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Combinatorial invariants computing the Ray–Singer analytic torsion

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Abstract: It is shown that for any piecewise-linear closed orientable manifold K of odd dimension there exits an invariantly defined metric on the determinant line of cohomology $\det(H^*(K; E))$, where E is an arbitrary flat bundle over K (here E is not required to be unimodular). The construction of this metric (called Poincaré–Reidemeister metric) is purely combinatorial; it combines the standard Reidemeister's construction with the Poincaré duality.

The main properties of the Poincaré–Reidemeister metric consist in the following: (a) the Poincaré–Reidemeister metric can be computed starting from any polyhedral cell decomposition of the manifold in purely combinatorial terms; (b) the Poincaré–Reidemeister metric coincides with the Reidemeister metric when the latter is correctly defined (i.e., when the bundle E is unitary or unimodular). (c) The construction of Ray and Singer, which uses zeta-function regularized determinants of Laplacians, produces the metric on the determinant of cohomology, which coincides (via the De Rham isomorphism) with the Poincaré–Reidemeister metric. This is the main result of the paper, showing that the Poincaré–Reidemeister metric computes combinatorially the Ray–Singer metric. (d) The Poincaré–Reidemeister metric behaves well with respect to natural correspondences between determinant lines which are discussed in the paper.

It is shown also that the Ray–Singer metrics on some relative determinant lines can be computed combinatorially (including the even-dimensional case) in terms of the metrics determined by correspondences.

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1. Introduction

Let K denote a closed odd-dimensional smooth manifold and let E be a flat vector bundle over K. In this situation the construction of Ray and Singer [13] gives a metric on the determinant line of the cohomology $\det H^*(M; E)$ which is a *smooth invariant of the manifold M and the flat bundle E*. (Note that if the dimension of K is even then the Ray–Singer metric depends on the choice of a Riemannian metric on K and of a Hermitian metric on E). The famous theorem which was proved by J. Cheeger [4] and W. Müller [10], states that assuming that the flat vector bundle E is *unitary* (i.e., E admits a flat Hermitian metric), the Ray–Singer metric coincides with the Reidemeister metric, which is defined using finite dimensional linear algebra by the combinatorial structure of K.

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The construction of the Reidemeister metric works also under a weeker assumption that the flat bundle E is *unimodular*, i.e., the line bundle $\det(E)$ admits a flat metric. In a recent paper W. Müller [11] proved that in this case the Ray–Singer metric again coincides with the Reidemeister metric.

Without the assumption that the flat bundle E is unimodular, the standard construction of Reidemeister metric is ambiguous (it depends on different choices made in the process of its construction). In this general situation J.-M. Bismut and W. Zhang [2] computed the deviation of the Ray–Singer metric from so called Milnor metric, cf. [2]; the result of their computation is given in a form of an integral of a Chern–Simons current which compensates the ambiguity of the Milnor metric and depends on the Riemannian metric on K and on the metric on E.

In this paper I show that for any piecewise-linear closed orientable manifold K of odd dimension there exits an invariantly defined metric on the determinant line of cohomology $\det(H^*(K; E))$, where E is an arbitrary flat bundle over K. The construction of this metric is purely combinatorial. I call this metric Poincaré–Reidemeister metric since it is defined by combining the standard Reidemeister's construction with the Poincaré duality.

The idea to use the Poincaré duality in order to construct an invariantly defined metric on the determinant line was prompted by the Theorem 4.1 of D. Burghelea, L. Friedlander, and T. Kappeler [3], expressing the analytic torsion through some torsion invariants (depending on the Riemannian metric) associated with a Morse function.

The main properties of the Poincaré–Reidemeister metric consist in the following:

(a) the Poincaré–Reidemeister metric can be computed starting from any polyhedral cell decomposition of the manifold and using purely combinatorial terms, cf. 4.3.

(b) the Poincaré–Reidemeister metric coincides with the Reidemeister metric when the latter is correctly defined;

(c) The construction of Ray and Singer, which uses zeta-function regularized determinants of Laplacians, produces the metric on the determinant of cohomology, which coincides (via the De Rham isomorphism) with the Poincaré–Reidemeister metric; this is the main result of the paper, formulated as Theorem 6.2;

(d) The Poincaré–Reidemeister metric behaves well with respect to natural correspondences between determinant lines which are discussed in Section 1, cf. Prop. 4.8.

Another interesting observation made here consists in finding that Ray–Singer metrics on some relative determinant lines can be computed combinatorially (including the even-dimensional case) in terms of the metrics determined by correspondences, cf. Theorems 5.1, 5.2 and 5.3.

The work technically is based on the fundamental Theorem (0.2) of J.-M. Bismut and W. Zhang [2] and uses actually a very special corollary of it which is proved by cancelling all complicated terms in the Bismut–Zhang theorem (cf. Theorem 5.1 below). Note, that even weeker Theorem 5.3 allows to *identify completely the Ray–Singer norm* (*cf. proof of Theorem* 6.2) *in terms of the Poincaré–Reidemeister norm in the odd-dimensional case*. Most of the paper consists in careful elementary analysis of the foundations: we tried to study separately different duality relations which exist among the determinant lines, duality between homology and cohomology and the Poincaré duality, and also correspondences induced by isomorphisms of the volume bundles.

Notations. K will denote a finite polyhedron given by its polyhedral cell decomposition (cf. [14]). We will consider flat vector bundles E over K; this we understand in the following way:

the structure group of E has been reduced to a descrete group. We will also identify flat vector bundles with the locally constant sheaves of their flat sections.

Given a polyhedral cell decomposition τ of K, one constructs the chain complex $C_*(M, \tau, E)$ as follows: the basis of $C_q(M, \tau, E)$ form pairs $(\Delta^q; s)$, where Δ^q is an oriented q-dimensional cell of τ and s is a flat section of E over Δ^q. The boundary operator

$$\partial : C_q(M, \tau, E) \to C_{q-1}(M, \tau, E)$$

is defined by the usual formula $\partial(\Delta^q; s) = \sum \epsilon[\Delta^q, \Delta^{q-1}](\Delta^{q-1}, s')$, where $\epsilon[\Delta^q, \Delta^{q-1}]$ is the sign ± 1 determined by the orientations and s' is the restriction of s on Δ^{q-1} and the sum runs over all $(q-1)$-dimensional cells $\Delta^{q-1} \subset \overline{\Delta^q}$.

We will denote by $T : \det(C_*(M, \tau, E)) \to \det(H_*(M; E))$ the canonical isomorphism between the determinant lines, cf. [1]. Sometimes we will write T_τ instead of T in order to emphasize dependance on the polyhedral cell decomposition τ.

Warning. In this paper, like in most of the literature, we neglect signs of isomorphism between determinant lines. To be more precise, we will work in the category whose objects are one-dimensional vector spaces (over $\mathbb{R}$ or over $\mathbb{C}$) and whose morphisms are linear maps, such that f and $-f$ are considered as representing the same morphism.

1. Correspondences between determinant lines

1.1. Suppose that we are given two flat vector bundles E and F (real or complex) over a finite polyhedron K and an isomorphism of the determinant flat line bundles $\phi : \det(E) \to \det(F)$. We will show in this section that these data determine canonically an isomorphism between the corresponding determinant lines

$$\hat{\phi} : \det H_*(K; E) \to \det H_*(K; F).$$

This map $\hat{\phi}$ will be called *correspondence between the determinant lines* determined by ϕ.

Let τ be a polyhedral cell decomposition of K. According to the definition above, the vector space $C_q(M, \tau, E)$ is the direct sum of the spaces of flat sections $\Gamma(\Delta; E)$, where Δ runs over all oriented q-dimensional cells of K. Thus one can identify the determinant line $\det C_q(K, \tau, E)$ with the tensor product of determinant lines $\bigotimes_\Delta \det \Gamma(\Delta; E)$. On the other hand, for any cell Δ we have the canonical isomorphism $\det \Gamma(\Delta; E) = \Gamma(\Delta; \det(E))$. Therefore we obtain the canonical isomorphisms

$$\det(C_*(K, \tau, E)) = \bigotimes_q \det(C_q(K, \tau, E))^{(-1)^q} = \bigotimes \Gamma(\Delta, \det(E))^{\epsilon(\Delta)}, \tag{1}$$

where $\epsilon(\Delta)$ denotes $(-1)^{\dim(\Delta)}$. We may consider the similar canonical decomposition for the flat bundle F as well and then one defines the map

$$\tilde{\phi} : \det(C_*(K, \tau, E)) \to \det(C_*(K, \tau, F)) \tag{2}$$

as the tensor product the maps $\Gamma(\Delta, \det(E)) \to \Gamma(\Delta, \det(F))$, induced by the map ϕ if the dimension $\dim(\Delta)$ is even, and induced by the contragradient map $\bar{\phi} : (\det(E))^* \to (\det(F))^*$ (inverse to the adjoint) if the dimension $\dim(\Delta)$ is odd. Finally we define the map $\hat{\phi}$ which appeares in the commutative diagram

$$\begin{array}{ccc} \det(C_*(K,\tau,E)) & \xrightarrow{\tilde{\phi}} & \det(C_*(K,\tau,F)) \\ {\scriptstyle T_\tau}\downarrow & & {\scriptstyle T_\tau}\downarrow \\ \det(H_*(K,E)) & \xrightarrow{\hat{\phi}} & \det(H_*(K,F)) \end{array} \tag{3}$$

Recall that the vertical maps T_τ are the canonical maps, cf.[1]. The above definition of $\hat{\phi}$ is justified by combinatorial invariance property, similar to combinatorial invariance of the Reidemeister torsion [6]:

1.2. Proposition (Combinatorial Invariance). *The map $\hat{\phi}$ does not depend on the polyhedral cell decomposition τ of K. More precisely, the above construction gives the same map $\hat{\phi}$ between the determinant lines for any pair of polyhedral cell decompositions of K having a common subdivision.*

Proof. Let τ' be a subdivision of the polyhedral cell decomposition τ. We have the natural inclusion of the chain complexes $i : C_*(K,\tau,E) \to C_*(K,\tau',E)$; let $D_*(\tau',\tau,E)$ denote the factor complex. It is acyclic and so there is correctly defined a volume

$$\alpha(\tau',\tau,E) \in \det(D_*(\tau',\tau,E)). \tag{4}$$

The spaces of chains of the complex $D_* = D_*(\tau',\tau,E)$ are the factor-spaces D_q which appear in the exact sequence

$$0 \to \bigoplus_\Delta \Gamma(\Delta,E) \xrightarrow{r} \bigoplus_{\Delta'} \Gamma(\Delta',E) \to D_q \to 0;$$

here Δ runs over all cells of τ which are not cells of τ' and Δ' runs over all cells of τ' which are not cells of τ. The map r is given by restriction of flat sections. Therefore we obtain

$$\det(D_*(\tau',\tau,E)) = \bigotimes_{\Delta'} \Gamma(\Delta',\det(E))^{\epsilon(\Delta')} \bigotimes_\Delta \Gamma(\Delta,\det(E))^{\epsilon(\Delta)+1}. \tag{5}$$

In the last formula again Δ runs over all cells of τ which are not cells of τ' and Δ' runs over all cells of τ' which are not cells of τ.

Now suppose that we are given two flat bundles E and F over the complex K and a bundle isomorphism $\phi : \det(E) \to \det(F)$. We may construct the corresponding factor-complexes $D_*(\tau',\tau,E)$ and $D_*(\tau',\tau,F)$ and consider their volume elements $\alpha(\tau',\tau,E) \in \det(D_*(\tau',\tau,E))$ and $\alpha(\tau',\tau,F) \in \det(D_*(\tau',\tau,F))$. From formula (5) for their determinants it is clear that ϕ defines an isomorphism $\tilde{\phi} : \det(D_*(\tau',\tau,E)) \to \det(D_*(\tau',\tau,F))$ which (as above) is the product of the maps $\Gamma(\Delta',\det(E)) \to \Gamma(\Delta',\det(F))$ and $\Gamma(\Delta,\det(E)) \to \Gamma(\Delta,\det(F))$ induced by ϕ or their contragradient maps.

We claim now that *the constructed map $\tilde{\phi}$ preserves the volume*; in other words,

$$\tilde{\phi}(\alpha(\tau',\tau,E)) = \alpha(\tau',\tau,F). \tag{6}$$

We will prove this fact assuming that τ' is obtained from τ by dividing one q-dimensional cell e of τ into two q-dimensional cells e_+ and e_- and introducing a $(q-1)$-dimensional cell e_0, cf. Figure 1. The general statement follows from this special case by induction.

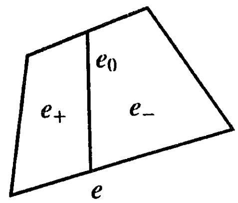

Fig. 1.

Under this our assumption the chain complexes $D_*(\tau', \tau, E)$ and $D_*(\tau', \tau, F)$ are particularly simple. They have only chains in dimensions $q-1$ and q and can be described as follows. The space $D_q = D_q(\tau', \tau, E)$ can be identified with $D_q = \Gamma(e_+, E)$, the $q-1$-dimensional chains D_{q-1} can be identified with $D_{q-1} = \Gamma(e_0, E)$ and the boundary homomorphism $\partial : D_q \to D_{q-1}$ can be identified with the restriction map $r : \Gamma(e_+, E) \to \Gamma(e_0, E)$. Thus the volume $\alpha(\tau', \tau, E) \in \det(D_*(\tau', \tau, E))$ is represented by the determinant of the restriction map r

$$\det(r) : \Gamma(e_+, \det(E)) \to \Gamma(e_0; \det(E)).$$

A similar description is valid for the bundle F. Now, the claim that $\tilde{\phi}(\alpha(\tau', \tau, E)) = \alpha(\tau', \tau, F)$ is precisely equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \Gamma(e_+, \det(E)) & \xrightarrow{r} & \Gamma(e_0, \det(E)) \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow \\ \Gamma(e_+, \det(F)) & \xrightarrow{r} & \Gamma(e_0, \det(F)) \end{array}$$

which is obvious. This finishes the proof that $\tilde{\phi}$ is volume preserving.

Let us now define a map

$$S_{\tau',\tau} : \det(C_*(K, \tau', E)) \to \det(C_*(K, \tau, E)). \tag{7}$$

We have the canonical identification

$$U : \det(C_*(K, \tau', E)) \to \det(C_*(K, \tau, E)) \otimes \det(D_*(K, \tau', \tau)).$$

For $x \in \det(C_*(K, \tau', E))$ set

$$S_{\tau',\tau}(x) = U(x)/\alpha(\tau', \tau, E). \tag{8}$$

Then, on one hand, we have the following commutative diagram

$$\begin{array}{ccc} \det(C_*(K, \tau', E)) & \xrightarrow{S_{\tau',\tau}} & \det(C_*(K, \tau, E)) \\ {\scriptstyle T_{\tau'}}\downarrow & & \downarrow{\scriptstyle T_\tau} \\ \det(H_*(K, E)) & \xrightarrow{=} & \det(H_*(K, E)). \end{array} \tag{9}$$

To justify (9) one may refer to [5]; Theorem 1.3 of [5] implies (9) immediately. On the other hand, for any bundle isomorphism $\phi : \det(E) \to \det(F)$ we have the commutative diagram

$$\begin{array}{ccc} \det(C_*(K,\tau',E)) & \xrightarrow{S_{\tau',\tau}} & \det(C_*(K,\tau,E)) \\ \tilde\phi\downarrow & & \tilde\phi\downarrow \\ \det(C_*(K,\tau',F)) & \xrightarrow{S_{\tau',\tau}} & \det(C_*(K,\tau,F)). \end{array} \tag{10}$$

It follows from the definition (8) and the fact established above (cf. (6)) that the induced map $\tilde\phi$: $\det(D_*(\tau',\tau,E)) \to \det(D_*(\tau',\tau,F))$ preserves the volumes: $\tilde\phi(\alpha(\tau',\tau,E)) = \alpha(\tau',\tau,F)$.

This proves independence of the constructed map $\hat\phi$ on the triangulation. □

1.3. Proposition (Functorial property). *Suppose that E, F, and G are three flat vector bundles over K and let $\phi : \det(E) \to \det(F)$ and $\psi : \det(F) \to \det(G)$ be two isomorphisms. Then*

$$\widehat{\psi\circ\phi} = \hat\psi\circ\hat\phi : \det(H_*(M;E)) \to \det(H_*(M;G)). \tag{11}$$

Proof. It is clear from the construction.

1.4. Remark. Suppose that $\phi : \det(E) \to \det(F)$ is an isomorphism of flat vector bundles. Let λ be a number. Then

$$\widehat{\lambda\phi} = \lambda^{\chi(K)}\hat\phi, \tag{12}$$

where $\chi(K)$ denotes the Euler characteristic of K.

It follows from this remark that under the assumption that $\chi(K) = 0$, the isomorphism $\hat\phi$ does not depend on ϕ (it only requires existence of an isomorphism $\det(E) \to \det(F)$).

1.5. Correspondces between determinant lines appear also in the following version.

Let E be a flat vector bundle (over the field $k = \mathbb{C}$ or $\mathbb{R}$) and let $\det_+(E)$ denote the flat bundle with fibre $\mathbb{R}_+$ obtained as follows. First consider the complement of the zero section of $\det(E)$ as a principal k^*-bundle; then form the associated bundle with fibre $\mathbb{R}_+$ corresponding to the action of k^* on $\mathbb{R}_+$ given by $\alpha\cdot x = |\alpha|\cdot x$ for $\alpha\in k^*$, $x\in\mathbb{R}_+$.

Suppose now that we have two flat vector bundles E and F over a polyhedron K and an isomorphism $\phi : \det_+(E) \to \det_+(F)$ of flat bundles. In terms of the flat bundles $\det(E)$ and $\det(F)$ this means that for simply-connected open sets $U \subset K$ ϕ determines an isomorphism $\Gamma(U,\det(E)) \to \Gamma(U,\det(F))$ up to multiplication by a number with norm 1. Repeating the construction of Section 1.1 we obtain that ϕ *determines a correspondence*

$$\hat\phi : \det H_*(K;E) \to \det H_*(K;F)$$

up to multiplication by a number with norm 1.

Yet another version of correspondences between the determinant lines can be constructed as follows.

1.6. Proposition. *Suppose that E and F are two flat vector bundles over a compact polyhedron K. Any flat section (metric) on the flat line bundle $\det(E)\otimes\det(F)$ determines canonically an element of (or a metric on, correspondingly) the product of the determinant lines*

$$\det(H_*(K,E)) \otimes \det(H_*(K,F)).$$

Proof. The construction is quite similar to the one described above. We first remark that the line $\det(H_*(K, E)) \otimes \det(H_*(K, E^*))$ can be identified with the product

$$\prod [\Gamma(\Delta, \det(E)) \otimes \Gamma(\Delta, \det(F))]^{\epsilon(\Delta)}$$

and each term of this product has a section (or a metric) induced by the data. Then one shows (using arguments similar to given above) that the constructed metric does not depend on the particular triangulation. □

1.7. Remark. The metric constructed in Proposition 1.6 can also be obtained using the standard construction of Reidemeister metric (cf. 4.5) by first observing that $\det(E \oplus F) = \det(E) \otimes \det(F)$ and so the data determine a flat metric on this bundle; then the standard construction produces a metric on the determinant line $\det(H_*(K, E \oplus F)$ which is canonically isomorphic to the product $\det(H_*(K, E) \otimes \det(H_*(K, F))$. It is easy to see that the constructed metric coincides with the one given by Proposition 1.6.

1.8. Corollary. *For any flat vector bundle E over the polyhedron K there is a canonical element in the line* $\det(H_*(K, E)) \otimes \det(H_*(K, E^*))$, *(defined up to a sign) where E^* denotes the dual flat vector bundle. This element determines a canonical metric (which will be denoted by $\langle\cdot\rangle_E$) on the above line.*

The next proposition describes the relation between two constructions which apeared in this section: between the correspondences $\hat{\phi}$ and the metrics $\langle , \cdot\rangle_E$ on the products of the determinant lines.

1.9. Proposition. *Suppose that E and F are two flat vector bundles over K and let $\phi : \det(E) \to \det(F)$ be an isomorphism of flat bundles. Denote by $\psi : \det(F^*) \to \det(E^*)$ the adjoint bundle isomorphism. Then for any $x \in \det(H_*(K, E))$ and $y \in \det(H_*(K, F^*))$ the following formula holds:*

$$\langle x \otimes \hat{\psi}(y)\rangle_E = \langle \hat{\phi}(x) \otimes y\rangle_F; \tag{13}$$

here $\hat{\phi} : \det H_*(K; E) \to \det H_*(K; F)$ and $\hat{\psi} : \det(H_*(K; F^*)) \to \det(H_*(K; E^*))$ are constructed as explained in 1.1 *and $\langle\cdot\rangle_E$ and $\langle\cdot\rangle_F$ denote the canonical metrics on the products* $\det H_*(K; E) \otimes \det H_*(K; E^*)$ *and* $\det H_*(K; F) \otimes \det H_*(K; F^*)$ *respectively, constructed in* 1.8. □

2. Correspondences between cohomological determinants

2.1. Let K be a finite polyhedron and let E be a (real or complex) flat vector bundle over K. To simplify our notations the determinant of the homology of K with coefficients in E will be denoted $L_\bullet(E) = \det(H_*(K, E))$. Similarly, we will denote by $L^\bullet(E) = \det(H^*(K, E))$ the determinant of the cohomology; the latter can be understood as the cohomology of the locally constant sheaf determined by E.

2.2. There is the canonical pairing

$$\{\cdot , \cdot\}_E : L_\bullet(E) \otimes L^\bullet(E^*) \to \mathbb{C} \tag{14}$$

which is determined by the evaluation maps $H_i(K; E) \otimes H^i(K; E^*) \to \mathbb{C}$. This gives a natural isomorphism

$$L^\bullet(E) \to L_\bullet(E^*)^*; \tag{15}$$

Using it we will reformulate the results of the previous section using cohomological determinants.

2.3. Suppose that E and F are two flat vector bundles and let $\phi : \det(E) \to \det(F)$ be an isomorphism of flat bundles. Consider the adjoint ϕ^* which acts $\phi^* : \det(F^*) \to \det(E^*)$; via the construction above, it induces the correspondence $\widehat{(\phi^*)} : L_\bullet(F^*) \to L_\bullet(E^*)$ and the adjoint of the latter gives the correspondence $L_\bullet(E^*)^* \to L_\bullet(F^*)^*$. Using the canonical isomorphism (14) we can interpret the last map as

$$\check{\phi} : L^\bullet(E) \to L^\bullet(F). \tag{16}$$

We will refer to it as to the *cohomological correspondence determined by* ϕ.

2.4. The cohomological correspondence has properties similar to the homological one. Namely,

(1) *If* $\chi(K) = 0$ *then* $\check{\phi} : L^\bullet(E) \to L^\bullet(F)$ *does not depend on* ϕ.

(2) *Suppose that* E, F, *and* G *are three flat vector bundles over* K *and let* $\phi : \det(E) \to \det(F)$ *and* $\psi : \det(F) \to \det(G)$ *be two isomorphisms. Then*

$$\psi \check{\circ} \phi = \check{\psi} \circ \check{\phi} : L^\bullet(E) \to L(G^\bullet).$$

(3) *Any flat section (metric) on the flat line bundle* $\det(E) otimes \det(F)$ *determines canonically an element (or a metric, correspondingly) on the product* $L^\bullet(E) \otimes L^\bullet(F)$. Indeed, any flat section (metric) on the line bundle $\det(E) \otimes \det(F)$ determines the obvious metric on the dual bundle $\det(E^*) \otimes \det(F^*)$ and the latter by the construction of Proposition 1.6 gives an element (metric, correspondingly) on the line $L_\bullet(E^*) \otimes L_\bullet(F^*)$. The latter clearly determines an element (or metric) on $L^\bullet(E) \otimes L^\bullet(F)$.

In particular, *for any flat line bundle E we obtain a canonical metric on the line* $L^\bullet(E) \otimes L^\bullet(E^*)$; *this metric will be denoted* $\langle \cdot \rangle_E$.

(4) There is also the relation similar to (13). *Suppose that E and F are two flat vector bundles over K and let* $\phi : \det(E) \to \det(F)$ *be an isomorphism of flat bundles. Denote by* $\psi : \det(F^*) \to \det(E^*)$ *the adjoint bundle isomorphism. Then we have the combinatorial correspondences* $\check{\phi} : L^\bullet(E) \to L^\bullet(F)$ *and* $\check{\psi} : L^\bullet(F^*) \to L^\bullet(E^*)$, *and for any* $x \in L^\bullet(E)$ *and* $y \in L^\bullet(F^*)$ *the following formula holds*

$$\langle x \otimes \check{\psi}(y) \rangle_E = \langle \check{\phi}(x) \otimes y \rangle_F. \tag{17}$$

3. Poincaré duality for determinant lines

In this section K *will denote a closed oriented piecewise-linear manifold of odd dimension.* E will denote a flat k-vector bundle over K, where k is $\mathbb{R}$ or $\mathbb{C}$.

It is well known that the Poincaré duality induces some isomorphisms between the determinant lines, cf. [7,15,2]. Our aim in this section is to introduce the notations and establish relations between the correspondences $\hat{\phi}$ and $\check{\phi}$ introduced in the previous sections and the isomorphisms determined by the Poincaré duality. The result of this section will be used in Section 4 to define the Poincaré–Reidemeister metric.

3.1. Since the orientation of the manifold K is supposed to be fixed, for any integer q we have a nondegenerate intersection form $H_q(K; E)\otimes H_{n-q}(K, E^*) \to k$, where $n = \dim K$ is supposed to be odd. The above pairing allows one to identify $H_q(K, E)$ with the dual of $H_{n-q}(K, E^*)$. Since q and $n - q$ are of opposite parity, it defines an isomorphism

$$\mathcal{D}_E : L_\bullet(E) \to L_\bullet(E^*). \tag{18}$$

3.2. Another description of the Poincaré duality map $\mathcal{D}_E$ can be given as follows. Consider a triangulation τ of K and the dual cell decomposition τ^*. There is the intersection pairing on the chain level $C_q(K, \tau, E) \otimes C_{n-q}(K, \tau^*, E^*) \to k$. Let Δ be a q-dimensional simplex of τ and let Δ^* be the dual $(n - q)$-dimensional cell of τ^*. Then the above intersection form splits into an orthogonal sum of partial pairings $\Gamma(\Delta, E) \otimes \Gamma(\Delta^*, E^*) \to k$, which assign to a pair of flat sections $s \in \Gamma(\Delta, E)$ and $s' \in \Gamma(\Delta^*, E^*)$ the number $\langle s'(x), s(x)\rangle \in k$, where x denotes the common point of Δ and Δ^*. The last pairing is non-degenerate and defines an isomorphism $\alpha_\Delta : \Gamma(\Delta, E) \to \Gamma(\Delta^*, E^*)^*$. Thus we obtain the maps

$$\det(\alpha_\Delta) : \Gamma(\Delta, \det(E)) \to \Gamma(\Delta^*, \det(E^*))^*. \tag{19}$$

We claim that the following diagram commutes

$$\begin{array}{ccc} \prod_\Delta \Gamma(\Delta, \det(E))^{\epsilon(\Delta)} & \xrightarrow{\bigotimes \det(\alpha_\Delta)^{\epsilon(\Delta)}} & \prod_{\Delta^*} \Gamma(\Delta^*, \det(E^*))^{\epsilon(\Delta^*)} \\ {\scriptstyle T_\tau}\downarrow & & \downarrow{\scriptstyle T_{\tau^*}} \\ L_\bullet(E) & \xrightarrow{\mathcal{D}_E} & L_\bullet(E^*)\,. \end{array} \tag{20}$$

This gives another characterization of the duality map $\mathcal{D}_E : L_\bullet(E) \to L_\bullet(E^*)$.

The claim is a special case of the following general algebraic remark. Let C be a chain complex of finite dimensional vector spaces and let n be an odd integer. Form a new complex $D = (D_j, \delta)$, where $D_j = C^*_{n-j}$ and $\delta : D_j \to D_{j-1}$ is the dual of $\partial : C_{n-j+1} \to C_{n-j}$. For any index i we have the canonical map $\alpha_i : C_i \to D^*_{n-i}$ (the identity) and $\det(\alpha_i) : \det(C_i) \to \det(D_{n-i})^*$. We obtain the following diagram

$$\begin{array}{ccc} \det(C) = \bigotimes \det(C_i)^{(-1)^i} & \xrightarrow{\bigotimes \det(\alpha_i)^{(-1)^i}} & \bigotimes \det(D_i)^{(-1)^i} = \det(D) \\ {\scriptstyle T_C}\downarrow & & \downarrow{\scriptstyle T_D} \\ \det(H_*(C)) & \xrightarrow{\Sigma} & \det(H_*(D)) \end{array} \tag{21}$$

where the vertical maps T_C and T_D are the canonical maps and the horizontal map Σ is determined by the obvious maps $H_i(C) \to H_{n-i}(D)^*$. The proof of this elementary fact can be obtained directly from the definitions.

3.3. Proposition. *The duality isomorphism $\mathcal{D}_E$ has the following properties*:

(1) *The map $\mathcal{D}_{E^*} : L_\bullet(E^*) \to L_\bullet(E^{**}) = L(E)$ is inverse to D_E.*

(2) *Let E and F be two flat vector bundles over K and let $\phi : \det(E) \to \det(F)$ be an isomorphism of flat bundles. Denote by $\psi : \det(F)^* = \det(F^*) \to \det(E^*) = \det(E)^*$ the map adjoint to ϕ. Then there are two combinatorial correspondences $\hat{\phi} : L_\bullet(E) \to L_\bullet(F)$ and*

$\hat{\psi} : L_\bullet(F^*) \to L_\bullet(E^*)$ *and the following diagram*

$$\begin{array}{ccc} L_\bullet(E) & \xrightarrow{\hat{\phi}} & L_\bullet(F) \\ {\scriptstyle \mathcal{D}_E}\downarrow & & {\scriptstyle \mathcal{D}_F}\downarrow \\ L_\bullet(E^*) & \xleftarrow{\hat{\psi}} & L_\bullet(F^*) \end{array} \tag{22}$$

commutes.

Proof. The first property above follows immediately from the symmetry of the intersection numbers. The second property follows from the second description of the map $\mathcal{D}_E$ given above and from the commutative diagram

$$\begin{array}{ccc} \Gamma(\Delta, \det(E)) & \xrightarrow{\det(\alpha_\Delta)} & (\Gamma(\Delta^*, \det(E^*)))^* \\ {\scriptstyle \phi}\downarrow & & {\scriptstyle \psi^*}\downarrow \\ \Gamma(\Delta, \det(F)) & \xrightarrow{\det(\alpha_\Delta)} & (\Gamma(\Delta^*, \det(F^*)))^*. \end{array}$$

This completes the proof. □

3.4. For any closed oriented manifold K of odd dimension and a flat vector bundle E over K there is the Poincaré duality map $\mathcal{D}_E : L^\bullet(E) \to L^\bullet(E^*)$ acting on cohomological determinant lines. It can be defined as follows. Using the canonical pairings (13), $\{\cdot, \cdot\}_E : L_\bullet(E) \otimes L^\bullet(E^*) \to \mathbb{C}$, and the similar pairing for the dual flat vector bundle E^*, we will define the map $\mathcal{D}_{E^*} : L^\bullet(E^*) \to L^\bullet(E^{**}) = L^\bullet(E)$ by the requirement

$$\{\mathcal{D}_E(x) \otimes \mathcal{D}_{E^*}(y)\}_{E^*} = \{x \otimes y\}_E \tag{23}$$

for any $x \in L_\bullet(E)$ and $y \in L^\bullet(E^*)$.

It is clear that the above duality map $\mathcal{D}_E : L^\bullet(E) \to L^\bullet(E^*)$ can also be described as the map between the determinant lines of the cohomology induced by the isomorphisms $H^i(K, E) \to (H^{n-i}(K, E^*))^*$, comming from the nondegenerate intersection forms $H^i(K, E) \otimes H^{n-i}(K, E^*) \to k$.

3.5. Note that if the dimension of the manifold K is even then the Poincaré duality determines an element of the line $\mathcal{D}_E \in L^\bullet(E) \otimes L^\bullet(E^*)$. It can be shown that $\langle \mathcal{D}_E \rangle_E = 1$; thus, in the even-dimensional case the Poincaré duality $\mathcal{D}_E$ determines the canonical pairing $\langle \cdot \rangle_E$. Compare [15].

4. Poincaré–Reidemeister metric

In this section K will denote an *odd-dimensional closed oriented manifold*. We will construct canonical metrics, which we call Poincaré–Reidemeister metrics, on the determinant lines $L_\bullet(E)$ and $L^\bullet(E)$ for any flat vector bundle.

4.1. Let E be a flat vector bundle over K. By Corollary 1.8 there is a canonical metric on the line $L_\bullet(E) \otimes L_\bullet(E^*)$ which we will denote by $\langle \cdot \rangle_E$. On the other hand, there is defined the duality map $\mathcal{D}_E : L_\bullet(E) \to L_\bullet(E^*)$, cf. 3.1.

Definition. The Poincaré–Reidemeister metric $\|\cdot\|^{PR}_{L_\bullet(E)}$ on the line $L_\bullet(E)$ is defined by

$$\|x\|^{PR}_{L_\bullet(E)} = \langle x \otimes \mathcal{D}_E(x)\rangle_E^{1/2} \tag{24}$$

for $x \in L(E)$.

First, we want to show that the Poincaré–Reidemeister metric behaves well with respect to the correspondences $\hat{\phi}$ between the determinant lines.

4.2. Proposition. *Suppose that K is a closed oriented manifold of odd dimension. Let E and F be two flat bundles over K and let $\phi : \det(E) \to \det(F)$ be an isomorphism of flat bundles. Let $\hat{\phi} : L_\bullet(E) \to L_\bullet(F)$ be the induced correspondence between the determinant lines, cf.* 1.1. *Then the correspondence $\hat{\phi}$ preserves the Poincaré–Reidemeister metrics: for any $x \in L_\bullet(E)$ we have*

$$\|\hat{\phi}(x)\|^{PR}_{L_\bullet(F)} = \|x\|^{PR}_{L_\bullet(E)} \tag{25}$$

where $\|\cdot\|^{PR}_{L_\bullet(E)}$ and $\|\cdot\|^{PR}_{L_\bullet(F)}$ denote the Poincaré–Reidemeister metrics on $L_\bullet(E)$ and $L_\bullet(F)$ correspondingly.

Proof. According to the definitions and using Prop. 1.9 and 3.3(2) we obtain $\|\hat{\phi}(x)\|^{PR,2}_{L_\bullet(F)} = \langle\hat{\phi}(x) \bigotimes \mathcal{D}_F\hat{\phi}(x)\rangle_F = \langle x \bigotimes \hat{\psi}\mathcal{D}_F\hat{\phi}(x)\rangle_E = \langle x \bigotimes \mathcal{D}_E(x)\rangle_E = \|x\|^{PR,2}_{L_\bullet(E)}$, where $\langle\cdot\rangle_E$ and $\langle\cdot\rangle_F$ denote the canonical metrics on the lines $L_\bullet(E) \otimes L_\bullet(E^*)$ and $L_\bullet(F) \otimes L_\bullet(F^*)$ respectively.

4.3. Now I will describe a recipe computing the Poincaré–Reidemeister metric from the combinatorial data.

Let K denote a closed oriented odd-dimensional manifold and let E be a flat vector bundle over K. Let τ be a triangulation of K. Given an element $\alpha \in \det C_*(K, \tau, E)$ our task is to compute the Poincaré–Reidemeister metric of the corresponding element $\|T_\tau(\alpha)\|^{PR}_{L_\bullet(E)}$, where $T_\tau : \det C_*(K, \tau, E) \to \det H_*(K; E)$ is the canonical isomorphism.

Let τ^* be the dual cell decomposition of K and let E^* be the dual flat vector bundle. Choose two arbitrary non-zero elements $\beta \in \det C_*(K, \tau^*, E)$ and $\gamma \in \det C_*(K, \tau^*, E^*)$. We are going to define three positive real numbers $[\alpha]/[\beta]$, $\langle\beta, \gamma\rangle$, and α/γ.

The first number $[\alpha]/[\beta] = \lambda > 0$ is defined by the requirement $T_\tau(\alpha) = \pm\lambda T_{\tau^*}(\beta)$, the equality taking place in the determinant line $\det H_*(K; E)$.

The second number is defined by

$$\langle\beta, \gamma\rangle = \prod_{\Delta^*}\{\beta_{\Delta^*}, \gamma_{\Delta^*}\}_{\Delta^*},$$

where the product is taken over all cells Δ^* of the dual cell decomposition τ^* and where $\beta = \bigotimes \beta_{\Delta^*} \in \bigotimes \Gamma(\Delta^*, \det(E))^{\epsilon(\Delta^*)}$ and $\gamma = \bigotimes \gamma_{\Delta^*} \in \bigotimes \Gamma(\Delta^*, \det(E^*))^{\epsilon(\Delta^*)}$. Here the brackets $\{\cdot, \cdot\}_{\Delta^*}$ denote the absolute value of the canonical evaluation pairing on

$$\Gamma(\Delta^*, \det(E))^{\epsilon(\Delta^*)} \otimes \Gamma(\Delta^*, \det(E^*))^{\epsilon(\Delta^*)}.$$

The third number α/γ is defined by the Poincaré duality: if $\alpha = \bigotimes \alpha_\Delta \in \Gamma(\Delta, \det(E))^{\epsilon(\Delta)}$ then

$$\alpha/\gamma = \prod_{(\Delta, \Delta^*)} (\alpha_\Delta : \gamma_{\Delta^*})$$

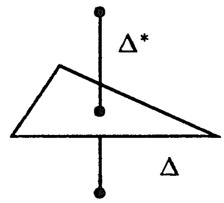

Fig. 2.

where the product is taken over all pairs (Δ, Δ^*) of mutually dual cells and for any such pair the number $(\alpha_\Delta : \gamma_{\Delta^*})$ is the absolute value of the evaluation at the common point of Δ and Δ^*, cf. Figure 2, of the flat sections $\alpha_\Delta \in \Gamma(\Delta, \det(E)^{\epsilon(\Delta)}$ and $\beta_{\Delta^*} \in \Gamma(\Delta, \det(E^*)^{\epsilon(\Delta^*)+1}$.

Now we can define the Poincaré–Reidemeister metric as the product of the constructed three numbers:

$$\|T_\tau(\alpha)\|^{PR,2}_{L_\bullet(E)} = [\alpha]/[\beta] \times \langle\beta, \gamma\rangle \times \alpha/\gamma \tag{26}$$

The above definition clearly does not depend on the choice of β and γ and the result coincides with the definition (24).

4.4. Proposition. *Suppose that flat vector bundle E is unimodular, i.e., there exists a flat metric on the bundle* $\det(E)$. *Then there is defined the standard Reidemeister metric on the determinant line $L_\bullet(E)$. We claim that in this case this Reidemeister metric coincides with the Poincaré–Reidemeister metric constructed above.*

4.5. Let us first recall the construction of the Reidemeister metric assuming that $\det(E)$ is unimodular. Let μ be a flat metric on the flat bundle $\det(E)$. Fix a polyhedral cell decomposition τ of K. For any cell $\Delta \subset K$ the flat metric μ defines an element $\mu_\Delta \in \det\Gamma(\Delta, E) = \Gamma(\Delta, \det(E))$, determined up to multiplication by a number with norm 1. The product of these μ_Δ's,

$$\mu_{\tau,E} = \prod_\Delta \mu_\Delta^{\epsilon(\Delta)}, \quad \epsilon(\Delta) = (-1)^{\dim(\Delta)}$$

defines an element $\mu_{\tau,E} \in \det C_*(K, \tau, E) = \prod \det\Gamma(\Delta, E)^{\epsilon(\Delta)}$ and thus it defines an element $T_\tau(\mu_{\tau,E}) \in L_\bullet(E)$ of the determinant line, determined again up to multiplication by a number with norm 1. The later element correctly defines a metric $\|\cdot\|_{L_\bullet(E)}$ on the determinant line $L_\bullet(E)$ by the requirement that $\|T_\tau(\mu_{\tau,E})\|_{L_\bullet(E)} = 1$. If one choses another flat metric $\mu' = \lambda\mu$ on $\det(E)$ then the constructed metric on $L_\bullet(E)$ does not change, since $\chi(K) = 0$ by the Poincaré duality.

One of the main properties of the Reidemeister metric is its *combinatorial invariance*. This means that for a pair of polyheral cell decompositions τ and τ' of K having a common subdivision we have the diagram

$$\mu_{\tau,E} \in \det(C_*(K, \tau, E)) \xrightarrow{T_\tau} \det H_*(K, E) \xleftarrow{T_{\tau'}} \det(C_*(K, \tau', E)) \ni \mu_{\tau',E}$$

and $T_\tau(\mu_{\tau,E}) : T_{\tau'}(\mu_{\tau',E})$ is a number of norm 1.

4.6. Proof of Proposition 4.4. Let μ and ν be a pair of mutually dual flat metrics on $\det(E)$ and $\det(E^*)$ correspondingly. Let τ be a triangulation of K and let τ^* be the dual cell decomposition.

Let $\mu_{\tau,E} \in L_\bullet(E)$ be the corresponding element as defined above in 4.5. To prove our statement we have to show that $\langle T_\tau(\mu_{\tau,E}) \otimes \mathcal{D}_E T_\tau(\mu_{\tau,E})\rangle_E = 1$.

First note that $T_\tau(\mu_{\tau,E}) = T_{\tau^*}(\mu_{\tau^*,E})$ by the combinatorial invariance property of the Reidemeister metric under subdivisions, cf. 4.5. On the other hand $\mathcal{D}_E T_{\tau^*}(\mu_{\tau^*,E}) = T_\tau(\nu_{\tau,E^*})$, as one directly checks using the definitions. Thus we have $\langle T_\tau(\mu_{\tau,E}) \otimes \mathcal{D}_E T_\tau(\mu_{\tau,E})\rangle_E = \langle T_\tau(\mu_{\tau,E}) \otimes T_\tau(\nu_{\tau,E^*})\rangle_E = 1$ completing the proof. □

4.7. Let us now define a similar *Poincaré–Reidemeister metric* (which we will denote $\|\cdot\|^{PR}_{L^\bullet(E)}$) on the cohomological determinant $L^\bullet(E)$. Here again K denotes a closed oriented odd-dimensional manifold and E a flat vector bundle over K.

Recall that there is a canonical norm $\langle\ ,\ \rangle_E$ on the line $L^\bullet(E) \otimes L^\bullet(E^*)$ (cf. 2.4(3)) and there is the Poincare duality map $\mathcal{D}_E : L^\bullet(E) \to L^\bullet(E^*)$ (cf. 3.4).

Definition. For $x \in L^\bullet(E)$ define

$$\|x\|^{PR,2}_{L^\bullet(E)} = \langle x \otimes \mathcal{D}_E(x)\rangle \tag{27}$$

Note that this Poincaré–Reidemeister metric on the cohomological determinant line is again a purely *combinatorial* invariant of K and E.

The cohomological version of the Poincaré–Reidemeister metric has properties similar to the homological one. We will formulate them without proof:

4.8. Proposition. (1) *If a flat bundle E is unimodular (i.e., the flat line bundle* $\det(E)$ *admits a flat metric) then the Poincaré–Reidemeister metric on $L^\bullet(E)$ coincides with the standard Reidemeister metric*;

(2) *Suppose that E and F are two flat bundles over K and let ϕ :* $\det(E) \to \det(F)$ *be an isomorphism of flat bundles. Let $\check\phi : L^\bullet(E) \to L^\bullet(F)$ denote the induced correspondence between the determinant lines, cf.* 2.3. *Then the correspondence $\check\phi$ preserves the Poincaré–Reidemeister metrics; in other words, for any $x \in L^\bullet(E)$ we have*

$$\|\check\phi(x)\|^{PR}_{L^\bullet(F)} = \|x\|^{PR}_{L^\bullet(E)}. \tag{28}$$

5. Ray–Singer norm of the combinatorial correspondence between the determinant lines

Let K denote a *closed oriented smooth manifold.* Suppose that we are given two flat vector bundles E and F over K and an isomorphism $\phi : \det(E) \to \det(F)$ between their flat determinant line bundles. As we have seen in Section 1, there is the correspondence $\check\phi : L^\bullet(E) \to L^\bullet(F)$ which is determined completely by the combinatorial structure of the initial data (K, E, F, ϕ). The correspondence $\check\phi$ determines a metric on the relative determinant line $L^\bullet(F) : L^\bullet(E)$, which is defined by the requirement that the norm of $\check\phi$ is 1.

In this section we use the main theorem of Bismut and Zhang [2, Theorem 0.2] to show that the Ray–Singer construction of analytic torsion produces the same metric on this relative determinant.

In the next section we will show that this theorem about the metrics on the relative determinants gives in the odd-dimensional situation an identification of the Ray–Singer metric with a combinatorially defined Poincaré–Reidemeister metric, defined in Section 4.

5.1. Theorem. *Let K be a closed oriented smooth manifold and let E and F be two flat vector bundles over K supplied with an isomorphism of flat bundles* $\phi : \det(E) \to \det(F)$. *Consider the correspondence* $\check{\phi} : L^\bullet(E) \to L^\bullet(F)$ *(cf. Section 1) constructed by means of a smooth polyhedral cell decomposition of K. Fix a Riemannian metric on K. Fix Hermitian metrics on the flat vector bundles E and F in such a way that the induced metrics on* $\det(E)$ *and* $\det(F)$ *are isomorphic via* $\phi : \det(E) \to \det(F)$. *There are then defined the Ray–Singer metrics* $\|\cdot\|^{RS}_{L^\bullet(E)}$ *and* $\|\cdot\|^{RS}_{L^\bullet(F)}$ *on the determinant lines* $L^\bullet(E)$ *and* $L^\bullet(F)$ *respectively. We claim that the relative Ray–Singer metric on the line* $\mathrm{Hom}(L^\bullet(E), L^\bullet(F)) = L^\bullet(F) \otimes L^\bullet(E)^*$ *coincides with the metric, determined on this line by the correspondence* $\check{\phi}$. *In other words, the correpondence* $\check{\phi}$ *preserves the Ray–Singer metrics: for any* $x \in L^\bullet(E)$ *the following formula holds*

$$\|\check{\phi}(x)\|^{RS}_{L^\bullet(F)} = \|x\|^{RS}_{L^\bullet(E)}. \tag{29}$$

Proof. The proof easily follows from Theorem (0.2) of Bismut and Zhang [2]. If we would have known that the cell decomposition associated with an arbitrary Morse function is a smooth polyhedral cell decomposition (in the sense of [14]) with respect to the smooth structure of K (which is not true in general) then we would be able to apply the theorem of Bismut and Zhang directly. Instead, we will proceed as follows.

Fix a smooth triangulation of K (cf. [12]) and consider its second derived subdivision (cf. [14]) which we will denote τ.

Choose a Riemanian metric on the manifold K. There is a Morse function f on K having the following properties:

(1) Any open simplex Δ of the triangulation τ contains a unique critical point p_Δ of f having index $\dim(\Delta)$;

(2) The unstable manifold of the critical point p_Δ coincides with Δ. Such function f can be constructed by considering the handle decomposition associated with the triangulation τ (cf. [14], Prop. 6.9), then by smoothing the corners of the handles (similarly to [9]) and then by using the standard correspondence between glueing handles, elementary cobordisms and the Morse functions having precisely one critical point, cf. [9].

The Thom–Smale complex associated with the function f is now identical to the simplicial chain complex of K with respect the triangulation τ.

For any simplex Δ of τ fix a volume element in $\det(\Gamma(\Delta, E)) = \Gamma(\Delta, \det(E))$. Then fix the volume element in $\det(\Gamma(\Delta, F)) = \Gamma(\Delta, \det(F))$ corresponding to the choice made for E under the bundle isomorphism $\phi : \det(E) \to \det(F)$. The above choices determine the metrics $\|\cdot\|^{M,X}_{L^\bullet(E)}$ and $\|\cdot\|^{M,X}_{L^\bullet(F)}$ via the canonical isomorphisms

$$L^\bullet(E) \to \det(C_*(K, \tau, E^*))^{-1} \tag{30}$$

(cf. (15)) and similarly for the bundle F; these metrics are called Milnor metrics in [2]. According to our general construction in Section 1, with the coherent choices (with respect to ϕ) as above made for the volume elements, we have

$$\|x\|^{M,X}_{L^\bullet(E)} = \|\check{\phi}(x)\|^{M,X}_{L^\bullet(F)} \tag{31}$$

for $x \in L^\bullet(E)$.

Let us apply Theorem (0.2) of Bismut and Zhang [2] twice: once for the metrics $\|\cdot\|^{RS}_{L^\bullet(E)}$ and

$\|\cdot\|^{M,X}_{L^\bullet(E)}$ on $L^\bullet(E)$ and the second time for the metrics $\|\cdot\|^{RS}_{L^\bullet(F)}$ and $\|\cdot\|^{M,X}_{L^\bullet(F)}$ on $L^\bullet(F)$. The right hand sides of the formula (0.8) of [2] in both cases will be the same because it depends only on the metric on the manifold K and the flat connection and the metric on the determinants $\det(E)$ and $\det(F)$. Thus, subtracting, we obtain

$$\frac{\|x\|^{RS}_{L^\bullet(E)}}{\|x\|^{M,X}_{L^\bullet(E)}} = \frac{\|y\|^{RS}_{L^\bullet(F)}}{\|y\|^{M,X}_{L^\bullet(F)}} \tag{32}$$

for any $x \in L^\bullet(E)$ and $y \in L^\bullet(F)$. Now, set $y = \check{\phi}(x)$; then from (30) we obtain $\|\check{\phi}(x)\|^{RS}_{L^\bullet(F)} = \|x\|^{RS}_{L^\bullet(E)}$ as stated. □

The following formulation is equivalent to Theorem 5.1.

5.2. Theorem. *Let K be a closed oriented smooth manifold and let E and F be two flat vector bundles over K such that the flat bundle* $\det(E) \otimes \det(F)$ *admits a flat metric μ. Then by* 2.4(3) *μ determines (combinatorially) a metric $\|\cdot\|_\mu$ on the line $L^\bullet(E) \otimes L^\bullet(F)$. Fix a Riemannian metric on K. Fix Hermitian metrics on the flat vector bundles E and F in such a way that the induced metrics on* $\det(E)\otimes\det(F)$ *is μ. There are then defined the Ray–Singer metrics $\|\cdot\|^{RS}_{L^\bullet(E)}$ and $\|\cdot\|^{RS}_{L^\bullet(F)}$ on the determinant lines $L^\bullet(E)$ and $L^\bullet(F)$ respectively. We claim that the product of these Ray–Singer metrics coincides with the combinatorial metric $\|\cdot\|_\mu$; more precisely for any $x \in L^\bullet(E)$ and $y \in L^\bullet(F)$ we have*

$$\|x\|^{RS}_{L^\bullet(E)} \cdot \|y\|^{RS}_{L^\bullet(F)} = \|x \otimes y\|_\mu.$$

We will formulate now a very special case of the above theorem, which will be used below. We will see later that this simple statement allows to identify completely the Ray–Singer metric in the odd-dimensional case.

5.3. Theorem. *Let K be a closed oriented manifold and let E be a flat vector bundle over K. By* 2.4(3) *there is a canonical metric $\langle\cdot\rangle_E$ on the line $L^\bullet(E) \otimes L^\bullet(E^*)$ which is determined by the combinatorial structure of K and E. Choose a Riemannian metrix on K and a Hermitian metric on the bundle E (which is not supposed to be flat). The latter determines a metric on the bundle E^*. The construction of Ray and Singer produces now the metrics $\|\cdot\|^{RS}_{L^\bullet(E)}$ and $\|\cdot\|^{RS}_{L^\bullet(E^*)}$ on the lines $L^\bullet(E)$ and $L^\bullet(E^*)$ correspondingly. Then their product is equal to the canonical combinatorial metric $\langle\cdot\rangle_E$ on $L^\bullet(E) \otimes L^\bullet(E^*)$.*

6. Ray–Singer metric coincides with the Poincaré–Reidemeister metric

We will prove in this section that for any closed oriented odd-dimensional manifold K and for any flat vector bundle E over K the Ray–Singer norm on the determinant line of the cohomology $L^\bullet(E)$ (constructed by using any Riemannian metric on K and a metric on E) coincides with the combinatorially defined Poincaré–Reidemeister metric. We will start with the following lemma.

6.1. Lemma. *Let K be an odd-dimensional manifold and let E be a flat vector bundle over K. Consider the map $\mathcal{D}_E : L^\bullet(E) \to L^\bullet(E^*)$ determined by the Poincare duality cf.* 3.4. *Fix a Riemannian metric on K and a metric on the bundle E; the latter determines a metric on E^*. Consider now the Ray–Singer metrics $\|\cdot\|^{RS}_{L^\bullet(E)}$ and $\|\cdot\|^{RS}_{L^\bullet(E^*)}$ on the lines $L^\bullet(E)$ and $L^\bullet(E^*)$*

determined by the choices made above. Then the map $\mathcal{D}_E$ preserves the Ray–Singer metrics: for any $x \in L^\bullet(E)$ we have $\|x\|^{RS}_{L^\bullet(E)} = \|\mathcal{D}_E(x)\|^{RS}_{L^\bullet(E^*)}$.

Proof. This is well-known, cf. [2], page 35.

6.2. Theorem. *Let K be a closed oriented smooth odd-dimensional manifold and let E be a flat vector bundle over K. Consider the Poincaré–Reidemeister metric $\|\cdot\|^{PR}_{L^\bullet(E)}$ on the determinant line of the cohomology $L^\bullet(E)$. Choose an arbitrary Riemannian metric on K and an arbitrary Hermitian metric on E and consider the metric $\|\cdot\|^{RS}_{L^\bullet(E)}$ given by the construction of Ray and Singer* [13]. *Then these two metrics coincide.*

Proof. For $x \in L^\bullet(E)$ we obtain

$$\begin{aligned} \|x\|^{PR,2}_{L^\bullet(E)} &= \langle x \otimes \mathcal{D}_E(x)\rangle_E && \text{by definition (27)} \\ &= \|x\|^{RS}_{L^\bullet(E)} \times \|\mathcal{D}_E(x)\|^{RS}_{L^\bullet(E)} && \text{by Theorem 5.3} \\ &= \|x\|^{RS}_{L^\bullet(E)} \times \|x\|^{RS}_{L^\bullet(E)} && \text{by Lemma 6.1.} \end{aligned}$$

This completes the proof. □

6.3. Remark. Theorem 6.2 together with Proposition 4.8(1) clearly generalize the theorem of W. Müller [11]. On the other hand, proof of Theorem 6.2 can be based on the theorem of Müller [11] instead of using theorem of J.-M. Bismut and W. Zhang [2]. Namely, if K is odd-dimensional then Theorem 5.3 follows from theorem of W. Müller since the bundle $E \oplus E^*$ is unimodular and the Reidemeister metric on $L^\bullet(E) \otimes L^\bullet(E^*)$ coincides with the canonical metric $\langle \cdot \rangle_E$. Using Proposition 4.8(2) one can obtain also Theorems 5.1 and Theorem 5.2 in the odd-dimensional case.

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