_i is the number of cells of the ith column of the polyomino, (1)

for every $i \in [1, k-1]$, a_i gives the way of gluing the columns i and $i+1$ namely, starting with the column $i+1$ glued on the

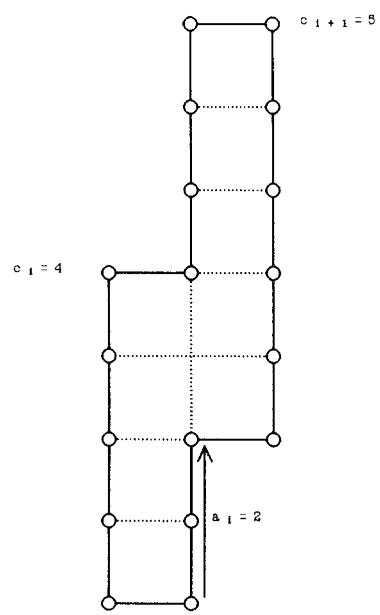


FIG. 2. Gluing column i and $i+1$. $a_i \geq 0$.

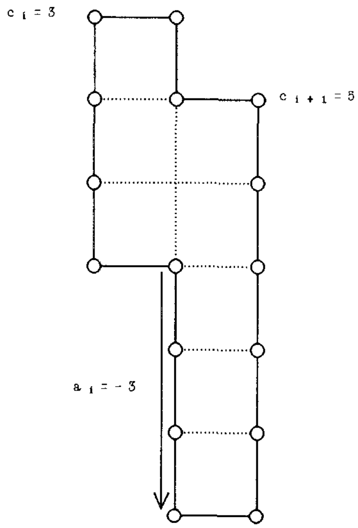


FIG. 3. Gluing column i and $i+1$. $a_i < 0$.

right of the ith column such that their two southmost east steps are on the same horizontal line then if $a_i \geqslant 0$ (resp. $a_i < 0$) the column $i+1$ is moved $|a_i|$ steps north (resp. south) (see Figs. 2 and 3). (2)

These sequences satisfy:

$$\text{for every } i \in [1, k-1], \qquad -c_{i+1} + 1 \leqslant a_i \leqslant c_i - 1. \tag{3}$$

The area of P is

$$a(P) = \sum_{i=1,\ldots,k} c_i. \tag{4}$$

For each polyomino P, we define the word $w = \mu(P)$ in $\{x, b, /\}^*$ using the following construction:

if $k = 1$ then $w = x^{c_1}/$, (5)

if $k \neq 1$ then $w = w_1/w_2/\cdots/w_k/$ with $w_k = x^{c_k}$, for every $i \in [1, k-1]$ if $a_i \geqslant 0$ then $w_i = x^{a_i} b x^{c_i - a_i}$ else $w_i = x^{c_i} b^{|a_i|+1}$. (6)

The map μ is a bijection between column-convex polyominoes and words of the following language:

DEFINITION 1. Let V be the language of the words of $\{x, b, /\}^*$ verifying:

$f = f_1/f_2/\cdots/f_k/$ with $k \geqslant 1$, (7)

for every $i \in [1, k]$, $f_i \in \{x, b\}^*$, $|f_i|_x \neq 0$ and $|f_k|_b = 0$, (8)

for every $i \in [1, k-1]$, $f_i = x^n b g_i$ with

$$\text{if} \quad |g_i|_x = 0 \text{ then } |f_{i+1}|_x > |g_i|, \qquad (9.1)$$

$$\text{if} \quad |g_i|_x \neq 0 \text{ then } |g_i|_b = 0. \qquad (9.2)$$

This definition results from the construction (5) and (6) and from the property (3).

Remark 2. The area of the polyomino P is clearly $|\mu(P)|_x$ and its number of columns is $|\mu(P)|_/$ (see Fig. 4).

We describe now a map τ which is the reserve bijection of μ.

Let f be a word of V, $f = f_1/f_2/\cdots/f_k/$. For every $i \in [1, k]$, we consider a column made with $|f_i|_x$ cells.

For every $i \in [1, k-1]$, let $f_i = x^n b g_i$. If $|g_i|_x = 0$ (resp. $|g_i|_x \neq 0$) then starting with the column $i+1$ glued on the right of the column i such that their southmost east steps are on the same horizontal line, we move the column $i+1$ $|g_i|$ south (resp. n north) steps.

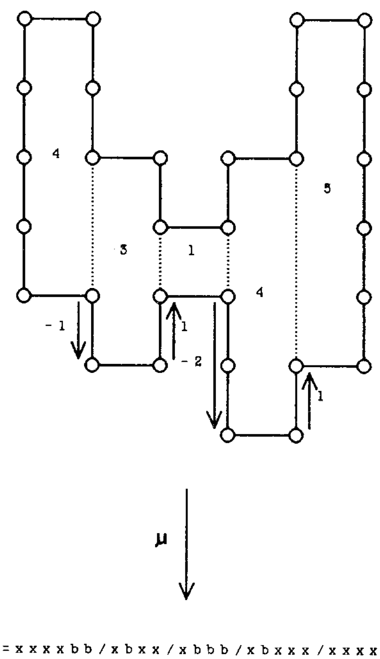


FIG. 4. Bijection μ between the column-convex polyominoes and the words of V.

Thus we obtain a column-convex polyomino $\tau(f)$ having area $\sum_{i=1,\ldots,k} |f_i|_x$ and k columns.

We introduce now auxiliary languages useful for the study of V. We denote by X the alphabet $\{x, b, /\}$.

Let L_1 be the set of words f of X^* satisfying

$$f = x^k/f_2 \qquad \text{with} \quad k \geq 0 \text{ and } f_2 \in V. \tag{10}$$

Let L_2 be the set of words f of X^* satisfying

$$f = \varepsilon \qquad \text{or} \qquad f = b^k/f_2 \quad \text{with} \quad k \geq 1,\ f_2 \in V \text{ and } xf_1/f_2 \in V. \tag{11}$$

Let L_3 be the set of words f of X^* satisfying

$$f = b^k/x^k \qquad \text{with} \quad k \geq 0. \tag{12}$$

Let L_4 be the set of words f of X^* satisfying

$$f = b^k/x^i b x^j \qquad \text{with} \quad k, i, j \geq 0 \text{ and } i + j = k. \tag{13}$$

The non-commutative generating function $\mathbf{V}$ satisfies the system of equations

$$\begin{aligned}
\mathbf{V} &= x\mathbf{V} + bx\mathbf{L}_1 + x\mathbf{L}_2,\\
\mathbf{L}_1 &= x\mathbf{L}_1 + /\mathbf{V},\\
\mathbf{L}_2 &= b\mathbf{L}_3\, x\mathbf{V} + b\mathbf{L}_4 xx\mathbf{L}_1 + /,\\
\mathbf{L}_3 &= b\mathbf{L}_3 x + /,\\
\mathbf{L}_4 &= \mathbf{L}_3 b + b\mathbf{L}_4 x.
\end{aligned} \tag{14}$$

The first equation is a consequence of the following fact. Every word f in V has one of the decompositions given below:

$f = x/$,

$f = xxh$ with h in V,

$f = xb^{k+1}/h$ with $k \geq 0$ and h in V; in this case $h = x^i bg$ with $i > k$ (see 6.1) and thus $b^{k+1}/h \in L_2 \backslash \{/\}$,

$f = bx^{k+1}/h$ with $k \geq 0$ and h in V (see 6.2) thus x^k/h is in L_1.

In the same manner, we show that the other equations come from the definitions of the languages L_1, L_2, L_3, L_4.

We define now the commutative power series

$$v(x, b, /) = \alpha(\mathbf{V}), \qquad l_1(x, b, /) = \alpha(\mathbf{L}_1),$$
$$l_2(x, b, /) = \alpha(\mathbf{L}_2), \qquad l_3(x, b, /) = \alpha(\mathbf{L}_3),$$
$$l_4(x\, b, /) = \alpha(\mathbf{L}_4).$$

Using the morphism ϕ defined on X by setting:

$$\phi(x) = x, \qquad \phi(/) = y, \qquad \phi(b) = \varepsilon,$$

we can obtain the generating function

$$G(x, y) = \sum_{n \geqslant 1} \sum_{m \geqslant 1} g_{n,m} x^n y^m$$

for the number $g_{n,m}$ of column-convex polyominoes having area n and m columns which is

$$G(x, y) = \phi(v(x, b, /)) = v(x, 1, y).$$

For simplicity let v, l_1, l_2, l_3, l_4 denote the respective images by ϕ of the series $v(x, b, /)$, $l_1(x, b, /)$, $l_2(x, b, /)$, $l_3(x, b, /)$, $l_4(x, b, /)$. These series are solutions of an commutative algebraic system coming from (11). Successively, we obtain

$$l_3 = \frac{y}{1-x},$$

$$l_4 = \frac{y}{(1-x)^2},$$

$$l_1 = \frac{yv}{1-x},$$

$$l_2 = \frac{xyv}{1-x} + \frac{x^2y^2v}{(1-x)^3} + y,$$

noting that

$$v = x(v + l_1 + l_2)$$

we have the following result

THEOREM 3. *The generating function for the number $g_{n,m}$ of column-convex polyominoes with area n and m columns is*

$$\sum_{n \geqslant 1} \sum_{m \geqslant 1} g_{n,m} x^n y^m = \frac{xy(1-x)^3}{(1-x)^4 - xy(1-x)^2(1+x) - x^3y^2}.$$

Remark 4. For $y=1$, we obtain the same result as Klarner [6]

$$\sum_{n\geq 1} g_n x^n = \frac{x(1-x)^3}{1-5x+7x^2-4x^3},$$

where g_n is the number of column-convex polyominoes with area n.

Remark 5. The parameter perimeter of a polyomino is not simply encoded in this bijection. Thus we introduce a new bijection in the following section.

2. The Coding of Convex-Polyominoes according to the Perimeter

In this section we give a bijection between column-convex polyominoes and words of a language that is close to the Dyck language. This bijection is an extension of the coding of parallellogram polyominoes given in [3].

Let d be the morphism from $\{x, \bar{x}, y, \bar{y}\}^*$ into $\{x, \bar{x}\}^*$ defined by

$$d(x) = d(y) = x, \qquad d(\bar{x}) = d(\bar{y}) = \bar{x}.$$

We define the language C which is defined as the set of the words w in $\{x, y, \bar{x}, \bar{y}\}^*$ verifying:

$d(f)$ is a Dyck word, (15)

f can be factorized in $f = f_1 x\bar{x} f_2 \cdots x\bar{x} f_k$ with $k \geq 2$ and $f_1 \in \{x\}^*$, $f_k \in \{\bar{x}\}^*$, and for every $i \in]1, k[$, (16)

$$f_i \in \{\bar{x}\}^* \{x\}^* \cup \{\bar{x}\}^* \{\bar{y}\}^* \bar{y} \cup y\{y\}^* \{x\}^* \cup \{\bar{y}\}^* \bar{y}y\{y\}^*.$$

Example 6. $w = xxxx\bar{x}\bar{y}\bar{y}yx\bar{x}\bar{x}\bar{y}x\bar{x}yyxx\bar{x}\bar{x}xxx\bar{x}\bar{x}\bar{x}\bar{x}\bar{x}$ is a word of the language C.

Now we prove the

Proposition 7. *There exists a bijection between column-convex polyominoes with perimeter $2n+2$ and words of C of length $2n$.*

Let w be a word of C of length $2n$. A factor $x\bar{x}$ (resp. x_1x_2 such that $d(x_1x_2) = \bar{x}x$) is called a *peak* (resp. *trough*) of w. We number the peaks (resp. troughs) from left to right. Let $k \geq 1$ be the number of peaks of w (thus w has $k-1$ throughs). The *height of the peak* in $w = fx\bar{x}g$ equals $\delta(f)+1$ with

$$\delta(f) = |d(f)|_x - |d(f)|_{\bar{x}}. \tag{17}$$

The *height of the trough* in $w=fx_1x_2g$ with $d(x_1x_2)=\bar{x}x$ equals $\delta(f)$.

Let $w=w_1x\bar{x}w_2x\bar{x}w_3$ be a word of C in which the two peaks are consecutive, the *colouring of the trough* between these peaks equals $\xi(w_2)$ with

$$\xi(w_2)=|w_2|_y+|w_2|_{\bar{y}}. \tag{18}$$

We associate to w the sequence of integers $C(w)=(c_1,\ c_2,..., c_k)$ satisfying:

$$\text{for every } i\in[1,k],\ c_i \text{ is the height of the } i\text{th peak.} \tag{19}$$

In the same way, we associate the sequence of integers $A(w)=(a_1, a_2,..., a_{k-1})$ verifying

for every $i\in[1,k]$, let q_i (resp. l_i) the height (resp. couloring) of the ith trough then $a_i=c_i-q_i-l_i$. (20)

These two sequences $C(w)$ and $A(w)$ satisfy (3). Indeed, for $i\in[1,k-1]$, consider $w=w_1x\bar{x}w_2x\bar{x}w_3$, where the first (resp. second) factor $x\bar{x}$ is the ith (resp. $(i+1)$th) peak of w.

If $w_2\in\{\bar{x}\}^*\{x\}^*\cup\{\bar{x}\}^*\{\bar{y}\}^*\bar{y}$ then

$$a_i=|w_2|_{\bar{x}} \qquad \text{and} \qquad a_i\geqslant 0,$$

but $d(w)$ is a Dyck word thus

$$\delta(w_1)\geqslant|w_2|_{\bar{x}} \qquad \text{and} \qquad a_i\leqslant c_i-1.$$

If $w_2\in y\{y\}^*\{x\}^*\cup\{\bar{y}\}^*\bar{y}y\{y\}^*$ then

$$a_i=-|w_2|_y \qquad \text{and} \qquad a_i<0,$$

but $d(w)$ is a Dyck word thus

$$|w_2|_y\leqslant\delta(w_1x\bar{x}w_2) \qquad \text{and} \qquad a_i\geqslant -c_i+1.$$

Using the construction (1) and (2), one can obtain from these two sequences a polyomino $P=\beta(w)$.

The perimeter of this polyomino $\beta(w)$ is given by

$$p=\sum_{i=1,k}(2+2c_i-q_{i-1}-q_i) \tag{21}$$

with

if $a_i\geqslant 0$ and $a_i+c_{i+1}\geqslant c_i$ then $q_i=c_i-a_i$,

if $a_i\geqslant 0$ and $a_i+c_{i+1}<c_i$ then $q_i=c_{i+1}-a_i$,

if $a_i < 0$ and $a_i + c_{i+1} \geqslant c_i$ then $q_i = c_i$,

if $a_i < 0$ and $a_i + c_{i+1} < c_i$ then $q_i = c_{i+1} + a_i$,

and the convention $q_0 = q_{k+1} = 0$.

q_i is the number of cells by which the ith and $(i+1)$th columns are glued. Also, it is the height of the ith trough of w. Thus, we obtained for a word of length $2n$ a polyomino $\beta(w)$ of perimeter $2n+2$ (see Fig. 5).

We describe now a map γ which is the reverse bijection of β.

Let P be a column-convex polyomino having for perimeter $2n+2$. Let $C(P)$ and $A(P)$ be the two sequences of integers associated by (1) to P. We define a word

$$w = w_1 x\bar{x} w_2 x\bar{x} \cdots x\bar{x} w_k$$

applying the rules

$$w_1 = x^{c_1 - 1},$$

$$w_k = \bar{x}^{c_k} - 1,$$

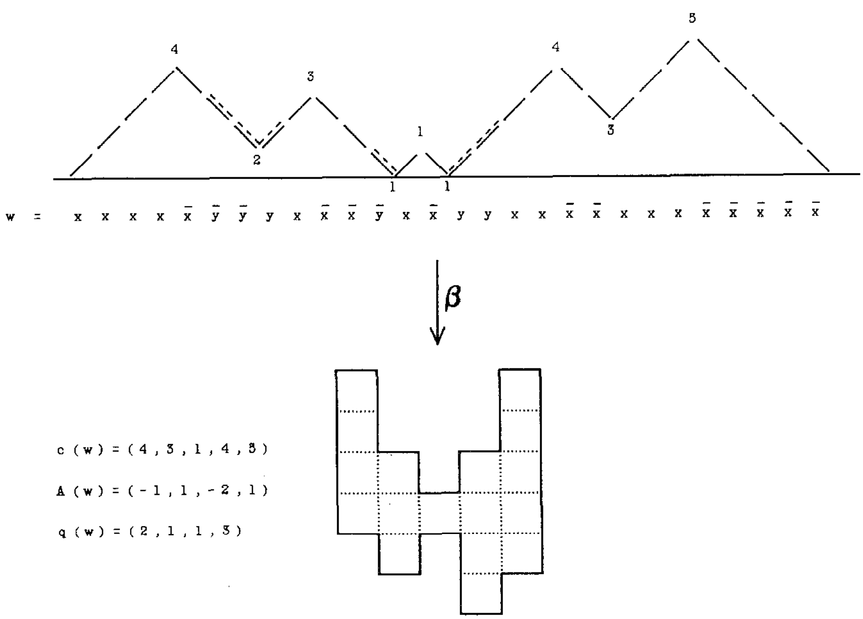


FIG. 5. The bijection β between column-convex polyominoes and words of C.

and for every $i \in]1, k[$,

$$\text{if } a_i \geqslant 0 \text{ and } a_i + c_{i+1} \geqslant c_i \text{ then } w_i = \bar{x}^{a_i} x^{a_i + c_{i+1} - c_i},$$

$$\text{if } a_i \geqslant 0 \text{ and } a_i + c_{i+1} < c_i \text{ then } w_i = \bar{x}^{a_i} \bar{y}^{c_i - c_{i+1} - a_i},$$

$$\text{if } a_i < 0 \text{ and } a_i + c_{i+1} < c_i \text{ then } w_i = \bar{y}^{c_i - c_{i+1} - a_i} y^{-a_i},$$

$$\text{if } a_i < 0 \text{ and } a_i + c_{i+1} \geqslant c_i \text{ then } w_i = y^{-a_i} x^{a_i + c_{i+1} - c_i}.$$

Clearly, w is a word of C of length $2n$. Let $\gamma(P)$ be this word. Then γ is the reciprocal bijection of β.

Remark 8. We can give an other definition for the bijection β. Let us consider the gluing of the column $i+1$ with the column i. Let $w = w_1 x\bar{x} \cdots x\bar{x} w_k$ be a word of C, and $i \in [1, k[$. There are five different cases according to the size of the columns.

If w_i is a *big valley* that is $w_i \in \{\bar{x}\}^* \, \bar{x}x \{x\}^* \cup \{\bar{y}\}^* \, \bar{y}y \{y\}^*$, thus we have $q_i < c_i$ and $q_i < c_{i+1}$ and there are two ways for gluing the columns according to one of the conditions $a_i > 0$ or $a_i < 0$ (see Fig. 6a).

If w_i is a *left* (resp. *right*) *small valley*, that is, $w_i \in \{\bar{x}\}^* \, \{\bar{y}\}^*$ (resp. $w_i \in \{y\}^* \, \{x\}^*$) thus $q_i = c_{i+1}$ (resp. $q_i = c_i$) and $c_{i+1} < c_i$ (resp. $c_i < c_{i+1}$) (see Fig. 6b (resp. 6c)).

If w_i is a *tiny valley* that is $w_i = \varepsilon$ thus $c_i = c_{i+1} = q_i$ (see Fig. 6d).

The gluing process is displayed on Fig. 6.

Remark 9. There is an other way for constructing the bijection γ. Let P be any column-convex polyomino having perimeter $2n+2$. We consider the point $S(P)$ (resp. $N(P)$) which is the southmost (resp. northmost) point among the westmost (resp. eastmost) points of P (see Fig. 7).

Then the polyomino P can be defined by two paths ω and η having only north, south, and east steps, beginning at the same point $S(P)$ and ending at the same point $N(P)$. These two paths do not intersect except at the end points. If the length of ω (resp. η) is q (resp. r) then

$$q + r = 2n + 2.$$

We take off the first (resp. last) step of ω (resp. η).

We construct a word of $\{x, y, \bar{x}, \bar{y}\}^*$ by following the two paths ω and η:

every north step of ω (resp. η) is coded by a letter x (resp. $\bar{x}$),

every south step of ω (resp. η) is coded by a letter $\bar{y}$ (resp. y),

every east step of ω (resp. η) is coded by a letter $\bar{x}$ (resp. x).

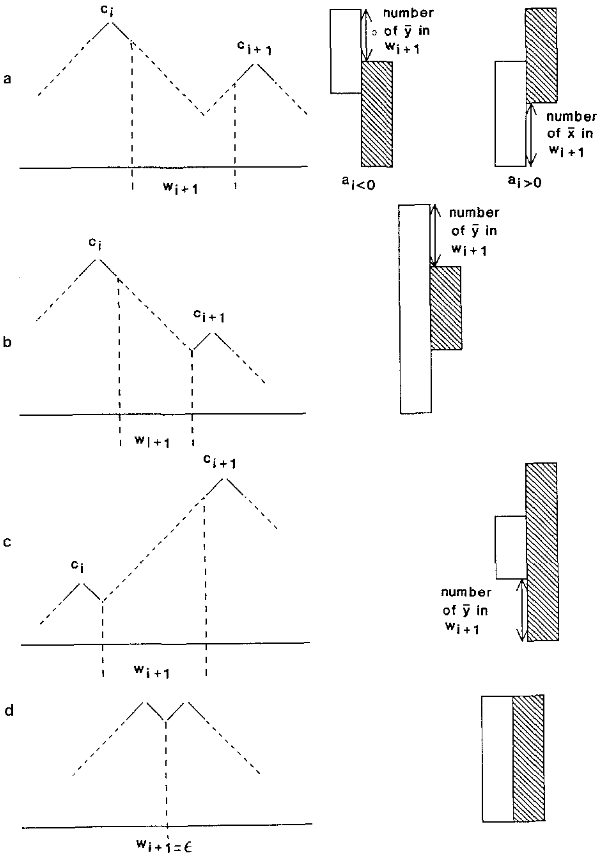
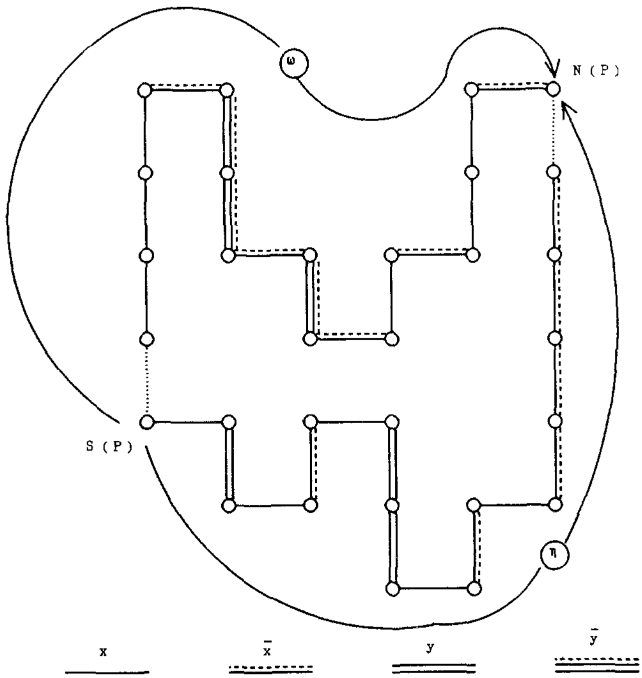


FIG. 6. An other definition for the bijection β.

We code the steps of the paths using the following rule: for every $i \in [1, k]$, we code first the north and south steps which are the right vertical border of the column $i-1$ and the left vertical border of the column i (we get a word w_i), and after we code the east steps of the column i. One can obtain by this construction some words having the form $w = w_1 x\bar{x} \cdots x\bar{x} w_k$ where each factor $x\bar{x}$ is associated with the pairs of east steps of η and ω. We make the convention that we write the letters in the words w_i in the following order: first $\bar{x}$ then $\bar{y}$ then y then x (see Fig. 7).

Remark 10. Let c_n be the number of column-convex polyominos then

$$c_{n+1} = \sum_{\substack{w \text{ dyck} \\ |w| = 2n}} \left(2^{bv(w)} \prod_{z \in SV} pc(z) \right)$$



$$w = x\ x\ x\ x\ \bar{x}\ \bar{y}\ \bar{y}\ y\ x\ \bar{x}\ \bar{x}\ \bar{y}\ x\ \bar{x}\ y\ y\ x\ x\ \bar{x}\ \bar{x}\ x\ x\ x\ \bar{x}\ \bar{x}\ \bar{x}\ \bar{x}\ \bar{x}$$

FIG. 7. An other definition for the bijection γ.

with if w is a dyck word then $bv(w)$ is the number of big valley of w, SV is the set of small valleys of w, and for any small valley u in w, $pc(u) = 1 + |u|$.

Remark 11. The area of the polyomino P is the sum of the height of the peaks of $\gamma(P)$.

Remark 12. The bijection β can be applied to the convex polyominoes. In [3], we gave a coding of convex polyominoes with some words of an algebraic language. In fact, we defined three different languages, three different coding, corresponding to three types of convex polyominoes. Using the bijection β, it is possible to give a single coding with a single algebraic language. Unfortunately, the equations of this language seem to be very hard to solve.

3. Generating Function according to the Perimeter

In this section we introduce auxiliary languages and give the generating function (in one variable) for the column-convex polyominoes.

The name of these languages is made with two letters giving their properties. The first letter (denoted L) is O or G or $\bar{G}$, the second (denoted R) is O or D or $\bar{D}$.

Let LR be such a language, the words f of LR are written on $\{x, \bar{x}, y, \bar{y}\}$ and verify:

(i) $d(f)$ is a Dyck word,

(ii) $f = f_1 x\bar{x} f_2 x\bar{x} \cdots x\bar{x} f_k$, $k \geqslant 2$ and for every $i \in]1, k[$,

$$f_i \in \{\bar{x}\}^* \{x\}^* \cup \{\bar{x}\}^* \{\bar{y}\}^* \bar{y} \cup y\{y\}^* \{x\}^* \cup \{\bar{y}\}^* \bar{y}y\{y\}^*. \quad (22)$$

Moreover

if $L = O$ (resp. $R = O$) then $f_1 \in \{x\}^*$ (resp. $f_k \in \{\bar{x}\}^*$),

if $L = G$ (resp. $R = D$) then $f_1 \in \{y\}^* \{x\}^*$ (resp. $f_k \in \{\bar{x}\}^* \{\bar{y}\}^*$),

if $L = \bar{G}$ (resp. $R = \bar{D}$) then $f_1 \in y\{y\}^*$ (resp. $f_k \in \{\bar{y}\}^* \bar{y}$).

Remark 13. We have thus defined 9 languages. The one coding the column-convex polyominoes is $C = OO$.

Lemma 14. *We can prove the equations*

$$\mathbf{OO} = x\bar{x} + x\bar{x}\,\mathbf{GO} + x\,\mathbf{OO}\,\bar{x} + x\,\mathbf{OO}\,\bar{x}\,\mathbf{OO} + x\,\mathbf{OD}\,\bar{y}x\bar{x}(\varepsilon + \mathbf{GO}) + x(\mathbf{OD} + \varepsilon)\,x\bar{x}\bar{y}\,\bar{\mathbf{G}}\mathbf{O} + x\,\mathbf{O}\bar{\mathbf{D}}\,\bar{y}\,\bar{\mathbf{G}}\mathbf{O}, \quad (23)$$

$$\mathbf{GO} = \mathbf{OO} + y\,\mathbf{GO}\,\bar{x} + y\,\mathbf{GO}\,\bar{x}\,\mathbf{OO} + y\,\mathbf{GD}\,\bar{y}x\bar{x}(\varepsilon + \mathbf{GO}) + y(\mathbf{GD} + \varepsilon)\,x\bar{x}\bar{y}\,\bar{\mathbf{G}}\mathbf{O} + y\mathbf{G}\bar{\mathbf{D}}\,\bar{y}\bar{\mathbf{G}}\mathbf{O}, \quad (24)$$

$$\mathbf{GD} = \mathbf{OD} + y\,\mathbf{GO}\,\bar{x} + y\,\mathbf{GO}\,\bar{x}\,\mathbf{OD} + y\,\mathbf{GD}\,\bar{y} + y\,\mathbf{GD}\,\bar{y}x\bar{x}(\varepsilon + \mathbf{GD}) + y(\varepsilon + \mathbf{GD})\,x\bar{x}\bar{y}\,\bar{\mathbf{G}}\mathbf{D} + y\,\mathbf{G}\bar{\mathbf{D}}\,\bar{y}\,\bar{\mathbf{G}}\mathbf{D}. \quad (25)$$

We prove only the equality (23); the others can be proved in the same way.

Proof of the Equality (23). Let f be a word of OO, f has a unique decomposition $f = xh\bar{x}g$ or $f = xh\bar{y}g$ with $d(f)$ and $d(g)$ are Dyck words.

Moreover, from (22), f has also a decomposition: $f = f_1 x\bar{x} f_2 x\bar{x} \cdots x\bar{x} f_k$ with $f_1 \in \{x\}^*$ and $f_k \in \{\bar{x}\}^*$.

Suppose that $f = xh\bar{x}g$ then

if $h=\varepsilon$ and $g=\varepsilon$ then $f=x\bar{x}$,

if $h=\varepsilon$ and $g\neq\varepsilon$ then $g=f_2x\bar{x}\cdots x\bar{x}f_k$ and $f_2\in\{y\}^*\{x\}^*$ thus $g\in GO$,

if $h\neq\varepsilon$ and $g=\varepsilon$ then $h=h_1x\bar{x}h_2x\bar{x}\cdots x\bar{x}h_k$ with $xh_1=f_1$ and $h_k\bar{x}=f_k$ thus $h\in OO$,

if $h\neq\varepsilon$ and $g\neq\varepsilon$ then $h=h_1x\bar{x}h_2x\bar{x}\cdots x\bar{x}h_i$ and $g=g_1x\bar{x}g_2x\bar{x}\cdots x\bar{x}g_{k-i}$, we have $f_1=xh_1\in\{x\}^*$, $f_i=h_i\bar{x}g_i\in\{\bar{x}\}^*\{x\}^*$, $f_k=g_{k-i}\bar{x}\in\{\bar{x}\}^*$, thus $h\in OO$ and $g\in OO$.

Suppose, now, that $f=xh\bar{y}g$ then $g\neq\varepsilon$, $h\neq\varepsilon$ and $g=g_1x\bar{x}g_2x\bar{x}\cdots x\bar{x}g_{k-i}$ and $h=h_1x\bar{x}h_2x\bar{x}\cdots x\bar{x}h_i$, with $f_1=xh_1$, $f_i=h_i\bar{y}g_i$, $f_k=g_{k-i}$:

if $g_i=\varepsilon$ then $f_i\in\{\bar{x}\}^*\{\bar{y}\}^*\bar{y}$ thus $h\in OD$ and $g=x\bar{x}$ or $g\in x\bar{x}GO$,

if $g_1\neq\varepsilon$, and $h_i=\varepsilon$ then $h=h_1x\bar{x}\cdots h_{i-1}x\bar{x}$ and $f_i=\bar{y}g_1\in\bar{y}y\{y\}^*$ thus $h=x\bar{x}$ or $h\in ODx\bar{x}$ and $g\in\bar{G}O$,

if $g_1\neq\varepsilon$ and $h_i\neq\varepsilon$ then $h_i\in\bar{y}\{\bar{y}\}^*$ and $g_1\in y\{y\}^*$ thus $h\in\overline{OD}$ and $g\in\bar{G}O$.

We have proved the equality (23).

Let $lr(x,\bar{x},y,\bar{y})$ be the series $\alpha(\mathbf{LR})$ and lr be the commutative series $lr(t,1,t,1)$.

Using Eqs. (23–25), we can write the following algebraic system in commutative variables:

$$
\begin{aligned}
oo &= t + t\,go + t\,oo + t\,oo^2 + t^2od(1+go) + t^2(1+od)\,\bar{g}o + t\,o\bar{d}\,\bar{g}o,\\
go &= oo + t\,go + t\,go\,oo + t^2gd(1+go) + t^2(1+gd)\,\bar{g}o + t\,g\bar{d}\,\bar{g}o, \qquad (26)\\
gd &= od + t\,go + t\,go\,od + t\,gd + t^2gd(1+gd) + t^2(1+gd)\,\bar{g}d + t\,g\bar{d}\,\bar{g}d.
\end{aligned}
$$

Using a symmetry argument, we get

$$od(x,\bar{x},y,\bar{y}) = go(x,\bar{x},y,\bar{y}) \qquad \text{and} \qquad o\bar{d}(x,\bar{x},y,\bar{y}) = \bar{g}o(x,\bar{x},y,\bar{y})$$

thus $od=go$ and $o\bar{d}=\bar{g}o$.

Moreover $\bar{G}O\cup\{x\bar{x}\}\cup x\bar{x}GO$ is in bijection with OO. Indeed, let $f\in\bar{G}O$ with $|f|=2n$,

$$f=f_1x\bar{x}\cdots x\bar{x}f_k \qquad \text{with} \quad f_1=y^{m+1},\ m\geq 0, \text{ and } f_k\in\{\bar{x}\}^*.$$

$g=x^{m+1}x\bar{x}f_2x\bar{x}\cdots x\bar{x}f_k$ is a word of OO, $|g|=|f|$ and g do not begin by a factor $x\bar{x}$. Moreover, all the words of OO which begin by a factor $x\bar{x}$ have the form $w=x\bar{x}w'$ with $w'=\varepsilon$ or $w'\in GO$. Thus we have

$$\bar{g}o = oo - t - t\,go. \qquad (27)$$

In the same way, there exists a bijection between $\bar{G}D \cup \{x\bar{x}\} \cup x\bar{x}GD$ and OD and then

$$\bar{g}d = od - t - t\,gd. \tag{28}$$

Using (26), (27), (28), we prove

PROPOSITION 15. *Let c be the serie $\sum_{n\geqslant 0} c_n t^n$ where c_n is the number of column-convex polyominoes with perimeter $2n+2$. Then c is the solution of the algebraic system:*

$$c = t + ts + tc + tc^2 + t^2 s(1+s)$$
$$+ t^2(1+s)(c-t-ts) + t(c-t-ts)^2, \tag{29}$$
$$s = c + ts + tcs + t^2 u(1+s)$$
$$+ t^2(1+u)(c-t-ts) + t(s-t-tu)(c-t-ts), \tag{30}$$
$$u = s + ts + ts^2 + tu + t^2 u(1+u)$$
$$+ t^2(1+u)(s-t-tu) + t(s-t-tu)^2. \tag{31}$$

Equation (30) is linear according to the variable u. So, we have

$$u = \frac{(s^2+s)t^2 + (1-2c)\,sx + s - c}{(s+1)t^2}.$$

Then we replace u in Eq. (31); using Macsyma we compute the resultant of this new equation and Eq. (29) eliminating the variable s. We get a new equation

$$t^6 p_1(c, t)\, p_2(c, t) = 0, \tag{32}$$

where $p_1(c, t)$ and $p_2(c, t)$ are the polynomials

$$p_1(c, t) = -4c^2 + (2t-5)c + t - 2,$$
$$p_2(c, t) = (2t^6 - 23t^5 + 38t^4 - 18t^3)c^4$$
$$+ (5t^6 + 40t^5 + 82t^4 - 68t^3 + 21t^2)c^3$$
$$+ (4t^6 - 30t^5 + 68t^4 - 70t^3 + 36t^2 - 8t)c^2$$
$$+ (t^6 - 10t^5 + 27t^4 - 32t^3 + 19t^2 - 6t + 1)c$$
$$-t^5 + 4t^4 - 6t^3 + 4t^2 - t.$$

The series for the column-convex polyomino cannot be solution of

$$p_1(c, t) = 0,$$

else it will have a Taylor series with complex coefficients. Thus it is one of the four solutions of

$$p_2(c, t) = 0. \tag{33}$$

Using the MACSYMA function *solve*, we can give an explicit expression for the four power series which are solution of this last equation. Then we compute the first coefficients of every one of the four solutions. Because of the complexity of these generating function, this last computation cannot be made by the mean of the MACSYMA function *taylor*. We have written a MACSYMA program which compute the Taylor series. Only two of them has integer coefficients. For only one the first coefficients agree with the coefficients obtained by direct enumeration. We get the following result:

THEOREM 16. *Let us consider the expressions*:

$$A = 18t^4(2t^3 - 23t^2 + 38t - 18)^2,$$
$$B = (t^2 - 38t + 1)(t - 1) - 6(t^2 - 6t + 1)\sqrt{-3t},$$
$$C = \frac{2(t-1)^9(t+1)^3 B}{3At^2},$$
$$D = -3t^2(t-1)^4(t+1)(11t^3 + 49t^2 - 439t + 171),$$
$$E = 2(t-1)^6(t+1)^2(t^2 + 10t + 1),$$
$$F = 81t^3(t-1)^5(t+1)^2(t^2 - 6t + 1)(t^3 - 79t^2 + 163t - 81),$$
$$G = -\frac{(t-1)(5t^3 - 25t^2 + 47t - 21)}{4t(2t^3 - 23t^2 + 38t - 18)},$$
$$H = \sqrt{\frac{2AC^{2/3} - DC^{1/3} + 2E}{C^{1/3}}}.$$

The generating function for the number c_n of column-convex polyominoes having perimeter $2n+2$ is:

$$\sum_{n \geq 0} c_n t^n = \frac{\sqrt{-(AC^{1/3} + D + EC^{-1/3}) - F}}{2\sqrt{AH}} - \frac{H}{2\sqrt{2A}} - G.$$

Remark 17. The first coefficients for the c_i are

1, 2, 7, 28, 122, 558, 2641,....

There is three different ways to check MACSYMA computations:

enumerating directly column-convex polyominoes for $n = 1, 2, \ldots, 7$,

setting the first coefficients of the Taylor expansion of the function defined by Theorem 16,

solving iteratively Eq. (33).

All the coefficients agreed for $n = 1, \ldots, 7$.

CONCLUSION

We have solved an enumerative problem by using algebraic languages methodology. It is significant to notice that this result would probably be very difficult to obtain without the use of languages methodology and MACSYMA.

Since this work was completed, some other problems have been solved using the languages methodology: the number of the so-called *directed lattice animals*, see Viennot [16], the number of *secondary structures of single-stranded nucleic acids* having a given complexity [15]. With convex polyominoes [3] and the coding of Cori and Vauquelin [2] for planar maps, we have five examples of a coding of a "planar picture" with words of an algebraic language.

A major problem would be to prove Theorem 3 using the bijection β of Proposition 7 and Remark 11. Then we would obtain a generating function according to the perimeter and the area of the polyomino.

It is intriguing that there exists exact formulae for some kind of polyominoes while only generating function are known for others. For polyominoes having a given perimeter, there exists an exact formula when they are convex [3] or *parallelogram* [11, 4, 3], and a generating function when they are column-convex. For polyominoes having a given area, there exists a simple generating function for column-convex [5, 6] and a complicated one for parallellogram [11, 4]. *Directed polyominoes* (*animals*) has a very simple exact formula according to the area [16]. Except some asymptotic results, there is nothing known about explicit formula for convex polyominoes having a given area and *directed polyominoes* having a given perimeter.

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