Interlacing Inequalities for Invariant Factors

R. C. Thompson

Institute for Applied Algebra
and Department of Mathematics
University of California at Santa Barbara
Santa Barbara, California 93106

Submitted by Hans Schneider

ABSTRACT

The following problems are solved in this paper: (1) characterization of the behavior of invariant factors of (generally rectangular) $\mathfrak{R}$-matrices under adjunction of rows; (2) characterization of the invariant factors of a square submatrix of a square $\mathfrak{R}$-matrix; (3) characterization of the relationship between the similarity invariants of a square $\mathfrak{F}$-matrix and those of a principal submatrix. Here $\mathfrak{R}$ is a commutative principal ideal ring, $\mathfrak{F}$ a field.

1. INTRODUCTION

The symbol $\mathfrak{R}$ will denote a principal ideal ring, that is, a commutative integral domain in which every ideal is principal. An important and well-known theorem [4, p. 62] asserts that a unimodular $m\times n$ matrix A over $\mathfrak{R}$, in which $m<n$, may always be augmented with a single row to obtain a unimodular $(m+1)\times n$ matrix B. Here A unimodular means that the greatest common divisor of its $m\times m$ subdeterminants is 1. This theorem is sometimes stated in the equivalent form that unimodular A may always be completed to a square unimodular matrix. Of course, not every $m\times n$ matrix A is unimodular. In general, a matrix over $\mathfrak{R}$ (whether rectangular or square) possesses *invariant factors*, namely, its diagonal elements when reduced under equivalence to Smith canonical form. Unimodularity occurs precisely when all invariant factors are 1. A natural question, therefore, is to determine the relationship between the invariant factors of a matrix A and those of a one row prolongation B. The first objective of this paper is to give the complete identificaiton of this relationship. As a special case we recover the results just stated concerning unimodular prolongations.

Two consequences flow easily from our identification of the behavior of invariant factors under a one row prolongation. The first of these is the

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characterization of the complete relationship between the invariant factors of a square $\mathcal{R}$-matrix and those of a submatrix. A second consequence, evolving out of the first, is the complete solution of a question in linear algebra that has been unresolved since the inception of elementary divisor theory in the 1800s. Let $\mathcal{F}$ be a field, let C be an $n \times n$ matrix over $\mathcal{F}$, and let A be a principal $(n-1)\times(n-1)$ submatrix of C. This long outstanding question is the determination of the relation between the similarity invariants of C and those of A. Several attempts to answer this question have been made in recent years, notably by de Oliviera [2]. We are able to answer this question in full, and indeed able to answer it for principal submatrices having deficiency higher than one.

An altogether unexpected and surprising feature of our results is a striking analogy between our invariant factor theorems on the one hand and the interlacing inequalities for eigenvalues of Hermitian matrices and singular values of arbitrary matrices on the other. [Recall that the singular values of a complex matrix A are the eigenvalues of the semidefinite Hermitian matrix $(AA^*)^{1/2}$.] Because of this striking analogy, we have incorporated the word "interlacing" in the title of this paper.

We remark that a recent most interesting paper of Carlson [1] gives a result rather closely related to our Theorems 1 and 2. Indeed, it is possible to deduce our Theorems 1 and 2 from Carlson's results by considering the matrix of relations on a finitely generated module.

2. ONE ROW PROLONGATIONS

Let A be an $m \times n$ matrix and B an $(m+1)\times n$ matrix, both over $\mathcal{R}$, with B containing A as its first m rows. We wish to identify the relationship between the invariant factors of A and those of B. We do not assume $m < n$; indeed, we will allow $m \geqslant n$. However, our discussion splits into two cases according as B has rank equal to or one more than the rank of A. In the first case we assume that $\operatorname{rank} B = \operatorname{rank} A + 1$, and in the second that $\operatorname{rank} B = \operatorname{rank} A$.

LEMMA 1.. *Let* $\operatorname{rank} A = k$ *and* $\operatorname{rank} B = k+1$. *Denote by*

$$h_1(A)|h_2(A)|\cdots|h_k(A)$$

and by

$$h_1(B)|h_2(B)|\cdots|h_k(B)|h_{k+1}(B)$$

the (nonzero) invariant factors of A and B, respectively, where the bar | denotes divides. (We include trivial invariant factors, i.e., those equal to 1.) *Then if B is a one row prolongation of A, the following "interlacing inequalities" must hold:*

$$h_1(B)|h_1(A)|h_2(B)|h_2(A)|\cdots|h_k(A)|h_{k+1}(B). \tag{1}$$

Proof. We have

$$B=\begin{bmatrix} A \\ b \end{bmatrix},$$

where b is a single row. After an equivalence we may diagonalize A; hence assume that B has the form

$$\begin{bmatrix} h_1(A) & & & & & & \\ & h_2(A) & & & & \mathbf{0} & \\ & & \ddots & & & & \\ & & & h_k(A) & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ b_1 & b_2 & \cdots & b_k & b_{k+1} & \cdots & b_n \end{bmatrix}. \tag{2}$$

After an equivalence on the last $n-k$ columns, we may take $b_{k+2}=\cdots=b_n=0$. Because $\operatorname{rank} B=\operatorname{rank} A+1$ we then see that $b_{k+1}\neq 0$.

Let p be a fixed prime element of $\mathfrak{R}$, and consider A and B as matrices over the ring $\mathfrak{R}_p$ consisting of fractions with denominator prime to p. Under this localization of $\mathfrak{R}$, all primes other than p become units, with the effect that A and B acquire new invariant factors comprising precisely the powers of p in the original invariant factors. This technique was used, for example, by Gerstein in [3]; effectively it permits the passage from invariant factors to elementary divisors. After this localization and trivial elementary operations on B to remove factors which now are units, we may assume that

$$h_i(A)=p^{\alpha_i}, \qquad 1\leqslant i\leqslant k,$$

$$h_i(B)=p^{\beta_i}, \qquad 1\leqslant i\leqslant k+1,$$

with

$$\begin{gathered}\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k,\\ \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_k \leqslant \beta_{k+1}.\end{gathered} \tag{3}$$

To prove (1) it will suffice to prove that

$$\beta_1 \leqslant \alpha_1 \leqslant \beta_2 \leqslant \alpha_2 \leqslant \cdots \leqslant \beta_k \leqslant \alpha_k \leqslant \beta_{k+1}. \tag{4}$$

For $1 \leqslant i \leqslant k+1$ let $d_i(B)$ be the ith determinantal divisor of B, namely the greatest common divisor of the $i \times i$ subdeterminants of B. It is well known that $d_i(B) = h_1(B) \cdots h_i(B)$; hence here

$$d_i(B) = p^{\beta_1 + \cdots + \beta_i}, \qquad 1 \leqslant i \leqslant k+1.$$

Evidently there is only one nonzero $(k+1)\times(k+1)$ minor, whence we may take

$$b_{k+1} = p^{\beta_1 + \cdots + \beta_{k+1} - \alpha_1 \cdots - \alpha_k}.$$

For fixed i, $1 \leqslant i \leqslant k$, the calculation of the i-rowed minors is straightforward. Rejecting minors obviously divisible by other minors, we find that

$$\begin{aligned} p^{\beta_1 + \cdots + \beta_i} = \big(& p^{\alpha_1 + \cdots + \alpha_i}, p^{\beta_1 + \cdots + \beta_{k+1} - \alpha_i - \cdots - \alpha_k}, b_1 p^{\alpha_2 + \cdots + \alpha_i},\\ & b_2 p^{\alpha_1 + \alpha_3 + \cdots + \alpha_i}, \ldots, b_i p^{\alpha_1 + \cdots + \alpha_{i-1}},\\ & b_{i+1} p^{\alpha_1 + \cdots + \alpha_{i-1}}, \ldots, b_k p^{\alpha_1 + \cdots + \alpha_{i-1}} \big). \end{aligned} \tag{5}$$

The parentheses on the right hand side here denote greatest common divisor. Ignoring units, we may take $b_i = p^{t_i}$, $1 \leqslant i \leqslant k$, where $t_i = +\infty$ if b_i should be zero. In order for the relation (5) to hold, it is necessary and sufficient that the following inequalities all hold, with at least one inequality being equality.

$$\beta_1 + \cdots + \beta_i \leqslant \begin{cases} \alpha_1 + \cdots + \alpha_i, & (6\mathrm{a})_i\\ \beta_1 + \cdots + \beta_{k+1} - \alpha_i - \cdots - \alpha_k, & (6\mathrm{b})_i\\ t_1 + \alpha_2 + \cdots + \alpha_i, & (6.1)_i\\ \alpha_1 + t_2 + \alpha_3 + \cdots + \alpha_i, & (6.2)_i\\ \cdots & \\ \alpha_1 + \cdots + \alpha_{i-1} + t_i, & (6.i)_i\\ \cdots & \\ \alpha_1 + \cdots + \alpha_{i-1} + t_k. & (6.k)_i \end{cases}$$

Denote these inequalities collectively as $(6)_i$. We first note that $\beta_1 \leqslant \alpha_1$ follows immediately from $(6a)_1$. Let $1 < i \leqslant k$; we wish to prove that $\beta_i \leqslant \alpha_i$. To do so, we exploit the equality in $(6)_{i-1}$ and the corresponding inequality in $(6)_i$.

If $(6a)_{i-1}$ is equality, we obtain $\beta_1 + \cdots + \beta_{i-1} = \alpha_1 + \cdots + \alpha_{i-1}$; in combination with $(6a)_i$, namely $\beta_1 + \cdots + \beta_i \leqslant \alpha_1 + \cdots + \alpha_i$, we see that $\beta_i \leqslant \alpha_i$.

If $(6b)_{i-1}$ is equality, we obtain $\beta_i + \cdots + \beta_{k+1} = \alpha_{i-1} + \cdots + \alpha_k$. From $(6b)_i$ we get $\beta_{i+1} + \cdots + \beta_{k+1} \geqslant \alpha_i + \cdots + \alpha_k$, whence $\beta_i \leqslant \alpha_{i-1} \leqslant \alpha_i$.

If $(6.1)_{i-1}$ is equality, we obtain $\beta_1 + \cdots + \beta_{i-1} = t_1 + \alpha_2 + \cdots + \alpha_{i-1}$, and from $(6.1)_i$ we have $\beta_1 + \cdots + \beta_i \leqslant t_1 + \alpha_2 + \cdots + \alpha_i$. Hence $\beta_i \leqslant \alpha_i$. Similarly we treat equality $(6.2)_{i-1}$ and inequality $(6.2)_i$, equality $(6.3)_{i-1}$ and inequality $(6.3)_i$, etc. Certain of these steps go by the route $\beta_i \leqslant \alpha_{i-1} \leqslant \alpha_i$.

This completes the proof that $\beta_i \leqslant \alpha_i$, $i = 1, \ldots, k$. Next, we prove that $\alpha_i \leqslant \beta_{i+1}$, for $i = 1, \ldots, k$. For $i = k$ this follows from $(6b)_k$. Therefore, suppose that $1 \leqslant i < k$. We shall exploit the equality in $(6)_{i+1}$ and the corresponding inequality in $(6)_i$.

If $(6a)_{i+1}$ is equality, we have $\beta_1 + \cdots + \beta_{i+1} = \alpha_1 + \cdots + \alpha_{i+1}$. From $(6a)_i$ we get $\beta_1 + \cdots + \beta_i \leqslant \alpha_1 + \cdots + \alpha_i$, whence $\alpha_i \leqslant \alpha_{i+1} \leqslant \beta_{i+1}$.

If $(6b)_{i+1}$ is equality, we have $\alpha_{i+1} + \cdots + \alpha_k = \beta_{i+2} + \cdots + \beta_{k+1}$. From $(6b)_i$ we get $\alpha_i + \cdots + \alpha_k \leqslant \beta_{i+1} + \cdots + \beta_i$, whence $\alpha_i \leqslant \beta_{i+1}$.

If $(6.1)_{i+1}$ is equality, we have $\beta_1 + \cdots + \beta_{i+1} = t_1 + \alpha_2 + \cdots + \alpha_{i+1}$. From $(6.1)_i$ we have $\beta_1 + \cdots + \beta_i \leqslant t_1 + \alpha_2 + \cdots + \alpha_i$, whence $\alpha_i \leqslant \alpha_{i+1} \leqslant \beta_{i+1}$. Similarly we handle equality $(6.j)_{i+1}$ and inequality $(6.j)_i$ for $j = 2, \ldots, k$. ■

The lemma is now proved. Returning to matrices over $\mathfrak{R}$, we next have

LEMMA 2.. *Assume that* (1) *holds. Then the matrix* (2) *with*

$$b_j = \frac{h_1(B) \cdots h_j(B)}{h_1(A) \cdots h_{j-1}(A)}, \qquad j = 1, \ldots, k+1,$$

$$b_j = 0 \qquad \text{for} \quad j > k+1$$

is integral, and its invariant factors are precisely $h_j(B)$ for $j = 1, \ldots, k+1$.

Proof. That B is integral is obvious from the divisibility conditions (1). Evidently the $(k+1)$st determinantal divisor of (2) is $h_1(B) \cdots h_{k+1}(B)$. For fixed i, $1 \leqslant i \leqslant k$, the ith determinantal divisor of (2) is

$$\Bigg(h_1(A)\cdots h_i(A), \frac{h_1(B)\cdots h_{k+1}(B)}{h_i(A)\cdots h_k(A)}, h_1(B)h_2(A)\cdots h_i(A),$$

$$h_1(B)h_2(B)h_3(A)\cdots h_i(A),\ldots,h_1(B)\cdots h_i(B),\ldots,\frac{h_1(B)\cdots h_k(B)}{h_i(A)\cdots h_{k-1}(A)}\Bigg).$$

All terms in this gcd being divisible by $h_1(B)\cdots h_i(B)$, and one term equaling $h_1(B)\cdots h_i(B)$, we see that the ith determinantal divisor of (2) is precisely $h_1(B)\cdots h_i(B)$, $1 \leqslant i \leqslant k$. From this it is clear that (2) has the claimed invariant factors. ■

Lemmas 1 and 2 together prove the following theorem.

THEOREM 1 (Interlacing inequalities for invariant factors). *An $m \times n$ $\mathfrak{R}$-matrix A of* rank *$k<n$ and invariant factors $h_1(A)|\cdots|h_k(A)$ may be augmented with a single row to obtain an $\mathfrak{R}$-matrix B of* rank *$k+1$ and invariant factors $h_1(B)|\cdots|h_{i+1}(B)$ if and only if the interlacing inequalities* (1) *hold.*

Next, we consider the case in which $\operatorname{rank} A = \operatorname{rank} B$. Setting $b_{k+1}=0$, $h_{k+1}(B)=0$ [or deleting b_{k+1} and $h_{k+1}(B)$ if $k=n$], and also deleting $(6\mathrm{b})_i$ for all i, the above arguments go through without change to establish

THEOREM 2. *An $\mathfrak{R}$-matrix A of* rank *k and invariant factors $h_1(A)|\cdots|h_k(A)$ may be augmented with a single row to obtain an $\mathfrak{R}$-matrix B of* rank *k and invariant factors $h_1(B)|\cdots|h_k(B)$ if and only if*

$$h_1(B)|h_1(A)|h_2(B)|h_2(A)|\cdots|h_k(B)|h_k(A). \tag{7}$$

REMARK. The condition (7) may be incorporated under (1) by taking $h_{k+1}(B)=0$. Indeed, by giving any $\mathfrak{R}$-matrix A an infinite set of invariant factors $h_1(A)|h_2(A)|\ldots$, only finitely many of which are nonzero, all reference to rank in Theorems 1 and 2 may be deleted, and (1) and (7) simply stated as

$$h_i(B)|h_i(A)|h_{i+1}(B), \qquad i=1,2,\ldots.$$

3. SUBMATRICES OF DEFICIENCY 1 IN A SQUARE MATRIX

Let C be an $n\times n$ $\mathfrak{R}$-matrix with invariant factors $h_1(C)|\cdots|h_n(C)$, some of which may be zero. Take A be to an $(n-1)\times(n-1)$ $\mathfrak{R}$-matrix with invariant factors $h_1(A)|\cdots|h_{n-1}(A)$, some of which may also be zero. We ask: When can C be constructed such that A is a submatrix?

THEOREM 3. *An n-square $\mathfrak{R}$-matrix C with invariant factors $h_1(C)|\cdots|h_n(C)$ exists containing as a submatrix an $(n-1)$-square $\mathfrak{R}$-matrix A with invariant factors $h_1(A)|\cdots h_{n-1}(A)$ if and only if*

$$\begin{aligned} &h_1(C)|h_1(A)|h_3(C),\\ &h_2(C)|h_2(A)|h_4(C),\\ &\cdots\\ &h_{n-2}(C)|h_{n-2}(A)|h_n(C),\\ &h_{n-1}(C)|h_{n-1}(A). \end{aligned} \tag{8}$$

Proof. Suppose that C exists, containing A. Take B to be the $n\times(n-1)$ submatrix of C containing A. By Sec. 2 we have

$$h_1(C)|h_1(B)|h_2(C)|h_2(B)|\cdots|h_{n-1}(C)|h_{n-1}(B)|h_n(C), \tag{9}$$

$$h_1(B)|h_1(A)|h_2(B)|h_2(A)|\cdots|h_{n-1}(B)|h_{n-1}(A). \tag{10}$$

The inequalities (9) and (10) clearly imply (8).

Now suppose that A is given, with its prescribed invariant factors, and that prescribed invariant factors for C are given such that (8) holds. We must demonstrate that C can be constructed. For this purpose we first specify elements $h_1(B),\ldots,h_{n-1}(B)$ of $\mathfrak{R}$. Set

$$h_i(B)=\operatorname{lcm}(h_i(C),h_{i-1}(A)),\qquad 1\leqslant i\leqslant n-1,$$

where lcm denotes least common multiple, and $h_0(A)$ is understood to be 1. The properties of the least common multiple immediately show that $h_1(B)|h_2(B)|\cdots|h_{n-1}(B)$, that $h_i(C)|h_i(B)$ for $1\leqslant i\leqslant n-1$, and that

$h_{i-1}(A)|h_i(B)$ for $i=2,\dots,n-1$. Using $h_{i-1}(A)|h_{i+1}(C)$, we see that

$$h_i(B)|\mathrm{lcm}(h_i(C),h_{i+1}(C))=h_{i+1}(C),$$

$1 \leqslant i \leqslant n-1$, and using $h_i(C)|h_i(A)$ we obtain

$$h_i(B)|\mathrm{lcm}(h_i(A),h_{i-1}(A))=h_i(A).$$

Thus $h_1(B),\dots,h_{n-1}(B)$ constitute a divisibility chain for which (9) and (10) hold. By Sec. 2 we may now prolong A by one row to obtain an $n\times(n-1)$ matrix B, and then prolong B by one column to obtain the desired matrix C. This completes the proof. ■

4. SIMILARITY INVARIANTS OF PRINCIPAL SUBMATRICES OF DEFICIENCY 1

Let $\mathfrak{F}$ be a field, and C an $n\times n$ matrix over $\mathfrak{F}$. The similarity invariants of C are then the invariant factors of the polynomial matrix $\lambda I - C$. Let A be an $(n-1)\times(n-1)$ matrix over $\mathfrak{F}$.

THEOREM 4. *Let the $n\times n$ $\mathfrak{F}$-matrix C have (similarity) invariant factors $h_1(C)|h_2(C)|\cdots|h_n(C)$, including trivial invariant factors. Let the $(n-1)\times(n-1)$ $\mathfrak{F}$-matrix A have (similarity) invariant factors $h_1(A)|\cdots|h_{n-1}(A)$, again including trivial invariant factors. Then A is a principal submatrix of some $\mathfrak{F}$ similarity transform of C if and only if*

$$\begin{aligned}&h_1(C)|h_1(A)|h_3(C),\\&h_2(C)|h_2(A)|h_4(C),\\&\dots\\&h_{n-2}(C)|h_{n-2}(A)|h_n(C),\\&h_{n-1}(C)|h_{n-1}(A),\end{aligned}\tag{11}$$

and

$$\begin{aligned}\mathrm{degree}(h_1(A)\cdots h_{n-1}(A))&=n-1,\\ \mathrm{degree}(h_1(C)\cdots h_n(C))&=n.\end{aligned}\tag{12}$$

Proof. Suppose that A is a principal submatrix of C. Then the characteristic matrix $\lambda I_{n-1}-A$ of A is a submatrix of the characteristic matrix $\lambda I_n - C$. Applying Theorem 3, with $\mathfrak{R}=\mathfrak{F}[\lambda]$, we see that the conditions (11) are necessary. By definition of invariant factors, (12) holds.

Now suppose that A is given, with its prescribed invariant factors. We wish to construct C such that it contains A as a principal submatrix and has the prescribed invariant factors. Take $\mathfrak{R}=\mathfrak{F}[\lambda]$, and apply Theorem 3 to $\lambda I - A$. Because the conditions (11) hold, we obtain an $n\times n$ $\mathfrak{F}[\lambda]$-matrix $C(\lambda)$ containing $\lambda I - A$ as the leading $(n-1)$-square principal submatrix, with $C(\lambda)$ having invariant factors $h_1(C)|\cdots|h_n(C)$. By elementary operations on $C(\lambda)$, adding polynomial multiples of columns $1,2,\dots,n-1$ to column n, and adding polynomial multiples of rows $1,2,\dots,n-1$ to row n (these operations do not change the invariant factors), we may step by step eliminate powers of λ in the last row and column, in descending order, and hence assume that

$$C(\lambda)=\left[\begin{array}{ccc|c} & & & -x_1 \\ & \lambda I - A & & \vdots \\ & & & -x_{n-1} \\ \hline -y_1 & \cdots & -y_{n-1} & z(\lambda) \end{array}\right],$$

where $x_1,\dots,x_{n-1}$, $y_1,\dots,y_{n-1}$, are constants. Since $\det C(\lambda)=h_1(C)\cdots h_n(C)$ has degree n, and is monic, it now follows that $z(\lambda)=\lambda-z'$ with z' constant. Thus $C(\lambda)=\lambda I - C$, where C is an $\mathfrak{F}$-matrix with A as the leading principal submatrix. Since $C(\lambda)$ has $h_1(C),\dots,h_n(C)$ as its invariant factors, evidently we have constructed the required matrix C. This ends the proof. ∎

5. COMPARISON WITH INTERLACING INEQUALITIES FOR EIGENVALUES AND SINGULAR VALUES

Let B be an $n\times n$ Hermitian matrix with eigenvalues $\beta_1\geqslant\cdots\geqslant\beta_n$, and let A be an $(n-1)\times(n-1)$ principal submatrix with eigenvalues $\alpha_1\geqslant\cdots\geqslant\alpha_{n-1}$. The Cauchy interlacing inequalities assert that

$$\beta_1\geqslant\alpha_1\geqslant\beta_2\geqslant\alpha_2\geqslant\cdots\geqslant\beta_{n-1}\geqslant\alpha_{n-1}\geqslant\beta_n. \tag{13}$$

It is further known that this set of inequalities comprises all generally valid relations between the eigenvalues of B and A. The resemblance of (13) to (1) is altogether striking. Now let C be an $n\times n$ complex matrix with singular values $\gamma_1\geqslant\cdots\geqslant\gamma_n$, and let A be an $(n-1)\times(n-1)$ submatrix (not necessarily principal) with singular values $\alpha_1\geqslant\cdots\geqslant\alpha_{n-1}$. Then it is known [6] that

$$\begin{aligned}
&\gamma_1\geqslant\alpha_1\geqslant\gamma_3,\\
&\gamma_2\geqslant\alpha_2\geqslant\gamma_4,\\
&\cdots\\
&\gamma_{n-2}\geqslant\alpha_{n-2}\geqslant\gamma_n,\\
&\gamma_{n-1}\geqslant\alpha_{n-1}.
\end{aligned}\tag{14}$$

It is further known that this set of inequalities comprises all generally valid relations between the singular values of C and those of A. We next remark that even more striking than the resemblance of (1) to (13) is the resemblance of (11) to (14). It would be of interest to explain these extraordinary analogies as two special cases of a theory sufficiently general to cover both.

6. SUBMATRICES OF DEFICIENCY MORE THAN 1 IN A SQUARE MATRIX

Let C be an $n\times n$ $\mathfrak{R}$-matrix with invariant factors $h_1(C)|\cdots|h_n(C)$, some of which may be zero. Let A be an $(n-k)\times(n-k)$ $\mathfrak{R}$-matrix with invariant factors $h_1(A)|\cdots|h_{n-k}(A)$, some of which also may be zero. We now extend the results in Sec. 3 by assuming that $1\leqslant k\leqslant n-1$ and asking: When can C be constructed such that A is a submatrix?

Assume first that C exists, containing A as a submatrix. Select a nested chain of submatrices of C,

$$A_{n-k}\subset A_{n-k+1}\subset\cdots\subset A_n,$$

with $A_{n-k}=A$ and $A_n=C$. Take the invariant factors of A_i to be

$$h_1(A_i)|\cdots|h_i(A_i),$$

and set $h_{i+1}(A_i)=h_{i+2}(A_i)=\cdots=0$, $i=n-k,n-k+1,\ldots,n$. Then $h_1(A_i),\ldots,h_i(A_i)$ interlace $h_1(A_{i+1}),\ldots,h_{i+1}(A_{i+1})$ in the sense specified by Eq. (8) in Theorem 3: $h_j(A_{i+1})|h_j(A_i)|h_{j+2}(A_{i+1})$. From this it easily follows that

$$\begin{aligned}
&h_1(C)|h_1(A)|h_{1+2k}(C),\\
&h_2(C)|h_2(A)|h_{2+2k}(C),\\
&\cdots\\
&h_{n-k}(C)|h_{n-k}(A)|h_{n+k}(C),
\end{aligned}\tag{15}$$

with

$$h_{n+1}(C)=h_{n+2}(C)=\cdots=h_{n+k}(C)=0. \tag{16}$$

The inequalities (15) [subject to (16)] therefore are a necessary condition for the existence of an n-square matrix C containing A as a submatrix of deficiency k such that both C and A have prescribed invariant factors. It turns out that these conditions are also sufficient.

THEOREM 5. *There exists an $n\times n$ $\mathfrak{R}$-matrix C having invariant factors $h_1(C)|\cdots|h_n(C)$ and containing an $(n-k)\times(n-k)$ submatrix A having invariant factors $h_1(A)|\cdots|h_{n-k}(A)$, if and only if the conditions* (15) (*with the agreement* (16)) *are satisfied.*

Proof. The proof is by induction on k, the case $k=1$ being Theorem 3. Assume $k>1$. We are given $h_1(A)|\cdots|h_{n-k}(A)$ and $h_1(C)|\cdots|h_n(C)$. We wish to construct elements $h_1(B)|\cdots|h_{n-k+1}(B)$ of $\mathfrak{R}$ such that (take $h_{n-k+2}(B)=0$)

$$h_i(B)|h_i(A)|h_{i+2}(B),\qquad i=1,\dots,n-k, \tag{17}$$

$$h_i(C)|h_i(B)|h_{i+2(k-1)}(C),\qquad i=1,\dots,n-k+1. \tag{18}$$

If these conditions can be satisfied, we may (by Theorem 3) embed A in an $(n-k+1)\times(n-k+1)$ $\mathfrak{R}$-matrix B having $h_1(B)|\cdots|h_{n-k+1}(B)$ as its invariant factors, and by the case $k-1$ of Theorem 5 embed B in an $n\times n$ $\mathfrak{R}$-matrix C having invariant factors $h_1(C)|\cdots|h_n(C)$. The proof, therefore, is simply a matter of specifying $h_1(B)|\cdots|h_{n-k+1}(B)$ such that (17) and (18) hold.

Using lcm to signify least common multiple, set

$$h_i(B)=\operatorname{lcm}(h_i(C),h_{i-2}(A)),\qquad i=1,2,\dots,n-k+1,$$

where it is understood that $h_{-1}(A)=h_0(A)=1$. It is a simple matter to check that $h_1(B)|\cdots|h_{n-k+1}(B)$ and that (17) and (18) both hold. It is necessary to use (15) in this verification. This completes the proof of Theorem 5. ∎

7. A LEMMA INVOLVING A SYSTEM OF LINEAR INEQUALITIES

Before generalizing Theorem 4 in the same way that Theorem 5 generalizes Theorem 3, we give a lemma that will be needed in the proof.

Let n and k be positive integers, with $2 \leqslant k < n$, and suppose that $\alpha_{k+1}(p),\dots,\alpha_n(p)$ and $\gamma_1(p),\dots,\gamma_n(p)$ are real numbers for finitely many values of a parameter p such that

$$\gamma_1(p) \leqslant \gamma_2(p) \leqslant \cdots \leqslant \gamma_n(p) \qquad \text{for each } p, \tag{19}$$

$$\alpha_{k+1}(p) \leqslant \alpha_{k+2}(p) \leqslant \cdots \leqslant \alpha_n(p) \qquad \text{for each } p, \tag{20}$$

$$\sum_p \{\gamma_1(p)+\gamma_2(p)+\cdots+\gamma_n(p)\} = n, \tag{21}$$

$$\sum_p \{\alpha_{k+1}(p)+\alpha_{k+2}(p)+\cdots+\alpha_n(p)\} = n-k, \tag{22}$$

$$\gamma_i(p) \leqslant \alpha_{i+k}(p), \qquad i=1,2,\dots,n-k, \quad \text{and all } p, \tag{23}$$

$$\alpha_i(p) \leqslant \gamma_{i+k}(p), \qquad i=k+1,\dots,n-k, \quad \text{and all } p. \tag{24}$$

[The condition (24) will be vacuous if $k+1>n-k$.] Then we have

Lemma 3. *Under the above conditions,*

$$\sum_p \left\{ \gamma_1(p)+\gamma_2(p)+\sum_{i=3}^{n-k+1} \max(\gamma_i(p),\alpha_{i+k-2}(p)) \right\} \leqslant n-k+1. \tag{25}$$

Proof. The proof is somewhat lengthy. We first reformulate the conclusion to be established, then organize a number of cases to be considered, and finally attack these cases one by one.

For fixed p, let $\mathcal{G}(p)$ be a collection of distinct indices chosen from $3,4,\dots,n-k+1$. The indices and indeed the number of indices in $\mathcal{G}(p)$ will generally be different for different p. It may happen that $\mathcal{G}(p)$ is empty. Furthermore the sets $\mathcal{G}(p)$ for different values of p do not need to be disjoint. We are going to prove that

$$\sum_p \left\{ \gamma_1(p)+\gamma_2(p)+\sum_{\substack{i=3\\ i\in\mathcal{G}(p)}}^{n-k+1} \gamma_i(p)+\sum_{\substack{i=3\\ i\notin\mathcal{G}(p)}}^{n-k+1} \alpha_{i+k-2}(p) \right\} \leqslant n-k+1. \tag{26}$$

By taking $\mathcal{I}(p)$ to be those indices for which $\gamma_i(p) \geqslant \alpha_{i+k-2}(p)$, $3 \leqslant i \leqslant n-k+1$, it is clear that (25) is a consequence of (26). It therefore suffices to prove (26).

We next write down an equality to be used in the proof of (26).

$$\sum_p \left\{ \gamma_1(p) + \gamma_2(p) + \sum_{i=3}^{n-k+1} \gamma_i(p) + \sum_{i=n-k+2}^{n} \gamma_i(p) \right\} = \sum_p \left\{ \sum_{i=3}^{n-k+1} \alpha_{i+k-2}(p) + \alpha_n(p) \right\} + k. \tag{27}$$

This equality follows from (21) and (22), each side equaling n. Rewriting (22), we have

$$\sum_p \left\{ \sum_{\substack{i=3 \\ i \in \mathcal{I}(p)}}^{n-k+1} \alpha_{i+k-2}(p) + \sum_{\substack{i=3 \\ i \notin \mathcal{I}(p)}}^{n-k+1} \alpha_{i+k-2}(p) + \alpha_n(p) \right\} = n-k.$$

Using this, the left hand side of (26) equals

$$\sum_p \left\{ \gamma_1(p) + \gamma_2(p) + \sum_{\substack{i=3 \\ i \in \mathcal{I}(p)}}^{n-k+1} \gamma_i(p) - \sum_{\substack{i=3 \\ i \in \mathcal{I}(p)}}^{n-k+1} \alpha_{i+k-2}(p) - \alpha_n(p) \right\} + n - k.$$

To prove (26) therefore amounts to proving that

$$\sum_p \left\{ \gamma_1(p) + \gamma_2(p) + \sum_{\substack{i=3 \\ i \in \mathcal{I}(p)}}^{n-k+1} \gamma_i(p) \right\} \leqslant \sum_p \left\{ \sum_{\substack{i=3 \\ i \in \mathcal{I}(p)}}^{n-k+1} \alpha_{i+k-2}(p) + \alpha_n(p) \right\} + 1, \tag{28}$$

and this inequality we shall deduce as a consequence of (27). More precisely, we are going to prove (28) by using (27), (19), (20), (23), and (24) in the following way. We shall add some of the inequalities (23) to (27), thereby obtaining an inequality with an increased number of terms, and also add some of the inequalities (24), thus causing cancellation of certain terms. In the inequality so obtained, we shall collapse clusters of terms into single terms by using (19) and (20), in such a manner that after this collapse we obtain k times the inequality (28). Cancellation of k will then yield (28).

As a first step, we fix p and organize those integers from $3,4,\ldots,n-k+1$ which are *not* in $\mathcal{G}(p)$ into "almost consecutive" strings, that is, strings of integers that would be consecutive except for isolated omissions of an integer, i.e., gaps of length one. Different almost consecutive strings of integers not in $\mathcal{G}(p)$ thus are to be separated by gaps of length at least two. An almost consecutive string therefore has the form

$$\begin{aligned}
&I+1, I+2,\ldots,I+i_1-1,\\
&I+i_1+1, I+i_1+2,\ldots,I+i_2-1,\\
&I+i_2+1, I+i_2+2,\ldots,I+i_3-1,\\
&\ldots\\
&I+i_{s-1}+1, I+i_{s-1}+2,\ldots,I+i_s-1.
\end{aligned} \tag{29}$$

No row of integers in (29) is to be vacuous, none of the integers (29) is in $\mathcal{G}(p)$, and all of the integers (29) lie between 3 and $n-k+1$ (inclusive). However, the integers

$$I-1 \quad \text{and} \quad I \tag{30$'$}$$

are in $\mathcal{G}(p)\cup\{1,2\}$, and the gaps in (29), namely,

$$I+i_1, I+i_2,\ldots,I+i_{s-1}, \tag{30}$$

are all in $\mathcal{G}(p)$. Furthermore, the almost consecutive string (29) cannot be lengthened by addition of terms not in $\mathcal{G}(p)$, gaps of length at most one permitted. Note that we form almost consecutive strings for each fixed p. Furthermore, an almost consecutive string (29), unless it is the highest (i.e., involving the largest integers), is separated from the following almost consecutive string by a gap of at least two, so that

$$I+i_s \quad \text{and} \quad I+i_s+1 \tag{30$''$}$$

are both in $\mathcal{G}(p)$.

For an almost consecutive string (29), therefore, the terms γ_i with i equal to the values (30$'$) and (30) all occur on the left hand side of (28), and the terms α_{i+k-2} with i equal to the values (30) and (30$''$) all occur in the right hand side of (28), except perhaps in the case of the highest almost consecutive string. For the highest almost consecutive string, the last sentence is still valid, except that the term with $i=I+i_s+1$ will not be present on the right hand side of (28) in just one case: that for which $I+i_s-1=n-k+1$.

We now have described indices belonging to an almost consecutive string. Next, we examine indices between almost consecutive strings. With (29) denoting a typical almost consecutive string, other than the last almost consecutive string, let

$$J+1, J+2, \text{ etc.},$$

be the integers in the immediately following almost consecutive string. Then

$$I+i_s, I+i_s+1, \ldots, J \tag{31}$$

is a string of at least two consecutive integers, all in $\mathcal{I}(p)$. The terms γ_i with i ranging over

$$I+i_s, I+i_s+1, \ldots, J-2 \tag{32$'$}$$

all occur on the left hand side of (28) as terms between almost consecutive strings and not attached to such a string, i.e., are not terms γ_i with i of the forms (30′) or (30). Furthermore, the terms α_{i+k-2} with i ranging over

$$I+i_s+2, I+i_s+3, \ldots, J \tag{32$''$}$$

all occur on the right hand side of (28), also as terms between almost consecutive strings and not attached to such a string, i.e., are not terms α_{i+k-2} with i of the forms (30) or (30″). Note that (32′) and (32″) comprise the same number of terms. Next, we consider indices preceding the first almost consecutive string. If the first almost consecutive string begins with $i=3$ (i.e., $I=2$), then no terms in (28) precede the terms associated as in (30), (30′) with this first almost consecutive string. If, however, the first almost consecutive string has $I>2$, then on the left hand side of (28) initial terms γ_i occur for i equal to

$$1, 2, \ldots, I-2, \tag{33$'$}$$

and on the right hand side initial terms α_{i+k-2} occur for i equal to

$$3, 4, \ldots, I. \tag{33$''$}$$

Note that (33′) and (33″) comprise the same number of terms.

Next, we examine trailing terms. Let now (29) be the last almost consecutive string. If $I+i_s-1<n-k+1$, then the terms γ_i in the left hand side of (28) with i following the last almost consecutive string are those for which i ranges over

$$I+i_s, I+i_s+1, \ldots, n-k \tag{34'}$$

and

$$n-k+1, \tag{35'}$$

whereas those terms α_{i+k-2} on the right hand side of (28) for which i follows and is not associated [as in (30″)] with the last almost consecutive string are those having i ranging over

$$I+i_s+2, I+i_s+3, \ldots, n-k+2. \tag{34''}$$

Note that (34′) and (34″) comprise the same number of terms (possibly zero terms). However, if $I+i_s-1=n-k+1$, then (34′) and (34″) do not occur as values of i yielding terms in (28).

We have now organized (for fixed p) the terms on the left hand side of (28) into sets, and as well organized the terms on the right hand side of (28) into sets. The unions of these sets cover all terms in (28) for this fixed p, each term covered just once (ignoring the $+1$ term).

We are now going to take the terms appearing in (27), and apply the procedures described below (28), in such a manner that we obtain k times the various terms just described in (28). We continue to hold p fixed. For notational simplicity we shall suppress dummy variable p.

Terms in (27) between almost consecutive strings are: on the left hand side terms γ_i with i given by (32′), and on the right hand side terms α_{i+k-2} with i given by (32″). These terms in (27) may be written as

$$\cdots+\sum_x \gamma_x+\cdots \leqslant \cdots+\sum_x \alpha_{x+k}+\cdots, \tag{35}$$

with x ranging over the indices (32′). Using (23), add $\gamma_x \leqslant \alpha_{x+k}$ a total of $k-1$ times for each x, thereby obtaining

$$\cdots+\sum_x k\gamma_x+\cdots \leqslant \cdots+\sum_x k\alpha_{x+k}+\cdots, \tag{36}$$

that is, we have succeded in obtaining the terms in (28) between almost consecutive strings times k.

Terms in (27) preceding the first almost consecutive string are: on the left hand side terms γ_i with i given by (33′), and on the right hand terms α_{i+k-2} with i given by (33″). These terms again have the form (35), x now ranging over (33′). Adding $\gamma_x \leqslant \alpha_{x+k}$ a total of $k-1$ times for each x, we again obtain (36), i.e., we obtain the terms in (28) preceding the first almost consecutive string times k.

Next, we examine trailing terms. First assume that in the last almost consecutive chain we have $I+i_s-1<n-k+1$. The trailing terms on the left hand side of (27) are terms γ_i with i ranging over the values (34′) and also terms γ_i with i ranging over

$$n-k+1, n-k+2, \ldots, n.$$

On the right hand side of (27) the trailing terms are terms α_{i+k-2} with i ranging over (34″). We may write these as

$$\cdots + \sum_x \gamma_x + \sum_{y=n-k+1}^{n} \gamma_y + \cdots \leqslant \cdots + \sum_x \alpha_{x+k} + \cdots,$$

x ranging over (34′). Adding $\gamma_x \leqslant \alpha_{x+k}$ a total of $k-1$ times for each x, and using $\gamma_y \geqslant \gamma_{n-k+1}$ for each of the k values of y, we obtain

$$\cdots + k\sum_x \gamma_x + k\gamma_{n-k+1} + \cdots \leqslant \cdots + k\sum_x \alpha_{x+k} + \cdots,$$

that is, we obtain all trailing terms in (28) times k. We postpone until below the treatment of the trailing terms in the case in which the last almost consecutive chain has $I+i_s-1=n-k+1$.

Next we take terms from (27) associated with an almost consecutive chain; if the almost consecutive chain is the last one, assume for the moment that $I+i_s-1<n-k+1$. The terms from (27) to be considered are

$$\begin{aligned}
\cdots + \gamma_{I-1} &+ (\gamma_I + \gamma_{I+1} + \cdots + \gamma_{I+i_1-1}) \\
&+ (\gamma_{I+i_1} + \gamma_{I+i_1+1} + \cdots + \gamma_{I+i_2-1}) \\
&+ \cdots \\
&+ (\gamma_{I+i_{s-1}} + \gamma_{I+i_{s-1}+1} + \cdots + \gamma_{I+i_s-1}) + \cdots \\
\leqslant \cdots &+ (\alpha_{I+k-1} + \alpha_{I+k} + \cdots + \alpha_{I+k+i_1-2}) \\
&+ (\alpha_{I+k+i_1-1} + \alpha_{I+k+i_1} + \cdots + \alpha_{I+k+i_2-2}) \\
&+ \cdots \\
&+ (\alpha_{I+k+i_{s-1}-1} + \alpha_{I+k+i_{s-1}} + \cdots + \alpha_{I+k+i_s-2}) + \alpha_{I+k+i_s-1} + \cdots. \qquad (37)
\end{aligned}$$

The numbers of terms within each bracketed cluster on each side are $i_1, i_2 - i_1, i_3 - i_2, \ldots, i_s - i_{s-1}$, respectively, corresponding bracketed clusters on opposite sides of the inequality sign having like numbers of terms. Since each row in (29) has at least one entry, we have

$$i_1 \geqslant 2, \quad i_2 - i_1 \geqslant 2, \quad i_3 - i_2 \geqslant 2, \quad \cdots, \quad i_s - i_{s-1} \geqslant 2.$$

The terms (37) all occur in (27). We wish to modify (37) to obtain the terms in (28) associated with this almost consecutive chain. That is, we wish to perform legitimate operations on (37) and obtain k times

$$\begin{aligned} &\cdots + \gamma_{I-1} + (\gamma_I) \qquad + (\gamma_{I+i_1}) \qquad + \cdots + (\gamma_{I+i_{s-1}}) \qquad + \cdots \\ &\quad \leqslant \cdots + (\alpha_{I+k+i_1-2}) + (\alpha_{I+k+i_2-2}) + \cdots + (\alpha_{I+k+i_s-2}) + \alpha_{I+k+i_s-1} + \cdots . \end{aligned} \tag{38}$$

Terminology "Sliding" or "collapsing" downwards will mean replacing a term γ_x with a smaller term γ_y, i.e., one having $y \leqslant x$; and "sliding" or "collapsing" upwards will mean replacing a term α_x with a larger term α_y, i.e., one having $y \geqslant x$. We shall often slide terms within the bracketed clusters in (37), and sometimes slide terms from one bracket into another bracket.

Generally speaking, some of the bracketed clusters in (37) have k or fewer terms and some more than k terms. A pair of corresponding bracketed clusters, e.g.,

$$\begin{aligned} &\cdots + (\gamma_{I+i_1} + \cdots + \gamma_{I+i_2-1}) + \cdots \\ &\qquad\qquad \leqslant \cdots + (\alpha_{I+k+i_1-1} + \cdots + \alpha_{I+k+i_2-2}) + \cdots, \end{aligned} \tag{39}$$

having k or fewer terms in each cluster (so that $i_2 - i_1 \leqslant k$) is to be handled as follows: add the valid inequality

$$\gamma_{I+i_1} \leqslant \alpha_{I+i_1+k}$$

to (39) exactly $k - (i_2 - i_1)$ times, the γ-term going into the left hand displayed cluster and the α-term into the right hand displayed cluster. Note that $I + i_1 + k \leqslant I + k + i_2 - 2$. The displayed clusters after incorporating these added terms contain exactly k terms (each). Collapse the terms in the left hand side cluster into the initial term γ_{I+i_1}, and collapse the terms in the right hand cluster onto the trailing term α_{I+k+i_2-2}, thereby obtaining the desired part

$$\cdots + k\gamma_{I+i_1} + \cdots \leqslant \cdots + k\alpha_{I+k+i_2-2} + \cdots$$

of (28) [or (38)] times k.

In this manner we handle any pair of corresponding clusters having k or fewer terms in each cluster. These pairs of corresponding clusters may therefore be ignored, and will be ignored, in the discussion of (37) to follow.

If every one of the clusters of terms shown in (37) has k or fewer terms, we complete the analysis pertaining to these terms by adding to (37) the inequality $\gamma_{I-1} \leqslant \alpha_{I+k+i_s-1}$ precisely $k-1$ times, thereby obtaining the remaining two terms shown in (38) times k. Note that this step is valid because $(I-1)+k \leqslant I+k+i_s-1$.

Suppose now that some of the clusters in (37) have more than k terms, and let these clusters be the following (the vertical line is inserted for typographical clarity to separate the two sides of the inequality):

$$\begin{array}{l|l}
\gamma_{I-1} & \leqslant \\
+\cdots+(\gamma_{I+\rho_1} \quad +\cdots+\gamma_{I+\rho_2-1}) & \quad \cdots+(\alpha_{I+k+\rho_1-1} \quad +\cdots+\alpha_{I+k+\rho_2-2}) \\
+\cdots+(\gamma_{I+\rho_3} \quad +\cdots+\gamma_{I+\rho_4-1}) & +\cdots+(\alpha_{I+k+\rho_3-1} \quad +\cdots+\alpha_{I+k+\rho_4-2}) \\
+\cdots+\cdots & +\cdots+\cdots \\
+\cdots+(\gamma_{I+\rho_{2t-1}}+\cdots+\gamma_{I+\rho_{2t}-1})+\cdots & +\cdots+(\alpha_{I+k+\rho_{2t-1}-1} \quad +\cdots+\alpha_{I+k+\rho_{2t}-2}) \\
 & \quad +\cdots+\alpha_{I+k+i_s-1}+\cdots .
\end{array} \tag{40}$$

The numbers of terms in the individual clusters here are $\rho_2-\rho_1>k$, $\rho_4-\rho_3>k,\ldots,$ $\rho_{2t}-\rho_{2t-1}>k$. Also $\rho_2 \leqslant \rho_3$, $\rho_4 \leqslant \rho_5,\ldots,$ $\rho_{2t-2} \leqslant \rho_{2t-1}$. Perhaps $\rho_1=0$.) Some of the clusters shown in (40) may even have $2k$ or more terms, and our first step will be to cancel enough terms in any such cluster to reduce it to precisely $2k-1$ terms. As a typical case, consider the first cluster pair shown in (40):

$$\cdots+(\gamma_{I+\rho_1}+\cdots+\gamma_{I+\rho_2-1})+\cdots$$
$$\leqslant \cdots+(\alpha_{I+k+\rho_1-1}+\cdots+\alpha_{I+k+\rho_2-2})+\cdots . \tag{41}$$

We have $\alpha_x \leqslant \gamma_y$ whenever $y \geqslant x+k$; this follows from (24), (19), (20). By adding the inequality $\alpha_x \leqslant \gamma_y$ to (40) [or (41)], one succeeds in canceling α_x from the right hand side and γ_y from the left hand side. Thus we may cancel any α-term against any desired γ-term that is at least k levels higher. We shall now use this device to cancel terms from (41) whenever $\rho_2-\rho_1 \geqslant 2k$, in such a manner that exactly $2k-1$ terms remain in each of the two clusters shown. Specifically:

$$\begin{array}{lll} \text{cancel} & \alpha_{I+k+\rho_1-1} & \text{against} \quad \gamma_{I+2k+\rho_1-1}, \\ & \cdots & \\ & \alpha_{I-k+\rho_2-1} & \text{against} \quad \gamma_{I+\rho_2-1}, \end{array}$$

thereby obtaining the following pair of corresponding clusters:

$$\begin{aligned} &\cdots + (\gamma_{I+\rho_1} + \cdots + \gamma_{I+\rho_1+2k-2}) + \cdots \\ \leqslant &\cdots + (\alpha_{I-k+\rho_2} + \cdots + \alpha_{I+k+\rho_2-2}) + \cdots . \end{aligned} \tag{42}$$

The clusters shown in (42) each have $2k-1$ terms. Do this cancellation on each cluster pair in (40) for which the clusters have $2k$ or more terms. Upon completion of this process, those cluster pairs to which this process did not apply are still as shown in (40), the remaining cluster pairs now take the form shown in (42), and always the number of terms in the clusters now at hand is $k+1$ or more, but not exceeding $2k-1$.

At this point we have converted (40) into the following:

$$\begin{array}{r|l} \cdots + \gamma_{I-1} + \cdots + (\delta_1) \quad \leqslant & \cdots + (\delta_1) \\ + \cdots + (\delta_2) \quad & + \cdots + (\delta_2) \\ + \cdots + \cdots \quad & + \cdots + \cdots \\ + (\delta_t) + \cdots & + \cdots + (\delta_t) + \cdots + \alpha_{I+k+i_s-1} + \cdots , \end{array} \tag{43}$$

the symbols $\delta_1, \delta_2, \ldots, \delta_t$ here indicating the number of terms in each cluster: $k < \delta_i \leqslant 2k-1$, $i = 1, 2, \ldots, t$.

Our next step is to cancel α-terms in one cluster of (43) against γ-terms in the immediately subsequent cluster, in such a way that every α-cluster (except the last α-cluster) comes to have exactly k terms. There are four cases to be considered here.

Case (i). We have in (43) an α-cluster and an immediately subsequent γ-cluster as displayed in (40), i.e., neither were involved in the canceling process above that reduced to $2k-1$ terms:

$$\begin{array}{r|l} & \leqslant \cdots + (\alpha_{I+k+\rho_1-1} + \cdots + \alpha_{I+k+\rho_2-2}) + \cdots \\ & \\ \cdots + (\gamma_{I+\rho_3} + \cdots + \gamma_{I+\rho_4-1}) + \cdots & \end{array}$$

Here $k<\rho_2-\rho_1\leqslant 2k-1$, $k<\rho_4-\rho_3\leqslant 2k-1$, $\rho_3\geqslant\rho_2$, $\delta_1=\rho_2-\rho_1$, $\delta_2=\rho_4-\rho_3$. We cancel

$$\begin{array}{lll} \alpha_{I+k+\rho_1-1} & \text{against} & \gamma_{I+\rho_4-\rho_2+\rho_1+k}, \\ & \cdots & \\ \alpha_{I+\rho_2-2} & \text{against} & \gamma_{I+\rho_4-1} \end{array}$$

[Note that $(I+k+\rho_1-1)+k<I+\rho_4-\rho_2+\rho_1+k$, $I+\rho_4-\rho_2+\rho_1+k>I+\rho_3$, $I+\rho_4-\rho_2+\rho_1+k\leqslant I+\rho_4-1$.] The effect of this is to reduce the δ_1 on the right hand side of (43) to k and the δ_2 on the left hand side to $\delta_2-\delta_1+k$. (Note: $0<\delta_2-\delta_1+k$, since $\delta_1\leqslant 2k-1$, $\delta_2>k$.)

Case (ii). In (43) we have an α-cluster as displayed in (40), i.e., not involved in the canceling process that reduced to $2k-1$ terms, and a γ-cluster that was involved in this canceling process:

$$\begin{array}{r|l} & \leqslant \cdots + (\alpha_{I+k+\rho_1-1}+\cdots+\alpha_{I+k+\rho_2-2})+\cdots \\ \cdots+(\gamma_{I+\rho_3}+\cdots+\gamma_{I+\rho_3+2k-2})+\cdots & \end{array}$$

Hence $\delta_1=\rho_2-\rho_1$, $k<\delta_1\leqslant 2k-1$, $\delta_2=2k-1$. Cancel as follows:

$$\begin{array}{lll} \alpha_{I+k+\rho_1-1} & \text{against} & \gamma_{I+\rho_3-\rho_2+\rho_1+3k-1}, \\ \cdots & & \\ \alpha_{I+\rho_2-2} & \text{against} & \gamma_{I+\rho_3+2k-2}. \end{array}$$

[Note that $(I+k+\rho_1-1)+k<I+\rho_3-\rho_2+\rho_1+3k-1$, $I+k+\rho_1-1\leqslant I+\rho_2-2$, $I+\rho_3<I+\rho_3-\rho_2+\rho_1+3k-1$.] The effect of this is to replace the δ_1 on the right hand side of (43) by k and the δ_2 on the left hand side by $\delta_2-\delta_1+k$.

Case (iii). The α-cluster in (43) participated in the reduction process to $2k-1$ terms, but the γ-cluster did not:

$$\begin{array}{r|l} & \leqslant \cdots + (\alpha_{I-k+\rho_2}+\cdots+\alpha_{I+k+\rho_2-2})+\cdots \\ \cdots+(\gamma_{I+\rho_3}+\cdots+\gamma_{I+\rho_4-1})+\cdots & \end{array}$$

Here $k<\rho_4-\rho_3\leqslant 2k-1$, $\delta_1=2k-1$, $\delta_2=\rho_4-\rho_3$. Cancel as follows:

$$\begin{array}{lll} \alpha_{I-k+\rho_2} & \text{against} & \gamma_{I+\rho_3}, \\ \cdots & & \\ \alpha_{I-2+\rho_2} & \text{against} & \gamma_{I+\rho_3+k-2}. \end{array}$$

(Note: $I+\rho_3+k-2<I+\rho_4-1$.) Again the effect is to replace the δ_1 in the right hand side of (43) by k and the δ_2 on the left hand side by $\delta_2-\delta_1+k$.

Case (iv). The α-cluster and the γ-cluster both participated in the reduction to $2k-1$ terms. Here we have

$$\begin{array}{r|l} & \leqslant \cdots + (\alpha_{I-k+\rho_2}+\cdots+\alpha_{I+k+\rho_2-2})+\cdots \\ \cdots+(\gamma_{I+\rho_3}+\cdots+\gamma_{I+\rho_3+2k-2})+\cdots & \end{array}$$

Here $\delta_1=2k-1$, $\delta_2=2k-1$. Cancel

$$\begin{array}{lll} \alpha_{I-k+\rho_2} & \text{against} & \gamma_{I+\rho_3}, \\ \cdots & & \\ \alpha_{I-2+\rho_2} & \text{against} & \gamma_{I+\rho_3+k-2}. \end{array}$$

Once more, the effect is to replace the δ_1 on the right hand side of (43) by k and the δ_2 on the left hand side by $\delta_2-\delta_1+k$.

After these cancellations, the chart (43) indicating the numbers of terms in the clusters becomes

$$\begin{array}{r|l} \cdots+\gamma_{I-1}+\cdots+(\delta_1) \qquad\qquad\qquad & \leqslant \quad \cdots+(k) \\ +\cdots+(\delta_2-\delta_1+k) & +\cdots+(k) \\ +\cdots+(\delta_3-\delta_2+k) & +\cdots+(k) \\ +\cdots+\cdots & +\cdots+\cdots \\ +\cdots+(\delta_{t-1}-\delta_{t-2}+k) & +\cdots+(k) \\ +\cdots+(\delta_t-\delta_{t-1}+k)+\cdots & +\cdots+(\delta_t)+\alpha_{I+k+i_s-1}+\cdots. \end{array} \tag{44}$$

Here, of course, $k<\delta_i\leqslant 2k-1$ for each i.

Our next step is to diminish some of the terms indicated on the left hand side of (44), in particular sliding terms in some of the clusters [except cluster (δ_1)] down into earlier clusters. We do this whenever a cluster on the left hand side has more than k terms, sliding some terms in it down into the immediately preceding cluster so that precisely k terms remain, repeating this process perhaps several times until no cluster [except (δ_1)] on the left hand side has more than k terms. To illustrate: if $\delta_4-\delta_3\geqslant 0$, then we slide terms so that

$$\begin{matrix} (\delta_3-\delta_2+k) & & (\delta_4-\delta_2+k) \\ & \text{becomes} & \\ (\delta_4-\delta_3+k) & & (k) \end{matrix},$$

and if $\delta_4-\delta_2\geqslant 0$ we slide further to convert

$$\begin{matrix} (\delta_2-\delta_1+k) & & (\delta_4-\delta_1+k) \\ (\delta_4-\delta_2+k) & \text{into} & (k) \\ (k) & & (k) \end{matrix}$$

If $\delta_4-\delta_1\geqslant 0$, further sliding converts (δ_1) to (δ_4) and $(\delta_1-\delta_1+\mathrm{k})$ to (k).

In this way, by sliding γ-terms downwards, some of the clusters shown on the left hand side come to have exactly k terms, and the rest [except the cluster that was (δ_1)] have fewer than k terms. Not writing out explicitly all those now having exactly k terms, (44) becomes

$$\begin{array}{rl|l} \cdots+\gamma_{l-1}+\cdots+(\delta_{r_1}) & \leqslant & \cdots+(k) \\ +\cdots+(\delta_{r_2}-\delta_{r_1}+k) & & +\cdots+(k) \\ +\cdots & & +\cdots \\ +\cdots+(\delta_{r_{\tau-1}}-\delta_{r_{\tau-2}}+k) & & +\cdots+(k) \\ +\cdots+(\delta_{r_\tau}-\delta_{r_{\tau-1}}+k) & & +\cdots+(k) \\ +\cdots+(k)+\cdots & & +\cdots+(\delta_t)+\alpha_{l+k+i_s-1}+\cdots . \end{array} \tag{45}$$

Here $r_1,\ldots,r_\tau$ are a subset of $1,\ldots,t$ with $r_\tau=t$, and $(\delta_{r_\tau}-\delta_{r_{\tau-1}}+k)$ is either opposite to or earlier than (δ_t). (Only the "earlier" case is displayed.) We assume that further sliding on the left is not possible. This means that

$$\begin{gathered} \delta_{r_2}-\delta_{r_1}<0, \quad \cdots, \quad \delta_{r_\tau}-\delta_{r_{\tau-1}}<0; \\ k<\delta_{r_1}\leqslant 2k-1, \quad \cdots, \quad k<\delta_{r_\tau}\leqslant 2k-1. \end{gathered} \tag{46}$$

Now slide $\delta_{r_1}-k$ terms γ from (δ_{r_1}) down onto γ_{l-1}, and if $(\delta_{r_\tau}-\delta_{r_{\tau-1}}+k)$ is earlier than (δ_t) slide α-terms upwards onto α_{l+k+i_s-1}, as follows:

slide $\delta_{r_1}-\delta_{r_2}$ terms α from the second cluster (k) upwards,
slide $\delta_{r_2}-\delta_{r_3}$ terms α from the third cluster (k) upwards,
. . .
slide $\delta_{r_{\tau-1}}-\delta_{r_\tau}$ terms α from the τth cluster (k) upwards,
slide δ_t-k terms α from (δ_t) upwards.

This converts the chart (45) into the following:

$$\begin{array}{ll|l}
\cdots+\gamma_{I-1}\,(\delta_{r_1}-k+1\text{ times})+\cdots+(k) & \leqslant & \cdots+(k)\\
\qquad\qquad+\cdots+(\delta_{r_2}-\delta_{r_1}+k) & & +\cdots+(\delta_{r_2}-\delta_{r_1}+k)\\
\qquad\qquad+\cdots & & +\cdots\\
\qquad\qquad+\cdots+(\delta_{r_\tau}-\delta_{r_{\tau-1}}+k) & & +\cdots+(\delta_{r_\tau}-\delta_{r_{\tau-1}}+k)\\
\qquad\qquad+\cdots+(k)+\cdots & & +\cdots+(k)\\
 & & +\cdots+\alpha_{I+k+i_s-1}\,(\delta_{r_1}-k+1\text{ times}).
\end{array} \tag{46}$$

However, if $(\delta_{r_\tau}-\delta_{r_{\tau-1}}+k)$ is opposite (δ_t), slide $\delta_{r_1}-k$ terms γ downwards from (δ_{r_1}) as before, and slide α-terms onto α_{I+k+i_s-1} in this manner: Slide

$\delta_{r_1}-\delta_{r_2}$ terms α from the second cluster (k) upwards,
$\cdots$
$\delta_{r_{\tau-2}}-\delta_{r_{\tau-1}}$ terms from the $(\tau-1)$st cluster (k) upwards,
$\delta_{r_{\tau-1}}-k$ terms from the τth cluster (δ_t) upwards.

This converts (45) into the same chart (46), with the next to last displayed line absent.

We next argue as before. Each pair of corresponding clusters now has the same number of terms, precisely k, or less than k [see (46)]. We may augment clusters having fewer than k terms until the number of terms is precisely k, as in the earlier part of the proof, then in every γ cluster slide down to the first γ-term in the cluster (thereby obtaining this term times k), or slide in each α-cluster up to the last α-term (thereby also obtaining this term times k). Since $\delta_{r_1}-k+1\leqslant k$ (because $\delta_{r_1}\leqslant 2k-1$), we may add the inequality

$$\gamma_{I-1}\leqslant\alpha_{I+k+i_s-1}$$

exactly $k-(\delta_{r_1}-k+1)$ times so as to obtain these two terms times k. After all this effort we obtain terms in (28) times k.

The only cases remaining in the proof of the lemma are (i) the examination of the trailing terms when for the last almost consecutive chain the equality

$$I+i_s-1=n-k+1 \tag{47}$$

holds, and (ii) the passage from (37) to k times (38) for the last almost consecutive chain when (47) holds. [When (47) holds, the term α_{I+k+i_s-1} is absent in (37) and (38).] We combine these two cases into a single case; thus in place of (37) we have

$$\begin{array}{rl|l} \cdots+\gamma_{I-1}+(\gamma_I+\cdots+\gamma_{I+i_1-1}) & \leqslant & \cdots+(\alpha_{I+k-1}+\cdots+\alpha_{I+k+i_1-2}) \\ +\cdots+(\gamma_{I+i_1}+\cdots+\gamma_{I+i_2-1}) & & +\cdots+(\alpha_{I+k+i_1-1}+\cdots+\alpha_{I+k+i_2-2}) \\ +\cdots & & +\cdots \\ +\cdots+(\gamma_{I+i_{s-1}}+\cdots+\gamma_{I+i_s-1}) & & +\cdots+(\alpha_{I+k+i_{s-1}-1}+\cdots+\alpha_{I+k+i_s-2}) \\ +\cdots+(\gamma_{n-k+2}+\cdots+\gamma_n). & & \end{array} \tag{48}$$

Note that $\gamma_{I+i_s-1}=\gamma_{n-k+1}$, $\alpha_{I+k+i_s-2}=\alpha_n$. Our objective is to add terms, cancel terms, and slide terms in (48) so as to obtain k times

$$\cdots+\gamma_{I-1}+\gamma_I+\gamma_{I+i_1}+\cdots+\gamma_{I+i_{s-1}}\leqslant\cdots+\alpha_{I+k+i_1-2}+\cdots+\alpha_{I+k+i_s-2}. \tag{49}$$

If every paired cluster of γ-terms and α-terms in (48) has k or fewer terms in each cluster, then we augment the clusters in each pair until k terms are enclosed, then slide the γ-terms in each such cluster onto its lowest γ-term, slide the α-terms in each such cluster onto its highest α-term, and finally slide the $k-1$ terms in the unpaired cluster $(\gamma_{n-k+2}+\cdots+\gamma_n)$ into γ_{I-1}. We then have obtained k times (49).

Thus assume that some of the clusters have $k+1$ or more terms. Set aside and treat as above pairs of clusters having k or fewer terms in each cluster. This leads us [in place of (40)] to

$$\begin{array}{rl|l} \cdots+\gamma_{I-1}+\cdots+(\gamma_{I+\rho_1}+\cdots+\gamma_{I+\rho_2-1}) & \leqslant & \cdots+(\alpha_{I+k+\rho_1-1}+\cdots+\alpha_{I+k+\rho_2-2}) \\ +\cdots+(\gamma_{I+\rho_3}+\cdots+\gamma_{I+\rho_4-1}) & & +\cdots+(\alpha_{I+k+\rho_3-1}+\cdots+\alpha_{I+k+\rho_4-2}) \\ +\cdots & & +\cdots \\ +\cdots+(\gamma_{I+\rho_{2t-1}}+\cdots+\gamma_{I+\rho_{2t}-1}) & & +\cdots+(\alpha_{I+k+\rho_{2t-1}-1}+\cdots+\alpha_{I+k+\rho_{2t}-2}) \\ +\cdots+(\gamma_{n-k+2}+\cdots+\gamma_n), & & \end{array} \tag{50}$$

where $\rho_2-\rho_1>k,\ldots,\ \rho_{2t}-\rho_{2t-1}>k$.

As before, we cancel in corresponding clusters when there are $2k$ or more terms in a cluster. Then [in place of (43)] we convert (50) to

$$\begin{array}{rl|l} \cdots+\gamma_{I-1}+\cdots+(\delta_1) & \leqslant & \cdots+(\delta_1) \\ +\cdots+(\delta_2) & & +\cdots+(\delta_2) \\ +\cdots & & +\cdots \\ +\cdots+(\delta_t) & & +\cdots+(\delta_t) \\ +\cdots+(k-1), & & \end{array} \tag{51}$$

the quantities shown in the parentheses being the numbers of enclosed terms. Here $k<\delta_i\leqslant 2k-1$ for each i. Next we cancel α-terms in a cluster on the right in (51) against γ-terms in the next cluster, so as to make each α-cluster have exactly k terms. The new feature here is that we do this cancellation as well on the last α-cluster (δ_t) and the trailing unpaired γ-cluster $(k-1)$. That this new cancellation is possible is seen by examining two cases.

(i) Cluster (δ_t) did not participate in the reduction to $2k-1$ terms, i.e., $k<\rho_{2t}-\rho_{2t-1}\leqslant 2k-1$:

$$\begin{array}{r|l} & \leqslant\cdots+(\alpha_{I+k+\rho_{2t-1}-1}+\cdots+\alpha_{I+k+\rho_{2t}-2}) \\ \cdots+(\gamma_{n-k+2}+\cdots+\gamma_n) & \end{array}$$

Here we cancel

$$\begin{array}{lll} \alpha_{I+k+\rho_{2t-1}-1} & \text{against} & \gamma_{n-\rho_{2t}+\rho_{2t-1}+k+1}, \\ \cdots & & \\ \alpha_{I-2+\rho_{2t}} & \text{against} & \gamma_n. \end{array}$$

Note that $(I+k+\rho_{2t-1}-1)+k\leqslant n-\rho_{2t}+\rho_{2t-1}+k+1$, since this amounts to $I+\rho_{2t}-1\leqslant n-k+1$, which follows from (47). Also $n-\rho_{2t}+\rho_{2t-1}+k+1\leqslant n$, because $\rho_{2t}-\rho_{2t-1}\geqslant k+1$, and $n-\rho_{2t}+\rho_{2t-1}+k+1\geqslant n-k+2$, since $\rho_{2t}-\rho_{2t-1}\leqslant 2k-1$.

(ii) Cluster (δ_t) did participate in the reduction to $2k-1$ terms. Here we must cancel in [see (42)]

$$\begin{array}{r|l} & \leqslant\cdots+(\alpha_{I-k+\rho_{2t}}+\cdots+\alpha_{I+k+\rho_{2t}-2}) \\ \cdots+(\gamma_{n-k+2}+\cdots+\gamma_n) & \end{array}$$

We cancel

$$\begin{array}{lll} \alpha_{I-k+\rho_{2t}} & \text{against} & \gamma_{n-k+2}, \\ \cdots & & \\ \alpha_{I-2+\rho_{2t}} & \text{against} & \gamma_n. \end{array}$$

Note that $(I-k+\rho_{2t})+k\leqslant n-k+2$, since $I+\rho_{2t}-1\leqslant n-k+1$.

In either case, therefore, we convert (51) to

$$\begin{array}{rl|l} \cdots+\gamma_{I-1}+\cdots+(\delta_1) & \leqslant & \cdots+(k) \\ +\cdots+(\delta_2-\delta_1+k) & & +\cdots+(k) \\ +\cdots & & +\cdots \\ +\cdots+(\delta_t-\delta_{t-1}+k) & & +\cdots+(k) \\ +\cdots+(2k-1-\delta_t) & & \end{array} \tag{52}$$

the symbols in the parentheses again indicating the numbers of terms enclosed. Now we slide γ-terms downwards as follows: from

$$\begin{array}{l} (\delta_t-\delta_{t-1}+k) \\ +(2k-1-\delta_t), \end{array}$$

slide to get

$$\begin{array}{l} ((2k-1-\delta_{t-1})+k) \\ +(0). \end{array}$$

Since $2k-1-\delta_{t-1}\geqslant 0$, we may slide $2k-1-\delta_{t-1}$ terms γ down to the next cluster to get

$$\begin{array}{l} ((2k-1-\delta_{t-2})+k) \\ +(k) \\ +(0). \end{array}$$

We may continue sliding this way, using $2k-1-\delta_i\geqslant 0$ for each i, so finally reaching

$$\begin{array}{rl|l} \cdots+\gamma_{I-1}+\cdots+(2k-1) & \leqslant & \cdots+(k) \\ +\cdots+(k) & & +\cdots+(k) \\ +\cdots & & +\cdots \\ +\cdots+(k) & & +\cdots+(k). \\ +\cdots+(0) & & \end{array}$$

Finally, sliding $k-1$ terms γ down onto γ_{I-1}, we get

$$\begin{array}{rl|l} k\gamma_{I-1}+\cdots+(k) & \leqslant & \cdots+(k) \\ +\cdots+(k)+ & & \cdots+(k) \\ +\cdots & & +\cdots \\ +\cdots+(k)+ & & \cdots+(k). \\ +\cdots+(0) & & \end{array} \tag{53}$$

Collapsing γ-terms downwards within each γ-cluster (k) in (53), and collapsing α-terms upwards within each α-cluster in (53), we obtain k times (49).

We have now succeeded in modifying (27) so that the γ- and α-terms remaining are precisely the γ- and α-terms in (28) times k. Canceling k, we obtain (28), as desired. The proof of Lemma 3 is now complete. ∎

8. SIMILARITY INVARIANTS OF PRINCIPAL SUBMATRICES OF DEFICIENCY EXCEEDING 1

Let $\mathcal{F}$ be a field, C an $n\times n$ matrix over $\mathcal{F}$, and A an $(n-k)\times(n-k)$ matrix, also over $\mathcal{F}$, with $1\leqslant k\leqslant n-1$. We now state a generalization of Theorem 4.

THEOREM 6. *Let the $n\times n$ $\mathcal{F}$-matrix C have (similarity) invariant factors $h_1(C)|\cdots|h_n(C)$, including trivial invariant factors. Let the $(n-k)\times(n-k)$ $\mathcal{F}$-matrix A have (similarity) invariant factors $h_1(A)|\cdots|h_{n-k}(A)$, again including trivial invariant factors. Then A is a principal submatrix of some $\mathcal{F}$ similarity transform of C if and only if the divisibility relations* (15) *hold (under the convention* (16)) *and*

$$\operatorname{degree}(h_1(C)\cdots h_n(C))=n,\qquad \operatorname{degree}(h_1(A)\cdots h_{n-k}(A))=n-k. \tag{54}$$

Proof. Since $\lambda I-A$ is a principal submatrix of $\lambda I-C$ whenever A is a principal submatrix of C, an application of Theorem 5 shows that the conditions (15) are indeed necessary. Obviously (54) must hold.

Conversely, suppose polynomials $h_1(C),\ldots,h_n(C),h_1(A),\ldots,h_{n-k}(A)$ are given satisfying the stated conditions. We must show that a matrix C exists with a matrix A as principal submatrix such that C and A have these given polynomials as invariant factors. The proof imitates that given for Theorem 5, and is by induction on k. The case $k=1$ is Theorem 4. Suppose $k>1$. We set [taking $h_{-1}(A)=h_0(A)=1$]

$$h_i(B)=\operatorname{lcm}(h_i(C),h_{i-2}(A)),\qquad i=1,2,\ldots,n-k+1.$$

Then we know that (17) and (18) hold, precisely as in the proof of Theorem 5. We wish to verify that

$$\operatorname{degree}(h_1(B)\ldots h_{n-k+1}(B))\leqslant n-k+1. \tag{55}$$

If (55) can be proved, then we argue as follows. We multiply $h_{n-k+1}(B)$ by linear factors until the degree inequality (55) becomes equality. With this new choice of $h_{n-k+1}(B)$, both (17) and (18) continue to hold [because $h_{n+k-1}(C)=0$]. Since (55) is equality and (17) and (18) hold, we can, by Theorem 4, construct an $\mathcal{F}$-matrix A with the prescribed invariant factors and contained as a principal submatrix in the $(n-k+1)\times(n-k+1)$ $\mathcal{F}$-matrix B, and by induction on k embed B as a principal submatrix of an $n\times n$ $\mathcal{F}$-matrix C having the prescribed invariant factors. Thus the proof will be complete once (55) is verified.

Before verifying (55), we make a notational change: In place of $h_1(A)|\cdots|h_{n-k}(A)$ write $h_{k+1}(A)|\cdots|h_n(A)$. Then we have

$$\begin{gathered} h_1(C)|h_{k+1}(A)|h_{1+2k}(C), \\ \cdots \\ h_{n-2k}(C)|h_{n-k}(A)|h_n(C), \\ h_{n-2k+1}(C)|h_{n-k+1}(A), \\ \cdots \\ h_{n-k}(C)|h_n(A). \end{gathered} \tag{56}$$

We also have $h_{k+1}(A)|\cdots|h_n(A)$, and

$$h_i(B)=\operatorname{lcm}(h_i(C),h_{i+k-2}(A)), \qquad i=1,\dots,n-k+1,$$

where it is understood that $h_{k-1}(A)=h_k(A)=1$. Furthermore

$$\operatorname{degree}(h_{k+1}(A)\dots h_n(A))=n-k.$$

Factor the polynomials $h_1(C),\dots,h_n(C)$, $h_{k+1}(A),\dots,h_n(A)$ into prime (linear) polynomials over the algebraic closure of $\mathcal{F}$, and let p be one of the finitely many linear polynomials appearing in these factorizations. Suppose that

p appears in $h_i(C)$ as p^{γ_i}, $1\leqslant i\leqslant n$,
p appears in $h_i(A)$ as p^{α_i}, $k+1\leqslant i\leqslant n$.

The divisibility relations satisfied by the $h_i(C)$ and the $h_i(A)$ then become (19), (20), (23), and (24), and the degree relations (54) are (21) and (22).

Because p appears in $h_i(B)$ as

$$p^{\max(\gamma_i, \alpha_{i+k-2})},$$

and because degree $p=1$, the degree inequality (55) to be proved is precisely (25). The Lemma in Sec. 7 now shows that this degree condition is satisfied. The proof of Theorem 6 is complete. ■

9. COMPARISON WITH SINGULAR VALUE INTERLACING INEQUALITIES FOR MINORS OF DEFECT 1 OR MORE

We cannot refrain from mentioning again the extraordinary and totally unexpected analogy between properties of invariant factors for minors and properties of singular values for minors. For minors of deficiency one, this was already noted in Sec. 5. For minors of deficiency k, we have

$$\begin{array}{ll}
\beta_1 \geqslant \alpha_1 \geqslant \beta_{1+2k}, & \beta_1 | \alpha_1 | \beta_{1+2k}, \\
\beta_2 \geqslant \alpha_2 \geqslant \beta_{2+2k}, & \beta_2 | \alpha_2 | \beta_{2+2k}, \\
\cdots & \cdots \\
\beta_{n-2k} \geqslant \alpha_{n-2k} \geqslant \beta_n, & \beta_{n-2k} | \alpha_{n-2k} | \beta_n, \\
\beta_{n-2k+1} \geqslant \alpha_{n-2k+1}, & \beta_{n-2k+1} | \alpha_{n-2k+1}, \\
\cdots & \cdots \\
\beta_{n-k} \geqslant \alpha_{n-k}, & \beta_{n-k} | \alpha_{n-k}. \\
\text{Singular values} & \text{Invariant factors}
\end{array} \tag{57}$$

Here the β's are the singular values or invariant factors of an $n\times n$ matrix B, and the α's are the singular values or invariant factors of an $(n-k)\times(n-k)$ matrix A, subject to the usual conditions $\beta_1 \geqslant \cdots \geqslant \beta_n$, $\alpha_1 \geqslant \cdots \geqslant \alpha_{n-k}$, or $\beta_1 | \cdots | \beta_n$, $\alpha_1 | \cdots | \alpha_{n-k}$. Both for unitary equivalence $B \to UBV$ with U, V unitary, and for unimodular equivalence $B \to UBV$ with U, V unimodular, the conditions (57) are necessary and sufficient for the embeddability of A as a submatrix of UBV for some U, V. And, of course, for matrices over a field and $V = U^{-1}$, the conditions (57) are necessary and sufficient subject to the obvious degree requirements.

The question of studying the relationship between the $\mathfrak{R}$-invariant factors of a matrix and those of a submatrix was raised by Dr. Morris Newman in the course of a series of lectures in Santa Barbara in the spring *of* 1976. *I am greatly indebted to Professor Newman for his penetrating and incisive suggestion, and his conjecture that the invariant factors of the submatrix should in some manner divide the invariant factors of the full matrix.*

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