Existence and global stability of traveling curved fronts in the Allen–Cahn equations

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Abstract

This paper is concerned with existence and stability of traveling curved fronts for the Allen–Cahn equation in the two-dimensional space. By using the supersolution and the subsolution, we construct a traveling curved front, and show that it is the unique traveling wave solution between them. Our supersolution can be taken arbitrarily large, which implies some global asymptotic stability for the traveling curved front.

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1. Introduction

Traveling wave solutions have been intensively studied in various mathematical models motivated by many chemical or physical studies on them. See [10] for traveling wave solutions in the Belousov–Zhabotinsky reactions, and see [5] for the filamentary vortex of the Ginzburg–Landau equation. Traveling wave solutions in many other systems are treated in [14], for instance.
In this paper, we consider a solution \( u(\cdot, \cdot, t) \in L^\infty(\mathbb{R}^2) \) of the Allen–Cahn equation
\[
\begin{align*}
  u_t &= \Delta u + f(u) \quad (x, y) \in \mathbb{R}^2, \quad t > 0, \\
  u(x, y, 0) &= u_0(x, y) \quad (x, y) \in \mathbb{R}^2.
\end{align*}
\] (1.1)

Here, a nonlinear smooth function \( f \) is of bistable type and a given initial function \( u_0 \) is bounded. The following is the standing assumptions on \( f \):

1. \( f(1) = 0, f(-1) = 0, \ f'(1) < 0 \) and \( f'(-1) < 0. \) (A1)
2. \( \int_{-1}^{1} f(s)ds > 0. \) (A2)
3. \( f(s) < 0 \) and \( f'(s) < 0 \) for \( s > 1. \ f(s) > 0 \) and \( f'(s) < 0 \) for \( s < -1. \) (A3)

A typical example of \( f \) is
\[
  f(u) = -(u + 1)(u + a)(u - 1),
\] (1.2)

where \( a \in (0, 1) \) is a given number.

Now, we state previous works of traveling wave solutions. Bonnet and Hamel [1] showed the existence of the traveling wave solutions when the reaction term \( f \) is of the "ignition temperature" type. Hamel and Monneau [8] shows the uniqueness of the traveling front of the corresponding singular limit problem. Hamel and Nadirashvili [9] studied traveling wave solutions when \( f \) is the Fisher-KPP type. Fife [6] studied the traveling wave solutions as a singular limit problem by putting a small coefficient to the diffusion term.

Under our assumptions on \( f \), the constant states \(-1 \) and \( 1 \) are asymptotically stable. The region of the state \( 1 \) is getting larger and larger and finally it covers the whole space. When the state \( 1 \) propagates, we can observe the characteristic profiles. In the one-dimensional space, one of the typical solutions is a traveling wave solution which never changes its shape. Setting \( u(x, t) = \Phi(\mu) \) and \( \mu = x - kt \), we have
\[
  -\Phi'' - k\Phi' - f(\Phi) = 0, \quad \Phi'(\mu) < 0 \quad (-\infty < \mu < \infty),
\]
\[
  \Phi(-\infty) = 1, \quad \Phi(+\infty) = -1.
\] (1.3)

For the nonlinearity (1.2), we have \( k = \sqrt{2}a \) and \( \Phi(\mu) = -\tanh(\mu/\sqrt{2}). \)

The Allen–Cahn equation (1.1) is related to the following interface equation given by
\[
  \mathcal{V} = H + k,
\] (1.4)

where \( \mathcal{V} \) is a normal velocity, \( H \) is the curvature, and \( k \) is a given constant. See [6] or [3] for example. For this interface model (1.4), traveling fronts have been studied by [2,5,11,12] and so on.
If the interface is represented by the graph $y = v(x, t)$, Eq. (1.4) is reduced to

$$v_t = \frac{v_{xx}}{1 + v_x^2} + k\sqrt{1 + v_x^2}, \quad x \in \mathbb{R}, \quad t > 0,$$

(1.5)

If the traveling front is represented by $\varphi(x) + ct$ for the suitable coordinate where $c$ is the speed of the traveling front. Then $\varphi(x)$ and $c$ satisfy

$$c = \frac{\varphi_{xx}}{1 + \varphi_x^2} + k\sqrt{1 + \varphi_x^2}.$$  

(1.6)

The following is the explicit expression for the traveling curved fronts of (1.6).

**Proposition 1.1.** (Ninomiya and Taniguchi [11, Propositions 1.1, 2.5]). For $c > k > 0$, there exists a unique solution $\varphi(x; c)$ of (1.5) with asymptotic lines $y = m_*|x|$, where

$$m_* \overset{\text{def}}{=} \frac{\sqrt{c^2 - k^2}}{k} > 0.$$  

The graph of $y = \varphi(x; c)$ can be parametrized by $\theta = \arctan \varphi_x(x; c)$ as

$$x(\theta; c) = \frac{\theta}{c} + \frac{2}{m_* c} \text{arctanh} \left( \frac{\sqrt{c+k}}{c-k} \tan \frac{\theta}{2} \right)$$

$$y(\theta; c) = \frac{1}{c} \log \left( \frac{2(c^2 - k^2)}{c(c \cos \theta - k)} \right) + \frac{m_*}{c} \arctan m_*.$$

for $\theta \in (-\arctan m_*, \arctan m_*)$. The graph is strictly convex with

$$\varphi_{xx}(x; c) > 0 \quad \text{for all} \quad x \in \mathbb{R}.$$  

The graph of $\varphi(x; c)$ is "V-shaped" and connects two asymptotes. The asymptotic stability of the traveling curved front in (1.5) is discussed in [5,12]. The traveling curved front is proved to be asymptotically stable, if the initial perturbation decays at infinity. See [12, Theorems 1.1 and 4.1]. By the observation for the interface model, we can expect that a "V-shaped" traveling wave solution of (1.1) exists.

We study traveling wave solutions. We assume that the solutions travel towards $y$-direction without loss of generality. We put

$$u(x, y, t) = w(x, y - ct, t), \quad z = y - ct.$$
Then, we obtain
\[
\begin{align*}
\frac{\partial w}{\partial t} - w_{xx} - w_{zz} - cw_z - f(w) = 0, \quad (x, z) \in \mathbb{R}^2, \quad t > 0, \\
 w|_{t=0} = u_0 \quad \text{in} \quad \mathbb{R}^2.
\end{align*}
\]

(1.7)

We denote the solution of (1.7) with \( w(x, z; 0; u_0) = u_0(x, z) \) by \( w(x, z; t; u_0) \).

We seek for \( v(x, z) \) with
\[
\mathcal{L}[v] \equiv -v_{xx} - v_{zz} - cv_z - f(v) = 0 \quad \text{in} \quad \mathbb{R}^2.
\]

(1.8)

The traveling wave with speed \( c \) in \( y \)-direction becomes a stationary solution of (1.7). Then, two functions \( \Phi(k(z \pm m_+ x)/c) \) satisfy (1.8) and these are called planar traveling fronts. Since the maximum of subsolutions is also a subsolution, it turns out that
\[
v^-(x, z) \overset{\text{def}}{=} \max \left\{ \Phi\left(\frac{k}{c}(z - m_+ x)\right), \Phi\left(\frac{k}{c}(z + m_+ x)\right) \right\}
\]

is a subsolution of (1.7). This \( v^-(x, z) \) is strictly monotone decreasing in \( z \).

Now, we are searching a traveling wave solution towards \( y \)-direction. If \( v^- \) is given as an initial condition, it seems to converge to a traveling curved front. Actually, we have the following theorem.

**Theorem 1.2 (Existence).** There exists a traveling wave solution \( u(x, y, t) = v_*(x, y - ct) \) of (1.1) with
\[
\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |v_*(x, z) - v^-(x, z)| = 0,
\]

\( v^-(x, z) < v_*(x, z) \).

The traveling wave solution \( v_* \) is asymptotically stable if the given perturbation decays at infinity.

**Theorem 1.3 (Stability).** Let \( u_0(x, y) \) satisfy
\[
\lim_{R \to \infty} \sup_{x^2 + y^2 > R^2} |u_0(x, y) - v^-(x, y)| = 0,
\]

(1.9)

\( v^-(x, y) \leq u_0(x, y) \).

(1.10)
Then the solution \( u(x, y, t; u_0) \) of (1.1) satisfies

\[
\lim_{t \to \infty} \|u(x, y, t; u_0) - v^*(x, y - ct)\|_{L^\infty(\mathbb{R}^2)} = 0.
\]

This theorem immediately implies the uniqueness of such traveling wave solutions.

**Corollary 1.4 (Uniqueness).** Let \( v(x, z) \) be a solution of (1.8) with

\[
\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |v(x, z) - v^-(x, z)| = 0,
\]

(1.11)

\[
v^-(x, z) \leq v(x, z) \text{ for } (x, z) \in \mathbb{R}^2.
\]

(1.12)

Then \( v(x, z) = v_*(x, z) \) holds true. Here \( v_*(x, z) \) is given by Theorem 1.2.

Hereafter, if \( v(x, z) \) is a solution of (1.8) and satisfies (1.11) and (1.12), then \( v(x, y - ct) \) is called a *traveling curved front*. See Figs. 1 and 2 for the shape of the traveling curved front.

**Remark 1.5.** Fife [6] showed the construction of traveling curved fronts by using the singular perturbation framework, which corresponds to our case when \( a \) in (1.2) is sufficiently close to 0. Theorem 1.2 studies the general case.
2. Existence of a traveling curved front

There exists a positive constant \( \delta_1 \) \((0 < \delta_1 < \frac{1}{4})\) with

\[-f'(s) \geq \kappa_1 \text{ for } s < -1 + 2\delta_1 \text{ or } s > 1 - 2\delta_1,\]

where

\[\kappa_1 \overset{\text{def}}{=} \frac{1}{2} \min\{-f'(-1), -f'(1)\} > 0.\]

Since \( \Phi(\mu) \) is monotone decreasing in \( \mu \), we can define positive constants \( A \) and \( B \) by

\[\Phi(-A) = 1 - \frac{\delta_1}{2}, \quad \Phi(B) = -1 + \frac{\delta_1}{2},\]

respectively. Then

\[-1 + \frac{\delta_1}{2} < \Phi(\mu) < 1 - \frac{\delta_1}{2}\]

is equivalent to \(-A < \mu < B\).

**Lemma 2.1.** (Brazhnik [2], Ninomiya and Taniguchi [11,12]) *There exist positive constants \( \gamma_1, \gamma_2, K_i \ (i = 1, \ldots, 4) \) and \( \mu_\pm \) with

\[
\max\{|\Phi'(\zeta)|, |\Phi''(\zeta)|\} \leq K_1 \exp(-\gamma_1|\zeta|),
\]

\[
\max\{|\varphi''(\zeta)|, |\varphi'''(\zeta)|\} \leq K_2 \sech(\gamma_2\zeta),
\]

\[K_3 \sech(\gamma_2\zeta) \leq \frac{c}{\sqrt{1 + \varphi'(\zeta)^2}} - k \leq K_4 \sech(\gamma_2\zeta),\]
\[
m_*|\xi| \leq \varphi(\xi),
\]
\[
\mu_- \leq \mu(\xi) \leq \mu_+ \]
for any \( \xi \in \mathbb{R} \) and \( \xi \in \mathbb{R} \), where
\[
\mu(\xi) \overset{\text{def}}{=} \frac{c(\varphi(\xi) - m_*|\xi|)}{c - k\sqrt{1 + \varphi'(\xi)^2}}.
\]

We note that
\[
\gamma_2 \overset{\text{def}}{=} cm_* = \frac{c\sqrt{c^2 - k^2}}{k} > 0
\]
and that the curvature of \( \zeta = \varphi(\xi) \) is calculated as
\[
\frac{\varphi''(\xi)}{(1 + \varphi'(\xi)^2)^{3/2}} = \frac{c}{\sqrt{1 + \varphi'(\xi)^2}} - k.
\]

**Theorem 2.2.** There exist a positive constant \( \varepsilon_0 \) and a positive function \( \alpha_0(\varepsilon) \) so that, for \( 0 < \varepsilon < \varepsilon_0 \) and \( 0 < \alpha \leq \alpha_0(\varepsilon) \),
\[
v^+(x, z; \varepsilon, \alpha) \overset{\text{def}}{=} \Phi \left( \frac{z - \varphi(\alpha x)}{\sqrt{1 + \varphi'(\alpha x)^2}} \right) + \varepsilon \text{sech}(\gamma_2 \alpha x),
\]
is a supersolution of (1.8) with
\[
\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |v^+(x, z; \varepsilon, \alpha) - v^-(x, z)| \leq 2\varepsilon, \quad (2.2)
\]
\[
v^-(x, z) < v^+(x, z; \varepsilon, \alpha) \quad \text{for} \quad (x, z) \in \mathbb{R}^2, \quad (2.3)
\]
\[
-(v^+)_z(x, z; \varepsilon, \alpha) > 0 \quad \text{for} \quad (x, z) \in \mathbb{R}^2. \quad (2.4)
\]

For simplicity, we often write \( v^+(x, z) \) instead of \( v^+(x, z; \varepsilon, \alpha) \). We prove this theorem in Section 3. For the meaning of \( (z - \varphi(\alpha x))/\sqrt{1 + \varphi'(\alpha x)^2} \), see Fig. 3.

Next, we prove the existence of the traveling curved front.

**Proof of Theorem 1.2.** Theorem 2.2 guarantees the existence of a supersolution \( v^+(x, z; \varepsilon, \alpha) \). By the definition of \( w(x, z, t; v^\pm) \), we have
\[
v^-(x, z) < w(x, z, t; v^-) < w(x, z, t; v^+) < v^+(x, z) \quad \text{for} \quad x, z \in \mathbb{R}, \quad t > 0. \quad (2.5)
\]
Fig. 3. The construction of the supersolution $v^+$.

Since $v^+$ is a supersolution and $v^-$ is a subsolution, $w(x, z, t; v^+)$ is monotone decreasing in $t$ and $w(x, z, t; v^-)$ is monotone increasing in $t$. The limit functions

$$v^*(x, z) \overset{\text{def}}{=} \lim_{t \to \infty} w(x, z, t; v^+),$$

$$v^*(x, z) \overset{\text{def}}{=} \lim_{t \to \infty} w(x, z, t; v^-),$$

exist in $L^\infty(R^2)$ with

$$\mathcal{L}[v^*] = 0, \quad \mathcal{L}[v^+] = 0,$$

$$v^-(x, z) \leq v^*(x, z) \leq v^*(x, z) \leq v^+(x, z; \varepsilon, \alpha) \quad \text{for} \ (x, z) \in R^2. \quad (2.7)$$

For the detail, see Sattinger [16].

Theorem 2.2 and (2.5) imply that, for any $0 < \varepsilon \leq \varepsilon_0$ and $0 < \alpha \leq \alpha_0(\varepsilon)$, there exist a positive constant $R$ and a supersolution $v^+(x, z; \varepsilon, \alpha)$ satisfying

$$v^-(x, z) \leq v^*_s(x, z) \leq v^+(x, z; \varepsilon, \alpha) \leq v^-(x, z) + 2\varepsilon \quad \text{when} \ x^2 + z^2 > R^2.$$

Thus, the limit function $v^*_s$ is a traveling curved front of (1.1). □

Since $\varepsilon \in (0, \varepsilon_0]$ can be taken arbitrarily small, (2.2) gives

$$\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |v^*(x, z) - v^*_s(x, z)| = 0. \quad (2.8)$$
By the definition, $v^*$ may depend on $\varepsilon$ and $\alpha$. Corollary 1.4 or Lemma 4.6 says that $v^*$ is independent of $\varepsilon$ and $\alpha$ and it equals $v_\ast$.

3. Proof of Theorem 2.2

Set

$$
\zeta \overset{\text{def}}{=} \alpha x,
$$

$$
\zeta' \overset{\text{def}}{=} \frac{z - \varphi(x)/x}{\sqrt{1 + \varphi'(x)^2}},
$$

$$
\sigma(\zeta) \overset{\text{def}}{=} \varepsilon \text{sech}(\sqrt{2} \zeta).
$$

By the chain rule, we have

$$
\zeta_x = -\frac{\alpha \varphi' \varphi''}{1 + \varphi'^2} - \frac{\varphi'}{\sqrt{1 + \varphi'^2}},
$$

$$
\zeta_{xx} = -\frac{\alpha^2 \varphi'''}{1 + \varphi'^2} + \frac{2 \varphi' \varphi''}{1 + \varphi'^2} \zeta + \frac{3 \alpha^2 \varphi'^2 \varphi'''}{(1 + \varphi'^2)^2} \zeta + \frac{\alpha (\varphi'^2 - 1) \varphi''}{(1 + \varphi'^2)^{3/2}}.
$$

Using these equalities and (1.3), we get

$$
\mathcal{L}[v^+] = -\frac{\Phi''(\zeta)}{1 + \varphi'(\zeta)^2} - (\Phi'(\zeta)\zeta)_x - \frac{c \Phi'(\zeta)}{\sqrt{1 + \varphi'(\zeta)^2}} - f(\Phi(\zeta) + \sigma(\zeta)) - \alpha^2 \sigma''(\zeta)
$$

$$
= \left(1 - \frac{1}{1 + \varphi'(\zeta)^2} - \zeta_x^2\right) \Phi''(\zeta) - \zeta_{xx} \Phi'(\zeta) + \left(k - \frac{c}{\sqrt{1 + \varphi'(\zeta)^2}}\right) \Phi'(\zeta)
$$

$$
+ f(\Phi(\zeta)) - f(\Phi(\zeta) + \sigma(\zeta)) - \alpha^2 \sigma''(\zeta)
$$

$$
= I_1 + I_2 + I_3 + I_4,
$$

where

$$
I_1 \overset{\text{def}}{=} \left(1 - \frac{1}{1 + \varphi'(\zeta)^2} - \zeta_x^2\right) \Phi''(\zeta),
$$

$$
I_2 \overset{\text{def}}{=} -\zeta_{xx} \Phi'(\zeta),
$$

$$
I_3 \overset{\text{def}}{=} f(\Phi(\zeta)) - f(\Phi(\zeta) + \sigma(\zeta)) - \alpha^2 \sigma''(\zeta),
$$

$$
I_4 \overset{\text{def}}{=} \left(k - \frac{c}{\sqrt{1 + \varphi'(\zeta)^2}}\right) \Phi'(\zeta).
$$
\[ I_3 \overset{\text{def}}{=} - \left( \frac{c}{\sqrt{1 + \varphi^2}} - k \right) \Phi'(\zeta), \]

\[ I_4 \overset{\text{def}}{=} - f(\Phi + \sigma) + f(\Phi) - \varphi^2 \sigma''. \]

By (3.1) and (3.2), we have

\[ I_1 = -\varphi \left\{ \left( \frac{\varphi ' \varphi ''}{1 + \varphi^2} \right)^2 \zeta^2 + \frac{2\varphi ' \varphi ''}{(1 + \varphi^2)^{3/2}} \right\} \Phi''(\zeta), \]

\[ I_2 = -\varphi \left\{ -\frac{\varphi^2 + \varphi ' \varphi ''}{1 + \varphi^2} \zeta + \frac{3\varphi ' \varphi ''^2}{(1 + \varphi^2)^2} \zeta + \frac{1}{(1 + \varphi^2)^{3/2}} \right\} \Phi'(\zeta), \]

By Lemma 2.1, we can easily show

\[ |I_1| \leq K_5 \varphi \operatorname{sech}(\gamma_2 \xi), \]

\[ |I_2| \leq K_6 \varphi \operatorname{sech}(\gamma_2 \xi), \]

\[ I_3 \geq -K_3 \Phi'(\zeta) \operatorname{sech}(\gamma_2 \xi) > 0, \]

for \(0 < \varphi \leq 1\). Assume

\[ 0 < \varepsilon < \varepsilon_0 \leq \frac{\delta_1}{2}. \]  \hspace{1cm} (3.3)

Lemma 2.1 implies

\[ I_4 \geq \kappa_1 \sigma - K_7 \varphi^2 \varepsilon \operatorname{sech}(\gamma_2 \xi) = (\kappa_1 - K_7 \varphi^2) \varepsilon \operatorname{sech}(\gamma_2 \xi), \]

when \(\Phi(\zeta) \leq -1 + \delta_1/2\) or \(\Phi(\zeta) \geq 1 - \delta_1/2\). Then we have \(L[v^+] \geq \kappa_1 \sigma/2 > 0\) if

\[ 0 < \varphi \leq \min \left\{ 1, \sqrt{\frac{K_1}{4K_7}}, \frac{K_1}{4(K_5 + K_6)} \varepsilon \right\}. \]  \hspace{1cm} (3.4)

If \(-1 + \delta_1/2 \leq \Phi(\zeta) \leq 1 - \delta_1/2\), namely, \(-A \leq \zeta \leq B\), then we have

\[ I_3 \geq K_3 \varphi \operatorname{sech}(\gamma_2 \xi), \]

\[ |I_4| \leq K_8 \sigma + K_7 \varphi^2 \varepsilon \operatorname{sech}(\gamma_2 \xi) \leq (K_8 + K_7 \varphi) \varepsilon \operatorname{sech}(\gamma_2 \xi). \]
where
\[ p \overset{\text{def}}{=} \min_{-A \leq \zeta \leq B} (-\Phi'(\zeta)) > 0. \tag{3.5} \]

Take \( \varepsilon \) and \( \alpha \) so small that
\[ \frac{1}{2} K_3 p > (K_5 + K_6)\alpha + K_8 \varepsilon + K_7 \alpha \varepsilon \tag{3.6} \]
holds. Then we have
\[ \mathcal{L}[v^+] \geq \frac{1}{2} \min\{\kappa_1 \varepsilon, K_3 p\} \operatorname{sech}(\gamma_2 \zeta) > 0 \quad \text{in } \mathbb{R}^2 \]
under (3.3), (3.4) and (3.6). Thus \( v^+ \) is a supersolution.

Next, we will show the inequality (2.3) in \( x \geq 0 \). For simplicity of notation, set
\[ \zeta_1 \overset{\text{def}}{=} \frac{k}{c} (z - m_* |x|), \quad \zeta_2 \overset{\text{def}}{=} \frac{z - m_* |x|}{\sqrt{1 + \varphi'(zx)^2}}. \]

Recall that
\[ v^+ - v^- = \Phi(\zeta) - \Phi(\zeta_1) + \sigma(\zeta), \quad \zeta = \frac{z - \varphi(\alpha x) / \alpha}{\sqrt{1 + \varphi'(zx)^2}} \]
and that
\[ m_* |x| \leq \frac{1}{\alpha} \varphi(\alpha x) \quad \text{for } 0 < \alpha < 1, \ x \in \mathbb{R}. \]

If \( \zeta \leq \zeta_1 \), then (2.3) holds true apparently. Assume that \( \zeta > \zeta_1 \). We have
\[ \zeta - \zeta_1 = \left( \frac{1}{\sqrt{1 + \varphi'(\zeta)^2}} - \frac{k}{c} \right) (z - m_* |x|) - \frac{\varphi(\zeta) - m_* |\zeta|}{\alpha \sqrt{1 + \varphi'(\zeta)^2}}. \]

This equality and Lemma 2.1 imply
\[ z - m_* |x| > \frac{\mu(\zeta)}{\alpha} \geq \frac{\mu_+}{\alpha} > 0. \tag{3.7} \]

Thus, we have
\[ 0 < \zeta_1 < \zeta \leq \zeta_2 \]
and
\[ v^+ - v^- = \Phi(\zeta) - \Phi(\zeta_2) + \Phi(\zeta_2) - \Phi(\zeta_1) + \sigma(\zeta) \]
\[ \geq \Phi(\zeta_2) - \Phi(\zeta_1) + \sigma(\zeta) \]
\[ = -\left(\frac{1}{\sqrt{1 + \phi'(\zeta)^2}} - \frac{k}{c}\right)(z - m_\ast|x|)|\Phi'(\theta\zeta_2 + (1 - \theta)\zeta_1)| + \sigma(\zeta) \]
\[ = -\left(\frac{1}{\sqrt{1 + \phi'(\zeta)^2}} - \frac{k}{c}\right)\frac{c\zeta_1}{k} |\Phi'(\theta\zeta_2 + (1 - \theta)\zeta_1)| + \sigma(\zeta) \]

for some \( \theta \in (0, 1) \). By (2.1), we see
\[ |\Phi'(\theta\zeta_2 + (1 - \theta)\zeta_1)| \leq K_1 \exp(-\gamma_1|\theta\zeta_2 + (1 - \theta)\zeta_1|) \leq K_1 \exp(-\gamma_1|\zeta_1|). \]

It follows from (3.7) that
\[ \zeta_1 = \frac{k}{c}(z - m_\ast|x|) \geq \frac{k\mu(\zeta)}{c\zeta} \geq \frac{k\mu_-}{c\zeta}. \]

Using (2.2) and the above inequalities, we have
\[ v^+ - v^- \geq -\left(\frac{1}{\sqrt{1 + \phi'(\zeta)^2}} - \frac{k}{c}\right)\frac{c\zeta_1}{k} \exp(-\gamma_1|\zeta_1|) + \varepsilon \text{sech}(\gamma_2 \zeta) \]
\[ \geq -\frac{K_1 K_4}{k\gamma_1^2} \left(\sup_{\zeta_0 > 0} \zeta_0 \exp(-\zeta_0)\right) \text{sech}(\gamma_2 \zeta) + \varepsilon \text{sech}(\gamma_2 \zeta) \]
\[ \geq -\frac{4K_1 K_4 c}{e^2 k^2 \gamma_1^2 \mu_-} \varepsilon \text{sech}(\gamma_2 \zeta) + \varepsilon \text{sech}(\gamma_2 \zeta). \]

Define \( \varepsilon_0 \) and \( \varepsilon_0(\varepsilon) \) by
\[ \varepsilon_0 \overset{\text{def}}{=} \min \left\{1, \frac{\delta_1}{2}, \frac{K_3 p}{4K_8} \right\} > 0, \]
\[ \varepsilon_0(\varepsilon) \overset{\text{def}}{=} \min \left\{1, \sqrt{\frac{\kappa_1}{4K_7}}, \frac{K_3 p}{4(K_5 + K_6 + K_7)}, \frac{\kappa_1}{4(K_5 + K_6)} e, \frac{e^2 k^2 \gamma_1^2 \mu_-}{4K_1 K_4 c} \right\} > 0. \]

Then, we see that \( v^+ \geq v^- \) for \( x \geq 0 \) if \( 0 < \varepsilon \leq \varepsilon_0 \) and \( 0 < \varepsilon \leq \varepsilon_0(\varepsilon) \).

In the case where \( x < 0 \), (2.3) can be also proved similarly.
Next we consider (2.2). The mean-value theorem implies
\[ v^+ - v^- = \Phi(\zeta) - \Phi(\zeta_1) + \sigma(\zeta), \]
\[ = \Phi'(\zeta_3)(\zeta - \zeta_1) + \sigma(\zeta), \]
where
\[ \zeta_3 \overset{\text{def}}{=} \theta\zeta + (1 - \theta)\zeta_1 = \theta \frac{z - \varphi(\zeta)/\alpha}{\sqrt{1 + \varphi'(\zeta)^2}} + (1 - \theta) \frac{k}{c} (z - m_*|x|) \]
with some \( \theta \in (0, 1) \). Since \( 0 < \sigma(\zeta) \leq \varepsilon \), we will show that, for some large \( R_1 > 0 \) and \( R_2 > 0 \),
\[ |\Phi'(\zeta_3)(\zeta - \zeta_1)| \leq \varepsilon \]
if \( |\zeta| \geq R_1 \) or \( |z| \geq R_2 \). By the definition of \( \zeta_3 \), we have
\[ z = \left( \zeta_3 + \frac{\theta \varphi(\zeta)}{\alpha\sqrt{1 + \varphi'(\zeta)^2}} + \frac{(1 - \theta)km_*|z|}{c\varphi(\zeta)} \right) \left( \frac{k}{c} + \theta \left( \frac{1}{\sqrt{1 + \varphi'(\zeta)^2}} - \frac{k}{c} \right) \right). \]
This implies
\[ \Phi'(\zeta_3)(\zeta - \zeta_1) \]
\[ = \left( \frac{1}{\sqrt{1 + \varphi'(\zeta)^2}} - \frac{k}{c} \right) \Phi'(\zeta_3)z \]
\[ - \frac{1}{\alpha} \left( \frac{\varphi(\zeta)}{\sqrt{1 + \varphi'(\zeta)^2}} - \frac{km_*|z|}{c} \right) \Phi'(\zeta_3). \]
By Lemma 2.1, we have
\[ \sup_{\zeta_3 \in \mathbb{R}} |\Phi'(\zeta_3)| \leq K_1, \quad \sup_{\zeta_3 \in \mathbb{R}} |\Phi'(\zeta_3)\zeta_3| \leq K_9 \]
with some positive constant \( K_9 \). It follows from (3.9)–(3.11) that if \( R_1 \) is so large, then
\[ \sup_{|\zeta| \geq R_1} |\Phi'(\zeta_3)(\zeta - \zeta_1)| \leq \varepsilon. \]
holds true. Next, we consider the case where $|\zeta| \leq R_1$ and $|z| \geq R_2$. By (3.8), we have

$$\liminf_{z \to \infty} \sup_{|\zeta| \leq R_1} |\zeta_3| = \infty. \quad (3.12)$$

Taking $R_2$ large enough and using (3.9)–(3.12), we have

$$\sup_{|z| \geq R_2} \sup_{|\zeta| \leq R_1} |\Phi'(\zeta_3)(\zeta - \zeta_1)| \leq \varepsilon.$$

If $R = \sqrt{R_1^2/\varepsilon^2 + R_2^2}$, we see (2.2). Finally, (2.4) follows immediately from (1.3) and the definition of $v^+(x, z; \varepsilon, \infty)$. This completes the proof. \(\square\)

4. Uniqueness of the traveling curved fronts

Let $w^{(j)}(x, z, t)$ be the solution of

$$w_t^{(j)} + \mathcal{L}[w^{(j)}] = 0 \quad \text{in } \mathbb{R}^2, \quad t > 0,$$

$$w^{(j)}(x, z, 0) = w_0^{(j)}(x, z) \quad \text{in } \mathbb{R}^2,$$

for $j = 1, 2$. We have the following fact.

**Lemma 4.1 (Continuity in initial data).** If

$$-2 \leq w_0^{(j)}(x, z) \leq 2 \quad \text{for } (x, z) \in \mathbb{R}^2, \quad j = 1, 2,$$

then

$$\sup_{(x, z) \in \mathbb{R}^2} |w^{(2)}(x, z, t) - w^{(1)}(x, z, t)| \leq e^{Mt} \sup_{(x, z) \in \mathbb{R}^2} |w_0^{(2)}(x, z) - w_0^{(1)}(x, z)|$$

where

$$M \overset{\text{def}}{=} \sup_{|s| \leq 2} |f'(s)|. \quad (4.1)$$

**Proof.** Let \(\hat{w}\) be a solution of

$$\hat{w}_t - \hat{w}_{xx} - \hat{w}_{zz} - c\hat{w}_z - M\hat{w} = 0 \quad \text{in } \mathbb{R}^2, \quad t > 0,$$

$$\hat{w}(x, z, 0) = \sup_{(x, z) \in \mathbb{R}^2} |w_0^{(2)}(x, z) - w_0^{(1)}(x, z)| \quad \text{in } \mathbb{R}^2.$$
Then we have $\hat{w} \geq 0$ and $\tilde{w} \equiv \hat{w} - (w^{(2)} - w^{(1)})$ satisfies

$$
\tilde{w}_t - \tilde{w}_{xx} - \tilde{w}_{zz} - c\tilde{w}_z - f'(\theta w^{(2)} + (1 - \theta)w^{(1)})\tilde{w} = (M - f'(\theta w^{(2)} + (1 - \theta)w^{(1)}))\hat{w} \geq 0 \quad \text{in } \mathbb{R}^2, \ t > 0,
$$

$$
\tilde{w}(x, z, 0) \geq 0 \quad \text{in } \mathbb{R}^2
$$

with some $\theta \in (0, 1)$. The maximum principle, see [15] for example, yields

$$
\begin{align*}
&w^{(2)}(x, z, t) - w^{(1)}(x, z, t) \leq \hat{w}(x, z, t) = e^{Mt} \sup_{(x,z) \in \mathbb{R}^2} |w_0^{(2)}(x, z) - w_0^{(1)}(x, z)| \\
&\text{for } (x, z) \in \mathbb{R}^2, \ t > 0. \quad \text{Similarly, we have}
\end{align*}
$$

$$
\begin{align*}
&w^{(1)}(x, z, t) - w^{(2)}(x, z, t) \leq \hat{w}(x, z, t) = e^{Mt} \sup_{(x,z) \in \mathbb{R}^2} |w_0^{(2)}(x, z) - w_0^{(1)}(x, z)| \\
&\text{for } (x, z) \in \mathbb{R}^2, \ t > 0. \quad \text{This completes the proof.} \quad \square
\end{align*}
$$

Let $w_1(t)$ be defined by

$$
\begin{align*}
w_1'(t) &= f(w_1(t)) \quad \text{for } t > 0, \\
w_1(0) &= \min \left\{ -1, \inf_{(x,z) \in \mathbb{R}^2} u_0(x, z) \right\} \leq -1.
\end{align*}
$$

Similarly, we define $w_2(t)$ by

$$
\begin{align*}
w_2'(t) &= f(w_2(t)) \quad \text{for } t > 0 \\
w_2(0) &= \max \left\{ 1, \sup_{(x,z) \in \mathbb{R}^2} u_0(x, z) \right\} \geq 1.
\end{align*}
$$

Then $w_1(t), w_2(t)$ are solutions to (1.7) with $w_1(0) \leq u_0(x, z) \leq w_2(0)$. The comparison principle yields

$$
\begin{align*}
w_1(t) \leq w(x, z, t) \leq w_2(t) \quad \text{for } (x, z) \in \mathbb{R}^2, \ t > 0. \quad (4.2)
\end{align*}
$$

Sending $t \to \infty$, we have

$$
-1 \leq \lim \inf_{t \to \infty} w(x, z, t) \leq \lim \sup_{t \to \infty} w(x, z, t) \leq 1 \quad \text{for } (x, z) \in \mathbb{R}^2.
$$
The strong maximum principle, see [7] for instance, implies the following strict inequalities.

**Lemma 4.2 (A priori estimate).** Let \( v_*(x, y), v^*(x, y) \) be as in (2.6). Then

\[
-1 < v_*(x, z) \leq v_*(x, z) < 1 \quad \text{for } (x, z) \in \mathbb{R}^2.
\]

holds true. Moreover, one has

\[
-(v_*)_z(x, z) > 0, \quad -(v^*)_z(x, z) > 0 \quad \text{for } (x, z) \in \mathbb{R}^2.
\]

**Proof.** We show the latter statement. Recall that \( w(x, z, t; v^\pm) \) is the solution of (1.7) with \( w(x, z, 0; v^\pm) = v^\pm(x, z) \). Then the derivative \( w_z(x, z, t; v^+) \) satisfies

\[
\left( \frac{\partial}{\partial t} - \Delta - c \frac{\partial}{\partial z} + f'(w(x, z, t; v^+)) \right) w_z(x, z, t; v^+) = 0 \quad \text{in } \mathbb{R}^2, \ t > 0
\]

\[
w_z(x, z, 0; v^+) \leq 0 \quad \text{in } \mathbb{R}^2.
\]

Then the maximum principle yields

\[
w_z(x, z, t; v^+) < 0 \quad \text{in } \mathbb{R}^2, \ t > 0.
\]

Sending \( t \to \infty \), we have \( (v^*)_z(x, z) \leq 0 \). The strong maximum principle says

\[
-(v^*)_z(x, z) > 0 \quad \text{in } \mathbb{R}^2.
\]

Since \( v^-(x, z) \) is also strictly monotone decreasing in \( z \), the same argument holds for \( v_*(x, z) \). \( \square \)

To show the uniqueness we prepare the following lemma.

**Lemma 4.3.** There exists a positive constant \( \gamma_3 \) with

\[
-(v_*)_z(x, z) \geq \gamma_3 \quad \text{when} \quad -1 + \delta_1 \leq v_*(x, z) \leq 1 - \delta_1,
\]

\[
-(v^*)_z(x, z) \geq \gamma_3 \quad \text{when} \quad -1 + \delta_1 \leq v^+(x, z) \leq 1 - \delta_1.
\] (4.3)

**Proof.** By (2.2) and (2.7), there exists a constant \( R \) with

\[
|v_*(x, z) - v^-(x, z)| \leq \frac{\delta_1}{2} \quad \text{for } x^2 + z^2 \geq R^2.
\]
Then, we have
\[
\{(x, z) \in \mathbb{R}^2 \mid x^2 + z^2 > R^2, \quad \frac{1}{2} + \delta_1 \leq v^+(x, z) \leq 1 + \frac{1}{2} \delta_1 \} \\
\subset \left\{(x, z) \in \mathbb{R}^2 \mid x^2 + z^2 > R^2, \quad -1 + \frac{\delta_1}{2} \leq v^-(x, z) \leq 1 + \frac{\delta_1}{2} \right\}
\]
\[
= \left\{(x, z) \in \mathbb{R}^2 \mid x^2 + z^2 > R^2, \quad -A \leq k c (z - m_\ast |x|) \leq B \right\}.
\]

For \( C > 0 \) we set
\[
\Omega_n \overset{\text{def}}{=} \{(x, z) \mid (x-n)^2 + (z - m_\ast |n|)^2 < C^2\},
\]
\[
v_n(x, z) \overset{\text{def}}{=} v_\ast(x-n, z - m_\ast |n|),
\]
\[
B_0 \overset{\text{def}}{=} \{(x, z) \in \mathbb{R}^2 \mid x^2 + z^2 < 4C^2\}
\]
(4.4)

for \( n = 0, \pm 1, \pm 2 \ldots \). Choose a positive constant \( C \) so large that
\[
\left\{(x, z) \mid x^2 + z^2 > R^2, \quad -A \leq k c (z - m_\ast |x|) \leq B \right\} \subset \bigcup_{n=-\infty}^{\infty} \Omega_n
\]
is valid. The function \( v_n \) is a solution of
\[
-\Delta v_n - c(v_n)z - f(v_n) = 0 \quad \text{in} \; B_0.
\]

Theorem 2.2 also says that
\[
\lim_{n \to \pm \infty} \sup_{(x, z) \in B_0} \left| v_n(x, z) - \Phi \left( \frac{k}{c} (z - m_\ast (\text{sgn} \; n)x) \right) \right| = 0.
\]

Using the interior \( L^q \) estimate for the second derivative of elliptic equations, see [7] for instance, we have
\[
v_n(x, z) \to \Phi \left( \frac{k}{c} (z - m_\ast x) \right) \quad \text{in} \; W^{2,q}(\Omega_0) \quad \text{as} \; n \to \infty,
\]
\[
v_n(x, z) \to \Phi \left( \frac{k}{c} (z + m_\ast x) \right) \quad \text{in} \; W^{2,q}(\Omega_0) \quad \text{as} \; n \to -\infty,
\]
for any $q > 1$. If we take $q > 2$ and choose a positive constant $\theta_*$ with

$$0 < \theta_* < 1 - \frac{2}{q},$$

this immediately implies

$$-(v_n)_z \rightarrow -\frac{k}{c} \Phi' \left( \frac{k}{c} (z - m_* x \sgn n) \right) \quad \text{in } C^0(\overline{\Omega_0}) \text{ as } n \rightarrow \pm \infty.$$ 

Thus there exists a large positive integer $n_0$ with

$$-(v_n)_z \geq \frac{1}{2} \inf_{(x,z) \in \Omega_0} \left\{ -\frac{k}{c} \Phi' \left( \frac{k}{c} (z - m_* |x|) \right) \right\}$$

for $|n| \geq n_0$. Defining

$$\gamma_3 \overset{\text{def}}{=} \min \left\{ \frac{1}{2} \inf_{(x,z) \in \Omega_0} \left\{ -\frac{k}{c} \Phi' \left( \frac{k}{c} (z - m_* |x|) \right) \right\}, \frac{p k_1}{c}, \right\}$$

we obtain (4.3) for $v_*$. Next, we will show the second inequality of (4.3). By the definition of $v^+(x, z; \varepsilon, \alpha)$ and $\varepsilon_0$, we have $0 < \varepsilon < \delta_1/2$ and

$$\{(x, z) \in R^2 \mid x^2 + z^2 > R^2, -1 + \delta_1 \leq v^+(x, z; \varepsilon, \alpha) \leq 1 - \delta_1\}$$

$$\subset \left\{ (x, z) \in R^2 \mid x^2 + z^2 > R^2, -1 + \frac{\delta_1}{2} \leq \Phi(\zeta) \leq 1 + \frac{\delta_1}{2} \right\}$$

$$= \left\{ (x, z) \in R^2 \mid x^2 + z^2 > R^2, -A \leq \zeta \leq B \right\}.$$

Then, we get

$$-(v^+_z)_z = -\Phi'(\zeta) \zeta_z = \frac{-\Phi'(\zeta)}{\sqrt{1 + \Phi'(\zeta)^2}}.$$
By (3.5) and Lemma 2.1, we have

\[-(v^+)_z \geq \frac{pk}{c}.\]

This completes the proof. □

**Lemma 4.4.** Let \( \bar{v} \) be a supersolution to (1.8) with

\[-\bar{v}_z(x, z) > 0, \quad -1 - \delta_1 < \bar{v}(x, z) < 1 + \delta_1 \quad \text{for all } (x, z) \in \mathbb{R}^2,\]

\[-\bar{v}_z(x, z) \geq \gamma_3 \quad \text{if} \quad -1 + \delta_1 \leq \bar{v}(x, z) \leq 1 - \delta_1.\]

Let \( \underline{v} \) be a subsolution to (1.8) with

\[-\underline{v}_z(x, z) > 0, \quad -1 - \delta_1 < \underline{v}(x, z) < 1 + \delta_1 \quad \text{for all } (x, z) \in \mathbb{R}^2,\]

\[-\underline{v}_z(x, z) \geq \gamma_3 \quad \text{if} \quad -1 + \delta_1 \leq \underline{v}(x, z) \leq 1 - \delta_1.\]

Then there exists a large positive constant \( \rho \) such that, for any \( \delta \in (0, \delta_1 / 2] \), \( w^+ \) and \( w^- \) defined by

\[w^+(x, z, t; \bar{v}) \overset{\text{def}}{=} \bar{v}(x, z - \rho \delta(1 - e^{-\beta t})) + \delta e^{-\beta t},\]

\[w^-(x, z, t; \underline{v}) \overset{\text{def}}{=} \underline{v}(x, z + \rho \delta(1 - e^{-\beta t})) - \delta e^{-\beta t}\]

are a supersolution and a subsolution of (1.7), respectively.

We remark that both \( v^* \) and \( v^- \) satisfy the assumption on \( v \).

**Proof.** We have

\[w^+_t + \mathcal{L}[w^+] = \delta \beta e^{-\beta t} (-\rho \bar{v}_z - 1) + \mathcal{L}[\bar{v} + \delta e^{-\beta t}] \geq \delta \beta e^{-\beta t} (-\rho \bar{v}_z - 1) - f (\bar{v} + \delta e^{-\beta t}) + f (\bar{v}) = \delta e^{-\beta t} \left( -\rho \beta \bar{v}_z - \beta - \int_0^1 f' (\bar{v} + s \delta e^{-\beta t}) \, ds \right),\]

\[w^-_t + \mathcal{L}[w^-] = -\delta e^{-\beta t} \left( -\rho \beta \underline{v}_z - \beta - \int_0^1 f' (\underline{v} - s \delta e^{-\beta t}) \, ds \right),\]
where $\bar{v} = \bar{v}(x, z - \rho \delta (1 - e^{-\beta t}))$ and $\bar{v} = v(x, z + \rho \delta (1 - e^{-\beta t}))$. For simplicity, let $v$ be either $\bar{v}$ or $\bar{v}$. By the assumption, we have

$$-\rho \beta v_z - \beta - \int_0^1 f'(v \pm s \delta e^{-\beta t}) \, ds \geq \beta \left( \gamma_3 \rho - 1 - \frac{M}{\beta} \right)$$

when $-1 + \delta_1 \leq v \leq 1 - \delta_1$. Here $M$ is as in (4.1). For $v < -1 + \delta_1$ or $v > 1 - \delta_1$, we have an estimate

$$-\rho \beta v_z - \beta - \int_0^1 f'(v \pm s \delta e^{-\beta t}) \, ds \geq \kappa_1 - \beta.$$

We choose $\beta$ so small and $\rho$ so large to get

$$0 < \beta < \kappa_1, \quad \rho > \frac{\beta + M}{\beta \gamma_3}. \quad (4.6)$$

This immediately implies

$$w^+_t + L[w^+] \geq 0, \quad w^-_t + L[w^-] \leq 0.$$

Thus, $w^+$ and $w^-$ are a supersolution and a subsolution, respectively. □

To show Theorem 1.3, we prepare the following two lemmas.

**Lemma 4.5.** Let $w(x, z, t)$ be the solution of (1.7) with (1.9). Then

$$\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |w(x, z, T) - v^*(x, z)| = 0$$

holds true for any fixed $T > 0$.

**Lemma 4.6.** Let $v_*$ and $v^*$ be as in (2.6). Then

$$v_*(x, z) \equiv v^*(x, z) \quad \text{for all} \, (x, z) \in \mathbb{R}^2.$$

holds true.

We state the proof of these lemmas later.

**Proof of Theorem 1.3.** The solution $w(x, z, t)$ of (1.7) satisfies (4.2). We will show that, for any $\varepsilon_* > 0$, there exists a positive constant $T_*$ with

$$\sup_{(x, z) \in \mathbb{R}^2} |w(x, z, t) - v_*(x, z)| \leq \varepsilon_*$$

for $t > T_*$. 

First, we choose $\delta$ small enough to satisfy

$$v^*(x, z - \rho \delta) \leq v^*(x, z) + \frac{\varepsilon_0}{3}, \quad 0 < \delta < \varepsilon_0$$

(4.7)

and fix it where $\varepsilon_0$ and $\rho$ are given in Theorem 2.2 and Lemma 4.4, respectively.

Next, we will choose a suitable supersolution. By the argument of (4.2), there exists $T_\delta > 0$ with

$$w(x, z, t; v^-) \leq w(x, z, t) < 1 + \frac{\delta}{2} \quad \text{for} \quad (x, z) \in \mathbb{R}^2, \quad t \geq T_\delta.$$

Lemma 4.5 shows that

$$w(x, z, T_\delta) \leq v^- (x, z) + \frac{\delta}{2} \quad \text{for} \quad x^2 + z^2 \geq R^2$$

for some $R > 0$. If $\varepsilon$ is small enough,

$$\zeta = \frac{z - \varphi(\zeta)/\sqrt{\varphi'(\zeta)^2}}{\sqrt{1 + \varphi'(\zeta)^2}} \leq \frac{k}{c} \left( R - \frac{\varphi(0)}{2\varepsilon} \right) \leq \Phi^{-1} \left( 1 - \frac{\delta}{2} \right)$$

for $x^2 + z^2 \leq R^2$. Namely, choosing $\varepsilon$ so small to satisfy

$$0 < \varepsilon < \min \left\{ \frac{\varphi(0)}{2(R - \min\{\Phi^{-1}(1 - \delta/2)c / k, 0\})}, \varepsilon_0(\delta) \right\},$$

we get

$$v^+(x, z) \geq 1 - \frac{\delta}{2} \quad \text{for} \quad x^2 + z^2 \leq R^2,$$

where $v^+(x, z) = v^+(x, z; \delta, \varepsilon)$. Combining these inequalities and (2.5), we obtain

$$w(x, z, T_\delta) < v^+(x, z) + \delta \quad \text{for} \quad (x, z) \in \mathbb{R}^2.$$

Then

$$w(x, z, t + T_\delta; v^-) \leq w(x, z, t + T_\delta) \leq w^+(x, z, t; v^+)$$

holds true for $t \geq 0$. Using the maximum principle again, we have

$$w(x, z, t + s + T_\delta; v^-) \leq w(x, z, t + s + T_\delta) \leq w(x, z, s; u^+)$$

(4.8)
holds true for \( t \geq 0 \) and \( s \geq 0 \) where
\[
u(x, z, t) \overset{\text{def}}{=} w^+(x, z, t; v^+).
\]
Since \( w(x, z, t; v^+) \) monotonically converges to \( v^*(x, z) \) as \( t \to \infty \), there exists a positive constant \( s_1 \) with
\[
\sup_{(x, z) \in \mathbb{R}^2} |w(x, z, s_1; v^+; \delta) - v^*(x, z - \rho \delta)| \leq \frac{\varepsilon_*}{3},
\]
where
\[
v^+; \delta(x, z) \overset{\text{def}}{=} v^+(x, z - \rho \delta).
\]
Lemma 4.1 implies
\[
\sup_{(x, z) \in \mathbb{R}^2} |w(x, z, s_1; u) - w(x, z, s_1; v^+; \delta)| \leq e^{ms_1} \sup_{(x, z) \in \mathbb{R}^2} |u(x, z) - v^+; \delta(x, z)| \tag{4.9}
\]
for any function \( u(x, z) \). We can take \( T_1 \) large enough to satisfy
\[
e^{ms_1} \sup_{(x, z) \in \mathbb{R}^2} |w^+(x, z, t; v^+) - v^+(x, z - \rho \delta)| \leq \frac{\varepsilon_*}{3} \tag{4.10}
\]
for \( t \geq T_1 \) by the definition of \( w^+ \). Combining (4.9) and (4.10) with \( u = u' \), we have
\[
|w(x, z, s_1; u') - v^*(x, z - \rho \delta)|
\]
\[
\leq |w(x, z, s_1; u') - w(x, z, s_1; v^+; \delta)| + |w(x, z, s_1; v^+; \delta) - v^*(x, z - \rho \delta)|
\]
\[
\leq \frac{2}{3} \varepsilon_*.
\]
for any \( t \geq T_1 \). Thus,
\[
w(x, z, t + s_1 + \delta) \leq w(x, z, s_1; u') \leq v^*(x, z - \rho \delta) + \frac{2}{3} \varepsilon_* \tag{4.11}
\]
holds true for \( t \geq T_1 \). By (4.7), (4.8), (4.11) and Lemma 4.6, we have
\[
w(x, z, t; v^-) \leq w(x, z, t) \leq v^*(x, z) + \varepsilon_* = v_+(x, z) + \varepsilon_*
\]
for \((x, z) \in \mathbb{R}^2\) and \(t \geq s_1 + T_1 + T_\delta\). Since \(v_\ast(x, z) = \lim_{t \to \infty} w(x, z, t; v^-)\), we have completed the proof. \(\square\)

5. Proof of key lemmas

Now we prove Lemma 4.5.

**Proof of Lemma 4.5.** For any fixed \(T > 0\), we prove that \(w(x, z, T)\) satisfies (1.11). We define

\[
V(x, z) = \Phi \left( \frac{z - \varphi(x)}{\sqrt{1 + \varphi'(x)^2}} \right).
\]

Lemma 2.1 implies

\[
\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |v^-(x, z) - V(x, z)| = 0.
\]

This relation and the assumption on the initial condition give

\[
\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |u_0(x, z) - V(x, z)| = 0.
\]

We define

\[
W(x, z, t) \overset{\text{def}}{=} w(x, z, t) - V(x, z).
\]

Then, we have

\[
W_t - W_{xx} - W_{zz} - c W_z - f(W + V) + f(V) = h(x, z).
\]

Here

\[
h(x, z) \overset{\text{def}}{=} -\mathcal{L}[V]
\]

satisfies

\[
\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |h(x, z)| = 0.
\]

Using

\[-f(W + V) + f(V) = -f'(V + \theta W)W\]
for some $0 < \theta(x, z, t) < 1$, we obtain

$$W_t - W_{xx} - W_{zz} - cW_z - f'(V + \theta W)W = h(x, z) \quad \text{for} \ (x, z) \in \mathbb{R}^2, \ t > 0,$$

$$W(x, z, 0) = W_0(x, z) \quad \text{for} \ (x, z) \in \mathbb{R}^2.$$ 

Here

$$W_0(x, z) \overset{\text{def}}{=} u_0(x, z) - V(x, z)$$

satisfies

$$\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |W_0(x, z)| = 0.$$ 

We define $\tilde{W}(x, z, t)$ by

$$\tilde{L}[\tilde{W}] \overset{\text{def}}{=} \tilde{W}_t - \tilde{W}_{xx} - \tilde{W}_{zz} - c\tilde{W}_z - M_*\tilde{W} = |h(x, z)| \quad \text{for} \ (x, z) \in \mathbb{R}^2, t > 0,$$

$$\tilde{W}(x, z, 0) = |W_0(x, z)| \quad \text{for} \ (x, z) \in \mathbb{R}^2.$$ 

Here, we define

$$M_* \overset{\text{def}}{=} \sup_{|s| \leq 2 + \|u_0\|_{L^\infty(\mathbb{R}^2)}} |f'(s)|.$$ 

The maximum principle yields

$$\tilde{W}(x, z, t) \geq 0 \quad \text{for} \ (x, z) \in \mathbb{R}^2, t \geq 0.$$ 

Then, we have

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} - c\frac{\partial}{\partial z} - f'(V + \theta W)\right)(\tilde{W} - W) = (M_* - f'(V + \theta W))\tilde{W} \geq 0$$

for $(x, z) \in \mathbb{R}^2, t > 0$ with the initial condition

$$(\tilde{W} - W)|_{t=0} \geq 0 \quad \text{for} \ (x, z) \in \mathbb{R}^2.$$ 

The maximum principle yields

$$W(x, z, t) < \tilde{W}(x, z, t) \quad \text{for} \ (x, z) \in \mathbb{R}^2, t > 0.$$
Applying the argument also to \((-W)\), we obtain

\[ |W(x, z, t)| < \tilde{W}(x, z, t) \quad \text{for } (x, z) \in \mathbb{R}^2, \ t > 0. \]

Let \(Z(x_1, x_2, t)\) be the kernel of \(\tilde{L}\), that is,

\[ Z(x_1, x_2, t) \equiv \frac{1}{4\pi t} \exp \left( -\frac{x_1^2 + (x + ct)^2}{4t} + M_*t \right). \]

Then one has

\[ \tilde{W}(x_1, x_2, t) = I(x_1, x_2, t) + J(x_1, x_2, t), \]

where

\[ I \equiv \int_0^t d\tau \int_{\mathbb{R}^2} Z(x_1 - \xi_1, x_2 - \xi_2, t - \tau) h(\xi_1, \xi_2) d\xi_1 d\xi_2, \]

\[ J \equiv \int_{\mathbb{R}^2} Z(x_1 - \xi_1, x_2 - \xi_2, t) W_0(\xi_1, \xi_2) d\xi_1 d\xi_2. \]

For an estimate for the kernel function, one has

\[ |Z(x_1, x_2, t)| \leq \frac{c_1}{t} \exp \left( -\frac{c_2 x_1^2 + x_2^2}{t} \right) \quad \text{for } 0 < t \leq T, \]

where \(c_1, c_2\) depend only on \(T\) and \(M_*\). For example, we can take

\[ c_1 = \frac{1}{4\pi} \exp \left( M_*T + \frac{c^2 T}{4} \right), \quad c_2 = \frac{1}{8}. \]

By direct calculations we have

\[ |J| \leq c_1 \int_{\mathbb{R}^2} e^{-c_2(\eta_1^2 + \eta_2^2)} |W_0(x_1 + \sqrt{\eta_1^2 + \eta_2^2})| d\eta_1 d\eta_2. \quad (5.1) \]

There exists \(0 < t_1 < t < T\) with

\[ I = t \int_{\mathbb{R}^2} Z(x_1 - x_2, \xi_1 - \xi_2, t - t_1) h(\xi_1, \xi_2) d\xi_1 d\xi_2. \]
Thus we get

$$|I| \leq c_1 T \int_{R^2} e^{-c_2 \left( \frac{\eta_1^2}{\lambda_1^2} + \frac{\eta_2^2}{\lambda_2^2} \right)} |h(x_1 + \sqrt{R-T} \eta_1, x_2 + \sqrt{R-T} \eta_2)| d\eta_1 d\eta_2. \quad (5.2)$$

Gathering (5.1) and (5.2), we obtain

$$\lim_{R \to \infty} \sup_{x_1^2 + x_2^2 > R^2} \tilde{W}(x_1, x_2, T) = 0$$

and thus

$$\lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |W(x, z, T)| = 0.$$

This complete the proof of Lemma 4.5. □

**Proof of Lemma 4.6.** We prove this lemma by contradiction using a similar argument as in [4,13]. Assume $v_* \neq v^*$. Then the strong maximum principle and

$$v_*(x, z) \leq v^*(x, z)$$

yield

$$v_*(x, z) < v^*(x, z) \quad \text{for all} \quad (x, z) \in R^2.$$

Take $\beta$ and $\rho$ as in (4.6). For any $0 < \delta < \delta_1/2$, we take $\lambda$ large enough so as to get

$$v^*(x, z) \leq v_*(x, z - \lambda) + \delta$$

using (2.8). Lemma 4.4 says that

$$w^+(x, z - \lambda, t; v_*) \equiv v_*(x, z - \lambda - \rho \delta(1 - e^{-\beta t})) + \delta e^{-\beta t}$$

is a supersolution. Letting $t \to \infty$ in

$$v^*(x, z) \leq v_*(x, z - \lambda - \rho \delta(1 - e^{-\beta t})) + \delta e^{-\beta t},$$

we obtain

$$v^*(x, z) \leq v_*(x, z - \lambda - \rho \delta).$$
Define

\[ A \overset{\text{def}}{=} \inf \{ \lambda | v^*(x, z) \leq v_*(x, z - \lambda) \}. \]

Then, we have \( A \geq 0 \) and

\[ v^*(x, z) \leq v_*(x, z - A). \tag{5.3} \]

Now, we show

\[
\lim_{R \to \infty} \sup_{|z - m_*,| \geq R} \max\{ |(v_*)_z(x, z - A)|, |(v^*)_z(x, z - A)| \} = 0. \tag{5.4}
\]

We show this equality only for \( v_* \), since the same argument is valid for \( v^* \). Assume the contrary, then there exist \( \varepsilon_1 > 0 \) and \( \{(x_n, z_n)\}_{n=1}^{\infty} \) with

\[
\lim_{n \to \infty} |z_n - m_*,| x_n || = \infty, \quad -(v_*)_z(x_n, z_n) \geq \varepsilon_1.
\]

For simplicity we assume \( \lim_{n \to \infty} (z_n - m_*,| x_n |) = \infty \). Define

\[ v_n(x, z) \overset{\text{def}}{=} v_*(x + x_n, z + z_n) \text{ in } B_0. \tag{5.5} \]

Because of \( \lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} |v_*(x, z) - v^- (x, z)| = 0 \), we have

\[
\lim_{n \to \infty} \sup_{B_0} |v_n(x, z) + 1| = 0,
\]

\[
\lim_{n \to \infty} \sup_{B_0} |f(v_n(x, z))| = 0.
\]

Applying the Schauder interior estimate to (5.5), we get

\[ v_n \to -1 \text{ in } C^{\theta_*}(\overline{\Omega_0}). \]

Here, \( \theta_* \) is as in (4.5) and \( B_0 \) is as in (4.4). Thus we get

\[
\lim_{n \to \infty} (v_n)_z(0, 0) = \lim_{n \to \infty} (v_*)_z(x_n, z_n) = 0,
\]

which contradicts the assumption. If we assume \( \lim_{n \to \infty} (z_n - m_*,| x_n |) = -\infty \), we get the same contradiction. Thus we proved (5.4).
We prove $A = 0$ by contradiction. Assume the contrary. Take $R_*$ so large that
\[ 2 \rho \sup_{|z-m_*|\geq R_*-\rho \delta_1} \max\{|(v_*)_z(x, z-A)|, |(v_*)_z(x, z-A)|\} < 1 \quad (5.6) \]
holds true. By (2.8) and the strong maximum principle, we have
\[ v_*(x, z) < v_*(x, z-A) \quad \text{in} \; D, \]
where
\[ D \overset{\text{def}}{=} \{(x, z) ||z - m_*|x|| \leq R_*\}. \]
Again by (2.8) we can choose a small positive constant $h$ with
\[ 0 < h < \min \left\{ \frac{\delta_1}{2}, \frac{A}{2 \rho} \right\}, \]
\[ v_*(x, z) < v_*(x, z-A+2\rho h) \quad \text{in} \; D. \]
In $\mathbb{R}^2 \setminus D$ we have
\[ v_*(x, z-A+2\rho h) - v_*(x, z-A) = 2\rho h \int_0^1 (v_*)_z(x, z-A+2s\rho h) \, ds \geq -h, \]
using (5.6). This implies
\[ v_*(x, z-A) \leq v_*(x, z-A+2\rho h) + h, \quad \text{in} \; \mathbb{R}^2 \setminus D. \]
Combining the inequalities in $D$ and $\mathbb{R}^2 \setminus D$, we get
\[ v_*(x, z) \leq v_*(x, z-A+2\rho h) + h \quad \text{in} \; \mathbb{R}^2. \]
Now
\[ w^{++}(x, z, t) \overset{\text{def}}{=} v_*(x, z-A+2\rho h - \rho h(1-e^{-\beta t})) + e^{-\beta t} \]
is a supersolution by Lemma 4.4. Since
\[ v^*(x, z) \leq w^{++}(x, z, 0), \]
holds, the comparison principle yields
\[ v^*(x, z) \leq w^{++}(x, z, t) \quad \text{for} \ t \geq 0. \]
By sending \( t \to \infty \), this inequality implies
\[ v^*(x, z) \leq v_*(x, z - A + \rho h), \]
which contradicts the definition of \( A \). Thus, \( A = 0 \) follows and \( v_* \equiv v^* \) holds true. This completes the proof. \( \Box \)

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References