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Bilinear Forms and (Hyper-) Determinants

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1. INTRODUCTION

In [5], Subsection 4.6, to every $\mathbf{C}$ n-linear form A it is associated a cer-

[illegible] useful in the study of the *hyperdeterminant* of A (in the sense of [4]). One finds among the above multihomogeneous components many instances of the Cayley–Koszul complexes introduced in [4].

It is the purpose of this paper to describe (with different degrees of completeness) the homology of $\mathbf{K}_A$, in the case $n = 2$. Such a description is of intrinsic interest and will hopefully lend itself to generalization to the cases $n \geqslant 3$. As far as possible, we work over any ground ring R_0, not just over $\mathbf{C}$.

The article is organized as follows. Section 2 (any R_0) contains some preliminaries together with the definition of the complexes $\mathbf{B}(a, b; A)$. Section 3 shows that over $\mathbf{C}$, the bihomogeneous components of $\mathbf{K}_A$ are expressible in terms of the complexes $\mathbf{B}(a, b; A)$. Section 4 (any R_0) completely describes all $H.(\mathbf{B}(a, b; A))$, under the assumption that A is a square and invertible matrix. Section 5 derives from Section 4 a description of $H.(\mathbf{K}_A)$ over $\mathbf{C}$, in case A is invertible, and provides some clues on $H.(\mathbf{K}_A)$, when A is not a square invertible matrix.

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2. PRELIMINARIES

Let R_0 be any commutative ring, F_0 and G_0 two finitely generated free R_0-modules of ranks f and g, respectively, and R the symmetric algebra

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$S(F_0^* \oplus G_0^*) \cong R_0[x_i, y_j]$, $1 \leqslant i \leqslant f$, $1 \leqslant j \leqslant g$. As usual, we think of $\{x_i\}$ and of $\{y_j\}$ as of bases of F_0^* and G_0^*, respectively, dual to some bases $\{f_i\}$ and $\{g_i\}$ of F_0 and G_0, respectively, fixed once and for all.

We denote by A the bilinear form $\sum_{i,j} a_{ij} x_i y_j \in R_0[x_i, y_j]$, and by F and G the R-modules $F_0 \otimes R$ and $G_0 \otimes R$, respectively. By abuse of notation, $f_i \otimes 1$ and $g_j \otimes 1$ will still be indicated by f_i and g_j, respectively.

Let us consider the maps

$$\psi: R \to F \oplus G, \qquad \text{by} \quad \psi(1) = \sum_i x_i f_i + \sum_j y_j g_j$$

and

$$\varphi: F \oplus G \to R, \qquad \text{by} \quad \varphi(f_i) = \frac{\partial A}{\partial x_i}, \qquad 1 \leqslant i \leqslant f,$$

$$\varphi(g_j) = -\frac{\partial A}{\partial y_j}, \qquad 1 \leqslant j \leqslant g.$$

We denote by M the cokernel of ψ.

Clearly, the composite $\varphi \circ \psi$ is zero. Therefore the morphism φ factors through M, inducing a map $\bar{\varphi}$: $M \to R$.

Definition 1. $\mathbf{K}_A$ is the Koszul complex

$$0 \to \Lambda^{f+g} M \to \Lambda^{f+g-1} M \to \cdots \to \Lambda^2 M \to M \xrightarrow[\bar{\varphi}]{} R.$$

In order to study $H.(\mathbf{K}_A)$, we make the following preparations.

For each $k \in \{1, ..., f+g\}$, consider the complex $\mathbf{\Lambda}^k \psi$, that is the Schur complex $\mathbf{L}_\lambda(\psi)$ (cf. [2]) with $\lambda = (k)$. Because of [2], Theorem V.1.17, $\mathbf{\Lambda}^k \psi$ is acyclic. But since ψ provides a finite presentation of M, [2], Proposition V.2.2, tells us that $\mathbf{\Lambda}^k \psi$ is in fact a resolution of $\Lambda^k M$. We wish to build a double complex $\mathbf{D}$ of the kind

$$0 \to \mathbf{\Lambda}^{f+g} \psi \to \mathbf{\Lambda}^{f+g-1} \psi \to \cdots \to \mathbf{\Lambda}^2 \psi \to \psi \to R, \tag{1}$$

where R is thought of as a complex concentrated in degree 0. It is easy to check that suitable arrows in (1) can be provided by truncations of the Koszul complex

$$0 \to \Lambda^{f+g}(F \oplus G) \to \cdots \to \Lambda^2(F \oplus G) \longrightarrow F \oplus G \xrightarrow[\varphi]{} R.$$

Explicitly, the double complex $\mathbf{D}$ looks like

$$
\begin{array}{ccccccccccc}
\Lambda^{f+g}(F\oplus G) & \longrightarrow & \Lambda^{f+g-1}(F\oplus G) & \longrightarrow & \cdots & \longrightarrow & \Lambda^{2}(F\oplus G) & \longrightarrow & F\oplus G & \underset{\varphi}{\longrightarrow} & R \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
\Lambda^{f+g-1}(F\oplus G) & \longrightarrow & \Lambda^{f+g-2}(F\oplus G) & \longrightarrow & \cdots & \longrightarrow & F\oplus G & \underset{\varphi}{\longrightarrow} & R & & \\
\uparrow & & \uparrow & & & & \uparrow & & & & \\
\Lambda^{f+g-2}(F\oplus G) & \longrightarrow & \Lambda^{f+g-3}(F\oplus G) & \longrightarrow & \cdots & \longrightarrow & R & & & & \\
\uparrow & & \uparrow & & & & & & & & \\
\vdots & & \vdots & & & & & & & & \\
\uparrow & & & & & & & & & & \\
R & & & & & & & & & &
\end{array}
$$

For later reference, we stipulate that rows are numbered from top to bottom, the row index is u, and the top row corresponds to $u=0$. As for columns, they are numbered right to left, the column index is v, and the rightmost column corresponds to $v=0$.

The idea we have in mind is to somehow study the bihomogeneous components of $\mathbf{K}_A$ in terms of those of $\mathbf{D}$.

One more ingredient is necessary. For every pair of nonnegative integers a and b, we define a complex $\mathbf{B}(a, b; A)$ of R_0-modules in the following manner.

For every $v=0, 1, \ldots,$ $(\mathbf{B}(a, b; A))_v = (L_{\mu'(v)}(F_0^* \oplus G_0^*))_{a+f-v,\, b+g-v}$, where the righthand side of the equality stands for the part of the Schur functor $L_{\mu'(v)}(F_0^* \oplus G_0^*)$ having F_0^*-content $a+f-v$ and G_0^*-content $b+g-v$, and $\mu'(v)$ is the conjugate partition of the hook $(a+b+1-v, 1^{f+g-1-v})$. We agree that when both $a+b+1-v$ and $f+g-1-v$ are 0, $(\mathbf{B}(a, b; A))_v=0$. (The only case of this kind with $a+f-v \geqslant 0$ and $b+g-v \geqslant 0$ corresponds to $a=g-1$ and $b=f-1$.)

As for the nonzero boundary morphisms, for each $v=f+g-1$, $f+g-2, \ldots,$ $\partial_v\colon (\mathbf{B}(a, b; A))_v \to (\mathbf{B}(a, b; A))_{v-1}$ is induced by $\widetilde{\partial}_v\colon \Lambda_{\mu'(v)}(F_0^* \oplus G_0^*) \to \Lambda_{\mu'(v-1)}(F_0^* \oplus G_0^*)$ such that

$$
\begin{aligned}
&\widetilde{\partial}_v(c_1 \otimes \cdots \otimes c_{a+b+1-v}) \\
&\qquad = (-1)^{v-1} \sum_{i,j} a_{ij}(x_i \wedge c_1 \otimes c_2 \otimes \cdots \otimes c_{a+b+1-v} \otimes y_j \\
&\qquad\qquad - y_j \wedge c_1 \otimes c_2 \otimes \cdots \otimes c_{a+b+1-v} \otimes x_i).
\end{aligned}
$$

(For notations and properties of Schur functors, cf. [2], Chapter II.)

One should remark that when $R_0 = \mathbf{C}$, if $a \geqslant g-1$ and $b \geqslant f-1$, and $(a-g+1, b-f+1) \neq (0,0)$, then $\mathbf{B}(a, b; A)$ coincides with the Cayley–Koszul complex $C^{\cdot}(m_1, m_2; A)$ of [4], where $m_1 = a-(g-1)$ and $m_2 = b-(f-1)$ (their term $C^p(m_1, m_2; A)$ corresponds to our $(\mathbf{B}(a, b; A))_{f+g-1-p}$).

3. THE BIHOMOGENEOUS COMPONENTS OF $\mathbf{K}_A$

Let $\mathbf{D}(a, b)$ denote the bihomogeneous component of $\mathbf{D}$ of bidegree (a, b), $a \geqslant 0$ and $b \geqslant 0$. In this section, we prove the following result.

PROPOSITION 2. *Assume that R_0 is a field of characteristic zero. Then for all pairs of nonnegative integers a and b, the complex induced by $\mathbf{D}(a, b)$ on the 0th homology modules of its columns precisely coincides with $\mathbf{B}(a, b; A)$, except for the following case: when $a = g$ and $b = f$, the complex obtained is $0 \to R_0 \to_{\chi} \mathbf{B}(a, b; A)$, with R_0 in degree $v = f+g$ and $\chi = 0$.*

Proof. Since $R = S(F_0^* \oplus G_0^*) \cong \sum_{a \geqslant 0,\, b \geqslant 0} (S_a F_0^* \otimes S_b G_0^*)$, for every fixed nonnegative a and b we can construct $\mathbf{D}(a, b)$ starting from $S_a F_0^* \otimes S_b G_0^*$, which is in coordinates $(u, v) = (0, 0)$. What we find in generic position (u, v) is

$$\sum_{s+t=v-u} (\Lambda^s F_0 \otimes \Lambda^t G_0) \otimes (S_{a-t-u} F_0^* \otimes S_{b-s-u} G_0^*).$$

(Notice that $v-u \geqslant 0$ since $\mathbf{D}$ is triangular.)

One should remark that we are using $\Lambda^k(F \oplus G) = \Lambda^k(F_0 \oplus G_0) \otimes_{R_0} R$ together with $\Lambda^k(F_0 \oplus G_0) \cong \sum_{s+t=k} \Lambda^s F_0 \otimes_{R_0} \Lambda^t G_0$.

Now, keeping in mind the filtration of $L_{\mu'(v)}(F_0^* \oplus G_0^*)$ described in [2], Theorem II.4.11, and the usual identifications $\Lambda^s F_0 \cong \Lambda^{f-s} F_0^*$ and $\Lambda^t G_0 \cong \Lambda^{g-t} G_0^*$, one can check that the boundary maps of the complex induced by $\mathbf{D}(a, b)$ on the 0th homology modules do coincide with those of $\mathbf{B}(a, b; A)$, and with χ when appropriate.

Thus the real point of the proof is showing that in bidegree (a, b), $H_0(\mathbf{\Lambda}^v \psi)$ yields $(\mathbf{B}(a, b; A))_v$. Since R_0 is a field of characteristic zero, we may assume that all the terms of $\mathbf{D}$ and of each $\mathbf{B}(a, b; A)$ are direct sums of irreducible $GL(F_0) \times GL(G_0)$-representations. Then for each v, it suffices to show that the irreducibles of $\sum_{s+t=v} (\Lambda^s F_0 \otimes \Lambda^t G_0) \otimes (S_{a-t} F_0^* \otimes S_{b-s} G_0^*)$, which are not canceled by those occurring in $\Lambda^{v-1}(F \oplus G)$, are precisely those belonging to $(\mathbf{B}(a, b; A))_v$, and to the isolated R_0 when appropriate.

Now Lemma 3 below states that when $v \leqslant f+g-1$, $H_0(\mathbf{\Lambda}^v\psi)$ always yields $(\mathbf{B}(a, b; A))_v$.

In Lemma 4 below, one reads that when $v = f+g$, $H_0(\mathbf{\Lambda}^v\psi)$ yields 0, except for $a=g$ and $b=f$, in which case one gets $S_0F_0^* \otimes S_0G_0^* = R_0$. (Recall that $(\mathbf{B}(a, b; A))_{f+g} = 0$ always.)

So the proof of the proposition is complete, once we prove the two lemmas. ▮

Lemma 3. *When $v \leqslant f+g-1$, $H_0(\mathbf{\Lambda}^v\psi)$ yields $(\mathbf{B}(a, b; A))_v$ in every bidegree (a, b).*

Proof. Let

$$(L_{(f-s,\,1^{a-t})}F_0^* \oplus L_{(f-s+1,\,1^{a-t-1})}F_0^*) \otimes S_{b-s}G_0^* \otimes \Lambda^{g-t}G_0^*$$

be abbreviated by

$$[(f-s,\, 1^{a-t}) \oplus (f-s+1,\, 1^{a-t-1}) \parallel (1^{b-s}) \otimes (g-t)].$$

Recalling $\Lambda^sF_0 \cong \Lambda^{f-s}F_0^*$, $\Lambda^tG_0 \cong \Lambda^{g-t}G_0^*$, and Pieri formula, one sees that for every bidegree (a, b), the *acyclic* complex $\mathbf{\Lambda}^v\psi$ looks as follows:

$$\begin{aligned}
&[(f-v,\, 1^a) \oplus \underline{(f-v+1,\, 1^{a-1})} \parallel (1^{b-v}) \otimes (g)] \qquad (v+1 \text{ terms})\\
&\oplus [(f-v+1,\, 1^{a-1}) \oplus \underline{(f-v+2,\, 1^{a-2})} \parallel (1^{b-v+1}) \otimes (g-1)]\\
&\oplus \ldots\ldots \oplus [(f-2,\, 1^{a-v+2}) \oplus \underline{(f-1,\, 1^{a-v+1})} \parallel (1^{b-2}) \otimes (g-v+2)]\\
&\oplus [(f-1,\, 1^{a-v+1}) \oplus \underline{(f,\, 1^{a-v})} \parallel (1^{b-1}) \otimes (g-v+1)]\\
&\oplus [(f,\, 1^{a-v}) \oplus \underbrace{(f+1,\, 1^{a-v-1})}_{0} \parallel (1^{b}) \otimes (g-v)]
\end{aligned}$$

$$\uparrow$$

$$\begin{aligned}
&[\underline{(f-v+1,\, 1^{a-1})} \oplus \overleftrightarrow{(f-v+2,\, 1^{a-2})} \parallel (1^{b-v}) \otimes (g)] \qquad (v \text{ terms})\\
&\oplus [\underline{(f-v+2,\, 1^{a-2})} \oplus \overleftrightarrow{(f-v+3,\, 1^{a-3})} \parallel (1^{b-v+1}) \otimes (g-1)]\\
&\oplus \ldots\ldots \oplus [\underline{(f-1,\, 1^{a-v+1})} \oplus \overleftrightarrow{(f,\, 1^{a-v})} \parallel (1^{b-2}) \otimes (g-v+2)]\\
&\oplus [\underline{(f,\, 1^{a-v})} \oplus \underbrace{(f+1,\, 1^{a-v-1})}_{0} \parallel (1^{b-1}) \otimes (g-v+1)]
\end{aligned}$$

$$\uparrow$$

$$\begin{aligned}
&[\,\underleftrightarrow{(f-v+2,\,1^{a-2})} \oplus (f-v+3,\,1^{a-3}) \parallel (1^{b-v}) \otimes (g)] \qquad (v-1 \text{ terms})\\
&\oplus [\,\underleftrightarrow{(f-v+3,\,1^{a-3})} \oplus (f-v+4,\,1^{a-4}) \parallel (1^{b-v+1}) \otimes (g-1)]\\
&\oplus \ldots\ldots \oplus [\,\underleftrightarrow{(f,\,1^{a-v})} \oplus \underbrace{(f+1,\,1^{a-v-1})}_{0} \parallel (1^{b-2}) \otimes (g-v+2)]
\end{aligned}$$

$$\uparrow$$

$$\vdots$$

$$\uparrow$$

$$\begin{aligned}
&[(f-2,\,1^{a-v+2}) \oplus \underline{\underline{(f-1,\,1^{a-v+1})}} \parallel (1^{b-v}) \otimes (g)] \qquad (3 \text{ terms})\\
&\oplus [(f-1,\,1^{a-v+1}) + \underline{\underline{(f,\,1^{a-v})}} \parallel (1^{b-v+1}) \otimes (g-1)]\\
&\oplus [(f,\,1^{a-v}) \oplus \underbrace{(f+1,\,1^{a-v-1})}_{0} \parallel (1^{b-v+2}) \otimes (g-2)]
\end{aligned}$$

$$\uparrow$$

$$\begin{aligned}
&[\underline{\underline{(f-1,\,1^{a-v+1})}} \oplus \underline{\underline{(f,\,1^{a-v})}} \parallel (1^{b-v}) \otimes (g)] \qquad (2 \text{ terms})\\
&\oplus [\underline{\underline{(f,\,1^{a-v})}} \oplus \underbrace{(f+1,\,1^{a-v-1})}_{0} \parallel (1^{b-v+1}) \otimes (g-1)]
\end{aligned}$$

$$\uparrow$$

$$[\underline{\underline{(f,\,1^{a-v})}} \oplus \underbrace{(f+1,\,1^{a-v-1})}_{0} \parallel (1^{b-v}) \otimes (g)] \qquad (1 \text{ term})$$

$$\uparrow$$

$$0$$

Notice that we have underlined the summands which cancel out.

Again recalling the filtration of [2], Theorem II.4.11, one sees that the summands which survive in the top summation add up precisely to $(\mathbf{B}(a, b; A))_v$, as required.

One should remark that when $v = a + f$, no F_0^* occurs in $\mathbf{\Lambda}^v \psi$, so that the above does not strictly apply. But it is easy to check that a similar pattern still holds, and the statement is true. ∎

LEMMA 4. *When* $v = f + g$, $H_0(\mathbf{\Lambda}^v \psi)$ *is equal to a copy of* R_0, *occurring in bidegree* $(a, b) = (g, f)$.

Proof. When either $a < g$ or $b < f$, $\mathbf{\Lambda}^{f+g}\psi$ is identically zero in bidegree (a, b), for one finds in degree u the term

$$\sum_{s+t=f+g-u} (\Lambda^s F_0 \otimes \Lambda^t G_0) \otimes (S_{a-t-u} F_0^* \otimes S_{b-s-u} G_0^*)$$
$$= \sum_{i+j=u} (\Lambda^{f-i} F_0 \otimes \Lambda^{g-j} G_0) \otimes (S_{a-g+j-u} F_0^* \otimes S_{b-f+i-u} G_0^*).$$

When $(a, b) = (g, f)$, $\mathbf{\Lambda}^{f+g}\psi$ is concentrated in degree $u=0$, and equals $S_0 F_0^* \otimes S_0 G_0^* = R_0$.

When $a \geqslant g$, $b \geqslant f$, and $(a, b) \neq (g, f)$, the complex $\mathbf{\Lambda}^{f+g}\psi$ in bidegree (a, b) looks like

$$[\underbrace{(0, 1^{a-g})}_{0} \oplus \underline{(1, 1^{a-g-1})} \parallel (1^{b-f}) \otimes (0)]$$
$$\uparrow$$
$$[\underbrace{(0, 1^{a-g})}_{0} \oplus \underset{\longleftrightarrow}{(1, 1^{a-g-1})} \parallel (1^{b-f-1}) \otimes (1)]$$
$$\oplus [\underline{(1, 1^{a-g-1})} \oplus \underset{\longleftrightarrow}{(2, 1^{a-g-2})} \parallel (1^{b-f}) \otimes (0)]$$
$$\uparrow$$
$$\vdots$$

(notations as in the proof of Lemma 3), and $H_0(\mathbf{\Lambda}^{f+g}\psi)$ turns out to be zero in the indicated bidegrees.

Also in this proof, the case $v = a + f$ is not covered by the above. But one can easily make the necessary adjustments. ▌

4. DISASSEMBLING THE COMPLEXES $\mathbf{B}(a, b; A)$

Proposition 2 has reduced the study of the homology of $\mathbf{K}_A$ to that of $H.(\mathbf{B}(a, b; A))$, $a \geqslant 0$ and $b \geqslant 0$, provided R_0 is a field of characteristic zero. We discuss $H.(\boldsymbol{B}(a, b; A))$ partly in this section and partly in the next one. We point out that the contents of this section are valid over every ground ring R_0.

THEOREM 5. *If $f = g$ and the matrix (a_{ij}) is invertible, then $\mathbf{B}(a, b; A)$ is always exact, except for the pairs $(a, b) = (h, h)$, where $0 \leqslant h \leqslant f-1$. When $h \in \{0, 1, ..., f-1\}$, $H.(\mathbf{B}(h, h; A))$ is equal to R_0 in (top) degree $v = 2h$.*

Proof. We begin by filtering $\mathbf{B}(a, b; A)$, $a \geqslant b$, in the following way:

$$\mathbf{0} \subseteq \mathbf{X}_0(a, b; A) \subseteq \mathbf{X}_1(a, b; A) \subseteq \cdots \subseteq \mathbf{X}_t(a, b; A)$$
$$\subseteq \cdots \subseteq \mathbf{X}_{f-1}(a, b; A) = \mathbf{B}(a, b; A);$$

recalling the filtration of $(\mathbf{B}(a, b; A))_v$ given in [2], Theorem II.4.11, by definition $(\mathbf{X}_t(a, b; A))_v$ is spanned by all tableaux of $(\mathbf{B}(a, b; A))_v$ whose F_0^*-part has first row of length $\geqslant f-t$. Clearly, the boundary morphism of $\mathbf{B}(a, b; A)$ is compatible with this definition. And the conditions $f=g$ and $a \geqslant b$ rule out the possibility that $\mathbf{B}(a, b; A)$ may contain tableaux with no F_0^*-part.

We denote $\mathbf{X}_t(a, b; A)/\mathbf{X}_{t-1}(a, b; A)$ by $\mathbf{Q}_t(a, b; A)$ (assuming that $\mathbf{X}_{-1}(a, b; A)=\mathbf{0}$). Notice that $\mathbf{Q}_t(a, b; A)=\mathbf{0}$, whenever $t \geqslant b+1$.

Explicitly, a nonzero $\mathbf{Q}_t(a, b; A)$ looks like

$$\begin{aligned} 0 &\rightarrow [(f-t, 1^{a+t+1-f-g}) \parallel (1^{b-t}) \otimes (1+t-f)] \\ &\rightarrow [(f-t, 1^{a+t+2-f-g}) \parallel (1^{b-t}) \otimes (2+t-f)] \\ &\rightarrow \cdots \rightarrow [(f-t, 1^{a+t+1}) \parallel (1^{b-t}) \otimes (g-1+t)] \\ &\rightarrow [(f-t, 1^{a+t}) \parallel (1^{b-t}) \otimes (g+t)], \end{aligned}$$

where we use the abbreviated notation introduced in the proof of Lemma 3.

Recalling the well known short exact sequence of modules

$$0 \rightarrow L_{(f-t+1,\, 1^{a+t-v-1})} F_0^* \rightarrow S_{a+t-v} F_0^* \otimes \Lambda^{f-t} F_0^* \rightarrow L_{(f-t,\, 1^{a+t-v})} F_0^* \rightarrow 0$$

(cf., e.g., [1], Theorem 3.3), one easily checks that for $b \geqslant 1$, there is a short exact sequence of complexes

$$\begin{aligned} \mathbf{0} &\rightarrow \mathbf{Q}_{t-1}(a-1, b-1; A)(-1) \\ &\rightarrow \Lambda^{f-t} F_0^* \otimes \mathbf{W}(a, A)(-t) \otimes S_{b-t} G_0^* \rightarrow \mathbf{Q}_t(a, b; A) \rightarrow \mathbf{0}, \end{aligned} \qquad (2)$$

where up to sign, $\mathbf{W}(a, A)$ coincides with the Schur complex $\mathbf{L}_{(1^a)}(G_0 \rightarrow_{(a_{ij})} F_0^*)$, which is independent of both b and t. (As usual, given a complex $\mathbf{Y}$ and an integer l, we denote by $\mathbf{Y}(l)$ the complex defined by $(\mathbf{Y}(l))_i = \mathbf{Y}_{l+i}$.)

By [2], Corollary V.1.15, the assumptions of the theorem imply that $\mathbf{W}(a, A)$ is exact for every $a \geqslant 1$. Hence the short exact sequence (2) says that the exactness of $\mathbf{B}(a, b; A)$ follows from that of $\mathbf{Q}_0(a-t, b-t; A) = \mathbf{X}_0(a-t, b-t; A)$, $0 \leqslant t \leqslant \min\{b, f-1\}$.

Since $\mathbf{X}_0(a-t, b-t; A) \cong \Lambda^f F_0^* \otimes \mathbf{W}(a-t; A) \otimes S_b G_0^*$ and $\mathbf{W}(a-t; A)$ is exact for $a-t \geqslant 1$, we are through provided $a-t \neq 0$. If $f-1<b$, clearly $t \leqslant f-1<b \leqslant a$ says $a \neq t$. If $f-1 \geqslant b$, $t \leqslant b \leqslant a$ says that $a=t$ can happen only when $a=b=t$. In fact $\mathbf{X}_0(0, 0; A)$ is the module $(L_{(f+g)}(F_0^* \oplus G_0^*))_{f,g} = \Lambda^f F_0^* \otimes \Lambda^g G_0^*$ in degree $v=0$, and obviously $H_\cdot(\mathbf{X}_0(0, 0; A)) = H_0(\mathbf{X}_0(0, 0; A)) \cong R_0$.

Summarizing, we have proved the exactness of $\mathbf{B}(a, b; A)$ whenever $a>b$, as well as the exactness of $\mathbf{B}(h, h; A)$ whenever $h \geqslant f$.

As for $a<b$, we notice that another filtration of $\mathbf{B}(a, b; A)$ exists, again given by [2], Theorem II.4.11, but ordering the basis $\{x_i\} \cup \{y_j\}$ of $F_0^* \oplus G_0^*$ by putting $y_1, ..., y_g$ before $x_1, ..., x_f$. Then (2) is replaced by a short exact sequence involving $\mathbf{L}_{(1^b)}(F_0 \to_{(a_{ij})^T} G_0^*)$, which is exact since the transpose matrix $(a_{ij})^T$ is invertible, too. So the argument given for $a>b$ translates into an argument for $b>a$, *mutatis mutandis.*

In order to complete the proof, we have to discuss $H.(\mathbf{B}(h, h; A))$, $h=0, 1, ..., f-1$. Clearly, $\mathbf{B}(0, 0; A)=\mathbf{X}_0(0, 0; A)$ and we have already seen that $H.(\mathbf{B}(0, 0; A))=H_0(\mathbf{B}(0, 0; A))=R_0$. As for $1 \leqslant h \leqslant f-1$, we examine the (nonzero) quotients $\mathbf{Q}_h(h, h; A)$, $\mathbf{Q}_{h-1}(h, h; A), ..., \mathbf{Q}_0(h, h; A)$. An argument involving (2) shows that the exactness of $\mathbf{Q}_{h-1}(h, h; A)$, ..., $\mathbf{Q}_0(h, h; A)$ follows from that of the complexes $\mathbf{X}_0(h, h; A)$, $h \geqslant 1$, which we have already established. But a similar argument for $\mathbf{Q}_h(h, h; A)$ breaks down at the very last step, for $\mathbf{B}(0, 0; A)$ gets involved. More precisely, the long exact homology sequence induced by (2), and the exactness of $\mathbf{L}_{(1^a)}(G_0 \to_{(a_{ij})} F_0^*)$ say that for every v,

$$\begin{aligned} H_v(\mathbf{Q}_h(h, h; A)) &= H_{v-1}(\mathbf{Q}_{h-1}(h-1, h-1; A)(-1)) \\ &= H_{v-2}(\mathbf{Q}_{h-1}(h-1, h-1; A)). \end{aligned}$$

Hence $H_v(\mathbf{Q}_h(h, h, A))=H_{v-2h}(\mathbf{Q}_0(0, 0; A))$ for every $v \in \{2h, 2h-1, ..., h\}$ (this is the range where $\mathbf{Q}_h(h, h; A)$ is not zero). Whenever $v<2h$, the righthand side of the last equality is 0, since $\mathbf{Q}_0(0, 0; A)$ is 0 in negative degrees. But $H_{2h}(\mathbf{Q}_h(h, h; A))=H_0(\mathbf{Q}_0(0, 0; A))=R_0$. ▮

We remark that if under the assumptions of Theorem 5, one sets $E \cong G_0 \xrightarrow[(a_{ij})]{\cong} F_0^*$, the definition of the complex $\mathbf{Q}_t(a, b; A)$ shows it to be isomorphic (up to $S_{b-t}G_0^*$) to the complex $C^{\cdot}_{\gamma:q}$ of [3], with $\gamma=(1^{a+f-t})$ and $q=f-t$. It then follows from [3], Corollary 3.4, that $H.(\mathbf{Q}_t(a, b; A))$ is concentrated in its leftmost position. Namely,

$$H.(\mathbf{Q}_t(a, b; A))=\begin{cases} 0 & \text{if } \; a \geqslant g+1 \\ H_{a+t}(\mathbf{Q}_t(a, b; A)) \cong \Lambda^{a+f-t}E & \text{if } \; a \leqslant g \end{cases} \tag{3}$$

(note that $\Lambda^{a+f-t}E \neq 0$ only if $a \geqslant t+1$). It is easy to derive from (3) another proof of Theorem 5 (one still needs to exchange the roles of F_0^* and G_0^* in the case $a<b$).

The reader will notice that Theorem 5 provides another proof of the following special case of [4], Theorem 2.1 (cf. the remark at the end of Section 2).

Corollary 6. *If (a_{ij}) is a square invertible matrix, it is exact every Cayley–Koszul complex $C^{\cdot}(m_1, m_2; A)$ with m_1 and m_2 two nonnegative integers not both* 0. ▮

5. THE HOMOLOGY OF $\mathbf{K}_A$

Throughout this section, we again assume that R_0 is a field of characteristic zero. Putting together Proposition 2 and Theorem 5, we have the following statement.

THEOREM 7. *Assume that R_0 is a field of characteristic zero, that $f = g$, and that (a_{ij}) is an invertible matrix. Then*

$$H_i(\mathbf{K}_A) = \begin{cases} R_0 & \text{if } \; i = 0, 2, ..., 2f-2, 2f \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The bihomogeneous components of $\mathbf{K}_A$ are equal to $\mathbf{B}(a, b; A)$, except for $a = g = f = b$, when one has $0 \to R_0 \to_\chi \mathbf{B}(f, f; A)$, $\chi = 0$.

$H.(\mathbf{B}(f, f; A)) = 0$ says that $H_{2f}(\mathbf{K}_A) = R_0$.

$H.(\mathbf{B}(h, h; A)) = H_{2h}(\mathbf{B}(h, h; A)) = R_0$, $0 \leqslant h \leqslant f-1$, says that $H_{2f-2}(\mathbf{K}_A) = H_{2f-4}(\mathbf{K}_A) = \cdots = H_0(\mathbf{K}_A) = R_0$. ▮

What about $H.(\mathbf{K}_A)$ when (a_{ij}) is not an invertible matrix (and possibly $f \neq g$)? We immediately have the following partial result.

PROPOSITION 8. *For every (a_{ij}), $H_{f+g}(\mathbf{K}_A) = R_0$ and $H_0(\mathbf{K}_A) = R/Im(\varphi)$.*

Proof. There is nothing to prove for H_0. (Notice that when (a_{ij}) is invertible, $Im(\varphi) = (x_1, ..., x_f, y_1, ..., y_g)$, so that $R/Im(\varphi) = R_0$.)

As for H_{f+g}, given the upper lefthand corner of $\mathbf{D}$:

$$\begin{array}{ccccc} \Lambda^{f+g}(F \oplus G) & \xrightarrow[\eta]{} & \Lambda^{f+g-1}(F \oplus G) & \longrightarrow & \cdots \\ \uparrow \zeta & & \uparrow \theta & & \\ \Lambda^{f+g-1}(F \oplus G) & \longrightarrow & \Lambda^{f+g-2}(F \oplus G) & \longrightarrow & \cdots, \\ \uparrow & & \uparrow & & \\ \vdots & & \vdots & & \end{array}$$

easy calculations show that $Im(\eta) \subseteq Im(\theta)$ and that $\zeta = \psi^*$; since $Im(\psi^*) = (x_1, ..., x_f, y_1, ..., y_g)$, it follows that $H_{f+g}(\mathbf{K}_A) = R_0$. ▮

One should remark that the proof of Proposition 8 is in fact independent of the assumption that R_0 is a field of characteristic zero.

Turning now to H_i, $1 \leqslant i \leqslant f+g-1$, we are unable to fully describe the situation. But the following statement indicates that the simple pattern of Theorem 7 no longer holds. (Such a pattern is in fact equivalent to (a_{ij}) being a square invertible matrix.)

Proposition 9. *$H_1(\mathbf{K}_A)=0$ if and only if (a_{ij}) is a square invertible matrix.*

Proof. If $H_1(\mathbf{K}_A)=0$, it follows that $H_1(\mathbf{B}(1,0;A))=0=H_1(\mathbf{B}(0,1;A))$. $\mathbf{B}(1,0;A)$ is simply $G_0\to_{(a_{ij})}F_0^*$, so that $\operatorname{Ker}(a_{ij})=0$. Up to sign, $\mathbf{B}(0,1;A)$ is $F_0\to_{(a_{ij})^T}G_0^*$, so that $\operatorname{Ker}(a_{ij})^T=0$, i.e., (a_{ij}) onto. ▮

We have a parallel result concerning $H_{f+g-1}(\mathbf{K}_A)$.

Proposition 10. *$H_{f+g-1}(\mathbf{K}_A)=0$ if and only if $(a_{ij})\neq 0$.*

Proof. If $(a_{ij})=0$, clearly

$$H_{f+g-1}(\mathbf{K}_A)=\Lambda^{f+g-1}M=\sum_{\substack{a\geqslant g-1,\,b\geqslant f-1\\(a,b)\neq(g-1,f-1)}} S_{a-g+1}F_0^*\otimes S_{b-f+1}G_0^*\neq 0.$$

Let us assume now that $(a_{ij})\neq 0$, and prove that ∂_{f+g-1} is injective for every (a,b) such that $a\geqslant g-1$, $b\geqslant f-1$, and $(a,b)\neq(g-1,f-1)$.

Let $m=a-g+1$ and $n=b-f+1$. Up to sign, ∂_{f+g-1} acts on a standard tableau as follows:

$$\begin{array}{l} i_1\\ \vdots\\ i_m\\ j_1\\ \vdots\\ j_n \end{array} \;\mapsto\; \Sigma_{i,j}\; a_{ij}\;\left(\begin{array}{ll} i_1 & j\\ \vdots & \\ i_m & \\ j_1 & \\ \vdots & \\ j_n & \\ i & \end{array} \;-\; \begin{array}{ll} i_1 & i\\ \vdots & \\ i_m & \\ j_1 & \\ \vdots & \\ j_n & \\ j & \end{array}\right)$$

($i_1\leqslant\cdots\leqslant i_m$, $j_1\leqslant\cdots\leqslant j_n$, and the F_0^*-indices come before the G_0^*-ones).

If $i>i_1$, the straightening law says that the above summation is equal to

$$\Sigma_{i,j}\; a_{ij}\;\left(\begin{array}{ll} i_1 & j\\ \vdots & \\ i & \\ \vdots & \\ i_m & \\ j_1 & \\ \vdots & \\ j_n & \end{array} \;-\; \begin{array}{ll} i_1 & i\\ \vdots & \\ i_m & \\ j_1 & \\ \vdots & \\ j & \\ \vdots & \\ j_n & \end{array}\right)\;;$$

but if $i \leqslant i_1$ (or $m = 0$), then it equals

$$\Sigma_{i,j} \quad a_{ij} \quad \left(\begin{array}{ll} i & j \\ i_1 & \\ i_2 & \\ \vdots & \\ i_m & \\ j_1 & \\ \vdots & \\ j_n & \end{array} \right) .$$

Let us order the standard bases involved, according to the lexicographic order of the sequences $(i_1, ..., i_m, j_1, ..., j_n)$ and of the sequences $(i_1, ..., i, ..., i_m, j_1, ..., j_n, j)$, $(i_1, ..., i_m, j_1, ..., j, ..., j_n, i)$, and $(i, i_1, ..., i_m, j_1, ..., j_n, j)$, but with the proviso that those of type $(i_1, ..., i_m, j_1, ..., j, ..., j_n, i)$ always come after those of the other two kinds. Then the matrix associated to ∂_{f+g-1} looks like

$$\left(\begin{array}{llll} a_{11} & 0 & 0 & \\ *_{21} & 0 & 0 & \\ \vdots & \vdots & \vdots & \\ *_{g1} & 0 & 0 & \\ *_{(g+1)1} & a_{11} & 0 & \\ *_{(g+2)1} & * & 0 & \text{et cetera} \\ \vdots & \vdots & \vdots & \\ *_{(2g)1} & * & 0 & \\ *_{(2g+1)1} & * & a_{11} & \\ *_{(2g+2)1} & * & * & \\ \vdots & \vdots & \vdots & \end{array} \right).$$

Since $(a_{ij}) \neq 0$, at least one entry is different from zero. We can assume that we have ordered the bases of F_0^* and G_0^* in such a way that $a_{11} \neq 0$. Then the matrix above says that ∂_{f+g-1} is indeed injective. ∎

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