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Every Semilinear Set is a Finite Union of Disjoint Linear Sets*

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ABSTRACT

We prove in this paper that every semilinear set is a finite union of disjoint linear sets, using elementary combinatorial-topological lemmas.

This paper gives a positive answer to an open problem proposed by Seymour Ginsburg in his book ([*2*], p. 195).

Let N denote the nonnegative integers and R denote the real numbers. For each integer $n \geqslant 1$ let $R^n = R \times \cdots \times R$ and $N^n = N \times \cdots \times N$ (n times). A *linear set* $L(c; p_1, ..., p_r)$ is a subset $\{x \mid x = c + \sum_{i=1}^{r} k_i p_i$ for nonnegative integers $k_i, i = 1, ..., r\}$ of N^n, where $c, p_1, ..., p_r$ are elements of N^n. We call c the *constant* and $p_1, ..., p_r$ the *periods* of the linear set. In particular we denote by $L(c)$ a linear set whose periods are all zero vectors (i.e. consisting of single element c) and regard as $r = 0$. A subset of N^n is called *semilinear* if it is a finite union of linear sets.

Let $L(c; p_1, ..., p_r) \subset N^n$ be a linear set. We say that the linear set is of *s-dimension*, if the periods $p_1, ..., p_r$ span a s-dimensional vector space in R^n, and that the linear set is *fundamental* if $s = r$. (i.e. $p_1, ..., p_r$ are linearly independent in R^n). In this paper the empty set is regarded also as a (-1)-dimensional fundamental linear set.

By Lemma A. 1. a ([*2*] p. 212), every linear set is a finite union of fundamental linear sets.

In fact, we prove the following:

THEOREM 1. *Let $A = L(c; p_1, ..., p_s)$ and $B = L(d; q_1, ..., q_t)$ be two fundamental linear sets. Then $A - B$ is a finite union of disjoint fundamental linear sets of dimension $\leqslant s$.*

THEOREM 2. *Every semilinear set is a finite union of disjoint fundamental linear sets.*

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* The author is grateful to the refree who pointed out that the result was obtained independently and perhaps at an earlier date by Eilenberg and Schutzenberger ([*1*]), but that the present method of proof differs from theirs.

Let us begin with notations relating with combinatorial topology, (see [3], [4]). Some of them are used only in the proof of Lemma 1.

A *s-simplex* ($s \geqslant 0$) S in R^n is the convex hull of $s + 1$ linearly independent points. We call the points *vertices*, and say that they *span* S. If S is any simplex, we shall use $I(S)$ to stand for the *open simplex* of interior points. We shall use $\bar{S}$ for the boundary of S. A simplex T spanned by a subset of the vertices is called a *face* of S, written $T < S$. Simplexes S, T are *joinable* if their vertices are linearly independent. If S, T are joinable we define the *join* ST to the simplex spanned by the vertices of both.

A (simplical) *complex* K in R^n is a finite collection of simplexes such that

(i) if $S \in K$, then all the faces of S are in K,

(ii) if S, $T \in K$, then $S \cap T$ is empty or a common face.

A *partition* πK of a complex K is a complex, each of whose open simplexes is contained in a single open simplex of K and which coincides with K as a point set.

The *star* and *link* of a simplex $S \in K$ are defined:

$$\mathrm{st}(S, K) = \{T; S < T\}, \qquad \mathrm{lk}(S, K) = \{T; ST \in K\}$$

Two complexes K, L in R^n are joinable provided:

(i) if $S \in K$, $T \in L$ then S, T are joinable

(ii) if S, $S' \in K$ and T, $T' \in L$, then $ST \cap S'T'$ is empty or a common face.

If K, L are joinable, we define the join $KL = K \cup L \cup [ST; S \in K, T \in L]$.

Choose a point P in the interior of a simplex $S \in K$. Let

$$\sigma K = (K - \mathrm{st}(S, K)) \cup P \cdot \bar{S} \cdot \mathrm{lk}(S, K)$$

Then σK is a partition of K, and we say σK is obtained from K by *starring* S (at P).

The operation of starring a simplex will be called a *simple subdivision*. The resultant of successive simple subdivisions will be called a *stellar subdivision*.

A *convex cell* S in R^n is a nonempty compact subset given by

linear equations $f_1 = 0, \ldots, f_r = 0$ and

linear inequalities $g_1 \geqslant 0, \ldots, g_s \geqslant 0$

A face T of S is a cell obtained by replacing some of the inequalities $g_i \geqslant 0$ by equations $g_i = 0$.

A *cell complex* K is a finite collection of cells such that

(i) if $S \in K$, then all the faces of S are in K

(ii) if S, $T \in K$, then $S \cap T$ is empty or a common face.

Let a and b be two points of R^n. Then $a * b$ is the line segment joining a and b, and $\overrightarrow{ab}$ is the half line through b starting with a. Let X be a subset of R^n such that

$X \cap a = \varnothing$. Then, by $a * X$ we denote the finite cone $\{y \mid y \in a * x$, where $x \in X\}$ and by $\mathbf{aX}$ the infinite cone $\{y \mid y \in \mathbf{ax}$, where $x \in X\}$.

Let $A = L(c; p_1, ..., p_s)$ be a fundamental linear set in N^n. By $\hat{A}$ we denote the subset $\{x \mid x = c + \sum_{i=1}^{s} x_i p_i$, where $x_i \geqslant 0, i = 1, ..., s$. are reals$\}$ of R^n.

Let $A = L(c; p_1, ..., p_s)$ and $B = L(c; q_1, ..., q_t)$ be two fundamental linear sets in N^n with the same constant c. If $s \geqslant 1$ and $t \geqslant 1$, let $S(A)$ and $S(B)$ be $(s-1)$-simplex and $(t-1)$-simplex which are intersections of $\hat{A}$ and $\hat{B}$ respectively with a suitable $(n-1)$-hyperplane.

The parts (1) and (2) of Lemma 1 are well known in combinatorial topology (see [3], [4]).

LEMMA 1. (1) *$S(A) \cap S(B)$ is subdivided into a simplicial complex K without introducing any more vertices, if it is not empty.*

(2) *Some stellar subdivision $\sigma S(A)$ gives partition πK such that πK is a subcomplex of $\sigma S(A)$.*

(3) *Let f be a vertex of K. Then there exists a point $d \neq c$ on* $\mathbf{cf}$ *such that the vector $d - c$ is a linear combination with nonnegative integer coefficients both of $p_1, ..., p_s$ and of $q_1, ..., q_t$.*

(4) *Let f be a vertex of $\sigma S(A)$*(in (2))*. Then there exists a point $d \neq c$ on* $\mathbf{cf}$ *such that $d - c$ is a linear combination of $p_1, ..., p_s$ with nonnegative integer coefficients.*

Proof. (1) If $S(A) \cap S(B) \neq \varnothing$, $S(A) \cap S(B)$ is a convex cell and also a cell complex. Order the vertices of the cell complex $S(A) \cap S(B)$. Write each cell X as a finite cone $X = x * Y$ where x is the first vertex. Subdivide the cells inductively in order of increasing dimension. The induction begins trivially with the vertices. From the inductive step, we have already defined the subdivision Y' of Y, and so define X' to be the finite cone $X' = x * Y'$. The definition is compatible with subdivision Z' of any face Z of X containing x, because since x is the first vertex of X, it is also the first vertex of Z. Therefore each cell and hence $S(A) \cap S(B)$ is subdivided into a simplicial complex of K.

(2) If a vertex x of K is in any open i-simplex $I(X)$ of $S(A)$ $(i > 0)$, we apply the simple subdivision (X, x) to $S(A)$. Repeating this process, we obtain a stellar subdivision of $S(A)$ having all the vertices of K as vertices. Let a partition of each i-simplex of K be a subcomplex of $\sigma S(A)$ for $i = 1, ..., k$. If there is a $(k+1)$-simplex of K which is not covered exactly by a subcomplex of $\sigma S(A)$, let there be precisely r such simplexes and let Y be any one of them. Then $I(Y)$ meets some open simplex $I(X)$ of $\sigma S(A)$ which is not contained in $I(Y)$. If the intersection $Y \cap X$ is not a single point, it is a convex domain, bounded by a polyhedron Π. Any vertex of Π which is on $\bar{Y}$, say y, is also a vertex of X. Otherwise y would be in some open i-simplex of X say $I(Z)$, where $i > 0$. Since a partition of $\bar{Y}$ is a subcomplex of $\sigma S(A)$, the open simplex $I(Z)$

would be contained in an open simplex of $\bar{Y}$ and would be altogether in Π. Then y would not be a vertex of Π. If all the vertices of Π were vertices of X, then Π would be the boundary of a simplex and $Y \cap X$ would be a simplex of X. This is not the case. Therefore at least one vertex of Π is not a vertex of X. Being a point of $I(Y)$, this vertex is the complete intersection of $I(Y)$ with some open simplex of X. Therefore at least one open simplex of $\sigma S(A)$ (e.g. $I(X)$ or an open simplex of $\bar{X}$) intersects $I(Y)$ in a single point without being contained in $I(Y)$. Let there be precisely m such open simplexes, and let $I(Z)$ be any one of them. If $z = Y \cap Z$, apply the simple subdivision (Z, z) to $\sigma S(A)$. Then each new open simplex (except the open simplex z) has z on its boundary, and being flat, meets $I(Y)$ in a flat l space, $l > 0$, if at all. Therefore the proof of (2) follows from induction on m, r and $h - k$, where h is the maximum dimensionality of the simplexes in K.

(3) Since f is a vertex of K, the half line $\mathbf{cf}$ is the complete intersection of $\mathbf{c\Delta}_1$ with $\mathbf{c\Delta}_2$, where Δ_1 and Δ_2 are faces of $S(A)$ and $S(B)$ respectively. Let $p'_{i_1},..., p'_{i_\mu}$ and $q'_{j_1},..., q'_{j_\nu}$ be the vertices of Δ_1 and Δ_2 respectively such that $p'_{i_\alpha} \in \mathbf{c}(\mathbf{c} + \mathbf{p}_{i_\alpha})$, $\alpha = 1,..., \mu$ and $q'_{j_\beta} \in \mathbf{c}(\mathbf{c} + \mathbf{q}_{j_\beta})$, $\beta = 1,..., \nu$. Then $\mathbf{cf}$ consists of the points

$$c + \sum_{\alpha=1}^{\mu} x_{i_\alpha} p_{i_\alpha} = c + \sum_{\beta=1}^{\nu} y_{j_\beta} q_{j_\beta},$$

where $(x_{i_1},..., x_{i_\mu}, y_{j_1},..., y_{j_\nu})$ are nonnegative solutions of the linear equation

$$c + \sum_{\alpha=1}^{\mu} x_{i_\alpha} p_{i_\alpha} = c + \sum_{\beta=1}^{\nu} y_{j_\beta} q_{j_\beta} \cdots (E)$$

Since every component of all vectors $p_{i_\alpha}, q_{j_\beta}$, $\alpha = 1,..., \mu$, $\beta = 1,..., \nu$ is nonnegative integer, there exists a solution $(x_{i_1},..., x_{i_\mu}, y_{j_1},..., y_{j_\nu})$ of (E) such that x_{i_α}, x_{j_β} $\alpha = 1,..., \mu$, $\beta = 1,..., \nu$ are nonnegative integers and are not all zero. Hence, there exist a point $d \neq c$ on $\mathbf{cf}$ such

$$d - c = \sum_{u=1}^{s} k_u p_u = \sum_{v=1}^{t} l_v q_v$$

where k_u, l_v are nonnegative integers for $u = 1,..., s$, $v = 1,..., t$.

(4) If f is a vertex of either K or $S(A)$ this proposition is clearly true. Hence it suffices to prove the result for a vertex f of $\sigma S(A)$ which is a vertex of neither K nor $S(A)$. Let $f_1,..., f_w$ be the order in which the vertices of $\sigma S(A)$ except those of $S(A)$ and K are introduced as in the proof of (2) of this lemma. We assume that $f_1,..., f_v$ $(v < w)$ satisfy the proposition of (4) and let $\sigma_1 S(A)$ be the stellar subdivision of $S(A)$ when f_v has been just introduced. Then $\mathbf{cf}_{v+1}$ is the complete intersection of $\mathbf{c\Delta}_1$ with $\mathbf{c\Delta}_2$ where Δ_1 is a simplex of $\sigma_1 S(A)$ and Δ_2 is a simplex of K. Let $g_1,..., g_\mu$ be the vertices of Δ_1 and let $h_1,..., h_\nu$ be the vertices of Δ_2. Then there are points $g'_i \neq c$ and

$h'_j \neq c$ on $\mathbf{cg}_i$ and $\mathbf{ch}_j$ respectively such that $g'_i - c$ and $h'_j - c$ are linear combinations of $p_1, \ldots, p_s$ with nonnegative integer coefficients for $i = 1, \ldots, \mu$ and $j = 1, \ldots, \nu$. Hence every component of $g'_i - c$ and $h'_j - c$ are nonnegative integers. On the other hand $\mathbf{cf}_{v+1}$ consists of the points

$$c + \sum_{i=1}^{\mu} x_i(g'_i - c) = c + \sum_{j=1}^{\nu} y_j(h'_j - c)$$

where $x_1, \ldots, x_\mu, y_1, \ldots, y_\nu$ are nonnegative. From the same argument of the proof of (3) there is a point $d_{v+1} \neq c$ on $\mathbf{cf}_{v+1}$ such that $d_{v+1} - c$ is a linear combination of $g'_i - c$ $i = 1, \ldots, \mu$ with nonnegative integer coefficients. Hence $d_{v+1} - c$ is also a linear combination of $p_1, \ldots, p_s$ with nonnegative integer coefficients. Therefore (4) follows from induction on $w - v$.

LEMMA 2. *Let Δ be an $(e-1)$-simplex with vertices $f_1, \ldots, f_e$ lying in $S(A) \cap S(B)$ such that there is a point $d_i \neq c$ on each half line $\mathbf{cf}_i$ such that the vector $d_i - c$ is a linear combination with nonnegative integer coefficients both of $p_1, \ldots, p_s$ and of $q_1, \ldots, q_t$. Then each of the following sets* (1), (2), (3), (4), (5) *and* (6) *is either a finite union of disjoint fundamental linear sets of e-dimension or the empty set.*

(1) $A \cap \mathbf{c\Delta}$	(2) $A \cap I(\mathbf{c\Delta})$
(3) $(A \cap B) \cap \mathbf{c\Delta}$	(4) $(A \cap B) \cap I(\mathbf{c\Delta})$
(5) $(A - B) \cap \mathbf{c\Delta}$	(6) $(A - B) \cap I(\mathbf{c\Delta})$

where $I(\mathbf{c\Delta})$ denotes $\mathbf{cI}(\Delta) - c$.

Proof. (1) By the assumption of the lemma each d_i is an element of $A \cap B$. For any element $x \in A \cap \mathbf{c\Delta}$, $x - c$ is written uniquely as $x - c = \sum_{i=1}^{e} x_i(d_i - c)$ with nonnegative rational coefficients x_i, $i = 1, \ldots, e$, since $d_1 - c, \ldots, d_e - c$ are linearly independent and $x - c$, $d_1 - c$ are all linear combinations of $p_1, \ldots, p_s$ with nonnegative integer coefficients. Let $\{c_1 = c, \ldots, c_\lambda\}$ be the set of all $x \in A \cap \mathbf{c\Delta}$ satisfying $0 \leqslant x_i < 1$ for $i = 1, \ldots, e$.
Now

$$x = c + \sum_{i=1}^{e} (x_i - [x])(d_i - c) + \sum_{i=1}^{e} [x_i](d_i - c)$$

where [ ] is Gaussian bracket. Moreover we may write

$$x - c = \sum_{i=1}^{e} x_i(d_i - c) = \sum_{u=1}^{s} k_u p_u$$

$$\sum_{i=1}^{e} [x_i](d_i - c) = \sum_{u=1}^{s} l_u p_u$$

where k_u, l_u are nonnegative integers. Since the point $c + \sum_{i=1}^{e} (x_i - [x_i])(d_i - c) = c + \sum_{u=1}^{s} (k_u - l_u) p_u$ lies in $\hat{A}$, $k_u - l_u$ is a nonnegative integer for $u = 1,\ldots, s$. Hence $c + \sum_{i=1}^{e} (x_i - [x_i])(d_i - c)$ is contained in A and equals to one of c_j, $j = 1,\ldots, \lambda$, say c_j. Therefore $x \in L_j = L(c_j ; d_1 - c,\ldots, d_e - c)$. Conversely each L_j, $j = 1,\ldots, \lambda$ is a set contained in $A \cap \mathbf{c}\Delta$. Since L_j, $j = 1,\ldots, \lambda$ have the same periods $d_1 - c,\ldots, d_e - c$ and different constants lying in the set

$$\left\{ y \mid y = c + \sum_{i=1}^{e} y_i(d_i - c), \quad \text{where} \quad \leqslant 0\, y_i < 1 \right\},$$

L_j, $j = 1,\ldots, \lambda$ are mutually disjoint. Therefore $A \cap \mathbf{c}\Delta$ is a finite union $\bigcup_{j=1}^{\lambda} L_j$ of disjoint fundamental linear sets with the same periods, $d_1 - c,\ldots, d_e - c$.

(2) Let $c_j = c + \sum_{i=1}^{e} y_{ji}(d_i - c)$ $(0 \leqslant y_{ji} < 1)$ be the constant of L_j, and let $i_1,\ldots, i_\mu$ be all i such that $y_{ji} = 0$. Then $c_j^* = c_j + (d_{i_1} - c) + \cdots + (d_{i_\mu} - c)$ is a point of $L_j \cap I(\mathbf{c}\Delta)$, and $L_j^* = L(c_j^* ; d_1 - c,\ldots, d_e - c)$ is a subset of $L_j \cap I(\mathbf{c}\Delta)$. These L_j^*, $j = 1,\ldots, \lambda$ are mutually disjoint. Let x be an element of $L_j \cap I(\mathbf{c}\Delta)$. Then

$$x = c_j + \sum_{i=1}^{e} k_i(d_i - c) = c + \sum_{i=1}^{e} (y_{ji} + k_i)(d_i - c)$$

where $y_{ji} + k_i > 0$ for $i = 1,\ldots, e$. Then $k_{i_\alpha} \geqslant 1$, since $y_{ji_\alpha} = 0$ for $\alpha = 1,\ldots, \mu$. Hence

$$x = c_j + (d_{i_1} - c) + \cdots + (d_{i_\mu} - c) + \sum_{\alpha=1}^{\mu} (k_{i_\alpha} - 1)(d_{i_\alpha} - c) + \sum_{h \neq i_1,\ldots,i_\mu} k_h(d_h - c)$$

$$= c_j^* + \sum_{\alpha=1}^{\mu} (k_{i_\alpha} - 1)(d_{i_\alpha} - c) + \sum_{h\neq i_1,\ldots,i_\mu} k_h(d_h - c) \in L_j^*$$

Thus $L_j \cap I(\mathbf{c}\Delta) = L_j^*$. Therefore $A \cap I(\mathbf{c}\Delta) = \bigcup_{j=1}^{\lambda} L_j^*$, completing the proof for (2).

(3) Any element $x \in (A \cap B) \cap \mathbf{c}\Delta$ can be written uniquely as

$$x = c + \sum_{i=1}^{e} x_i(d_i - c)$$

where x_i, $i = 1,\ldots, e$ are nonnegative rationals. Let $\{z_1 = c,\ldots, z_\nu\}$ be the set of all $x \in (A \cap B) \cap \mathbf{c}\Delta$ satisfying $0 \leqslant x_i < 1$ for $i = 1,\ldots, e$. Then $\{z_1,\ldots, z_\nu\}$ is a subset of $\{c_1,\ldots, c_\lambda\}$. Hence we may assume that $z_1 = c_1,\ldots, z_\nu = c_\nu (\nu \leqslant \lambda)$. Since $x = c + \sum_{i=1}^{e} (x_i - [x_i])(d_i - c) + \sum_{i=1}^{e}[x_i](d_i - c)$ and $c + \sum_{i=1}^{e} (x_i - [x_i])(d_i - c)$ is one of z_h, $h = 1,\ldots, \nu$, say $z_\mu = c_\mu$, $x \in L_\mu$. Therefore $(A \cap B) \cap \mathbf{c}\Delta = \bigcup_{h=1}^{\nu} L_h$, completing the proof for (3).

(4) Since

$$(A \cap B) \cap I(\mathbf{c}\Delta) = \{(A \cap B) \cap \mathbf{c}\Delta\} \cap \{A \cap I(\mathbf{c}\Delta)\}$$
$$= \Big(\bigcup_{h=1}^{\nu} L_h\Big) \cap \Big(\bigcup_{j=1}^{\lambda} L_j^*\Big) = \bigcup_{h=1}^{\nu} L_h^*$$

the proof for (4) is complete.

(5) Since

$$(A - B) \cap \mathbf{c}\Delta = A \cap \mathbf{c}\Delta - (A \cap B) \cap \mathbf{c}\Delta$$
$$= \bigcup_{j=1}^{\lambda} L_j - \bigcup_{h=1}^{\nu} L_h = \bigcup_{j=\nu+1}^{\lambda} L_j$$

the proof for (5) is complete. Note that $(A - B) \cap \mathbf{c}\Delta = \varnothing$, if $\nu = \lambda$.

(6) Since

$$(A - B) \cap I(\mathbf{c}\Delta) = A \cap I(\mathbf{c}\Delta) - (A \cap B) \cap I(\mathbf{c}\Delta)$$
$$= \bigcup_{j=1}^{\lambda} L_j^* - \bigcup_{h=1}^{\nu} L_h^* = \bigcup_{j=\nu+1}^{\lambda} L_j^*$$

the proof for (6) is complete. Note that $(A - B) \cap I(\mathbf{c}\Delta) = \varnothing$, if $\nu = \lambda$.

LEMMA 3. *Let $A = L(c; p_1, ..., p_s)$ and $B = L(c; q_1, ..., q_t)$ be two fundamental linear sets with the same constant c. Then $A \cap B$ is a finite union of disjoint fundamental linear sets of dimension* $\leqslant \min(s, t)$.

Proof. If either A or B is 0-dimensional, $A \cap B$ is either one element or empty. Thus the lemma is trivial. Thus we may assume that $s \geqslant 1$ and $t \geqslant 1$. If $S(A) \cap S(B) = \varnothing$, $A \cap B = c$ and the lemma is trivial. Therefore we may assume that $S(A) \cap S(B)$ is a convex cell and by Lemmas 1(1) it can be subdivided into a simplicial complex K without introducing any more vertices. Let Δ be any simplex of K and $f_1, ..., f_e$ be the vertices of Δ. By Lemma 1 (3) there exists a point $d_i \neq c$ on $\mathbf{c}\mathbf{f}_i$ such that $d_i - c$ is a linear combination both of $p_1, ..., p_s$ and of $q_1, ..., q_t$ with non-negative integer coefficients. Then by Lemma 2(4), $(A \cap B) \cap I(\mathbf{c}\Delta)$ is a finite union of disjoint fundamental linear sets of dimension $\leqslant e$. Since each element of $A \cap B$ except c lies in $I(\mathbf{c}\Delta)$ for a single simplex Δ of K, $A \cap B$ is a union of the fundamental linear sets lying in $\bigcup_{\Delta \in K}(\mathbf{c}\Delta)$ and $L(c)$. These fundamental linear sets are mutually disjoint and of dimension $\leqslant \min(s, t)$. Thus the proof of Lemma 3 is complete.

Lemma 4. *Let $A = L(c; p_1, ..., p_s)$ and $B = L(c; q_1, ..., q_t)$ be two fundamental linear sets of dimension $\geqslant 1$ with the same constant c. Then $(A - B) \cap (\hat{A} \cap \hat{B})$ is a finite union of disjoint fundamental linear sets of dimension* $\leqslant \min(s, t)$.

Proof. By the same argument as lemma 3 we can apply Lemma 1 (3) and Lemma 2 (6). Therefore $(A - B) \cap I(\mathbf{c}\Delta)$ is a finite union of disjoint fundamental linear sets for any simplex Δ of K. Since each element of $(A - B) \cap (\hat{A} \cap \hat{B})$ lies in $I(\mathbf{c}\Delta)$ for a single simplex Δ of K, $(A - B) \cap (\hat{A} \cap \hat{B})$ is a union of the fundamental linear sets lying in $\bigcup_{\Delta \in K} I(\mathbf{c}\Delta)$. These fundamental linear sets are mutually disjoint and are of dimension $\leqslant \min(s, t)$. Thus the proof of Lemma 4 is complete.

Lemma 5. *Let $A = L(c; p_1, ..., p_s)$ and $B = L(c; q_1, ..., q_t)$ be two fundamental linear sets of dimension $\geqslant 1$ with the same constant c. Then $A \cap (\hat{A} - \hat{B})$ is a finite union of disjoint fudamental linear sets of dimension $\leqslant s$.*

Proof. By Lemma 1 (2), (4) and the special case of Lemma 2 (2) (where $B = A$), $A \cap I(\mathbf{c}\Delta)$ is a finite union of disjoint fundamental linear sets for each open simplex $I(\Delta)$ of $\sigma S(A) - \pi K$. Therefore $A \cap (\hat{A} - \hat{B})$ is a finite union of the fundamental linear sets lying in $I(\mathbf{c}\Delta)$ over all the open simplex $I(\Delta)$ of $\sigma S(A) - \pi K$. These fundamental linear sets are mutually disjoint and of dimension $\leqslant s$, because $S(A)$ is $(s - 1)$-simplex. Thus the proof of Lemma 5 is complete.

Lemma 6. *Let $A = L(c; p_1, ..., p_s)$ be a fundamental linear set. Put*

$$A^* = L\left(c + \sum_{i=1}^{s} k_i p_i \,; p_1, ..., p_s\right),$$

where k_i, $i = 1, ..., s$ are nonnegative integers. Then A is a finite union $A^* \cup \bigcup_{f=1}^{\mu} A_f$ of disjoint fundamental linear sets, where A_f, $f = 1, ..., \mu$ are of dimension $\leqslant s - 1$

Proof. Consider the following family of $(s - r)$-dimensional fundamental linear subsets of A,

$$L\{(j_1, ..., j_r), (l_{j_1}, ..., l_{j_r})\} = L(c + \sum_{i \neq j_1, ..., j_r} k_i p_i + \sum_{\alpha=1}^{r} l_{j_\alpha} p_{j_\alpha} ;$$
$$\text{all the } p_i \text{ except for } p_{j_\alpha}, \alpha = 1, ..., r)$$

for $r = 0, ..., s$, all the r combinations $(j_1, ..., j_r)$ of $(1, 2, ..., s)$ and integers $0 \leqslant l_{j_\alpha} \leqslant k_{j_\alpha} - 1$, $\alpha = 1, ..., r$. When we put $r = 0$, the linear set above is A^*. For each element $x = c + \sum_{i=1}^{s} x_i p_i$ of A, x belongs to $L\{(j_1, ..., j_r), (x_{j_1}, ..., x_{j_r})\}$ if and only if j_α, $\alpha = 1, ..., r$ are all the index i such that $0 \leqslant x_i \leqslant k_i - 1$. In particular, those linear sets are mutually disjoint, and A is the union of the family. Let denote by A_f, $f = 1, ..., \mu$ the fundamental linear sets considered except A^*. Then $A = A^* \cup \bigcup_{f=1}^{\mu} A_f$, where A_f, $f = 1, ..., \mu$ are fundamental linear sets of dimension $\leqslant s - 1$.

THEOREM 1. *Let* $A = L(c; p_1, \ldots, p_s)$ *and* $B = L(d; q_1, \ldots, q_t)$ *be two fundamental linear sets. Then* $A - B$ *is a finite union of disjoint fundamental linear sets of dimension* $\leqslant s$.

Proof. By Lemma 4 and Lemma 5 it is assumed that A and B have different constants, i.e. $c \neq d$. We shall prove the theorem by induction on the pair (s, t). If $s = 0$, then A consists of one element c. Hence $A - B$ is either empty or $L(c)$. Thus the theorem is true for any t, if $s = 0$. If $t = 0$, and $s \neq 0$, then $A = L(c; p_1, \ldots, p_s)$ and $B = L(d)$. If $A \cap B = \varnothing$ then $A - B = A$ and the theorem is true. If $A \cap B = B = L(d)$, we may write $d = c + \sum_{i=1}^{s} h_i p_i$ where, h_i $i = 1, \ldots, s$ are nonnegative integers. Put $h_i + 1 = k_i$, then by Lemma 6, $A = A^* \cup \bigcup_{f=1}^{\mu} A_f$, where A_f, $f = 1, \ldots, \mu$ are fundamental linear sets of dimension $\leqslant s - 1$. Moreover $B = L(d)$ is one of A_f of 0-dimension say A_α. Hence $A - B = A^* \cup \bigcup_{f \neq \alpha} A_f$, completing the proof for $t = 0$, and $s \neq 0$. Now we assume that the theorem is true for the pair (s', t') such that either $s' < s, t' \leqslant t$ or $s' \leqslant s$, $t' < t$. Let us prove the theorem assuming $s \geqslant 1$, $t \geqslant 1$. If $A \cap B = \varnothing$, then $A - B = A$ and we have nothing to prove. Therefore we assume that $A \cap B$ contains an element, say $z = c + \sum_{i=1}^{s} k_i p_i = d + \sum_{j=1}^{t} l_j q_j$. Consider the fundamental linear sets $A^* = L(z; p_1, \ldots, p_s)$, $B^* = L(z; q_1, \ldots, q_t)$. Then, by lemma 6, A is a finite union $A^* \cup \bigcup_{f=1}^{\mu} A_f$ of disjoint fundamental linear sets, where A_f, $f = 1, \ldots, \mu$ are of dimension $\leqslant s - 1$. The same is true for B, and $B = B^* \cup \bigcup_{g=1}^{\nu} B_g$ where B_g, $g = 1, \ldots, \nu$ are of dimension $\leqslant t - 1$. Therefore

$$\begin{aligned} A \cap B &= \Big(A^* \cup \bigcup_{f=1}^{\mu} A_f\Big) \cap B \\ &= (A^* \cap B) \cup \Big(\bigcup_{f=1}^{\mu} A_f \cap B\Big) \\ &= (A^* \cap B^*) \cup \Big(A^* \cap \bigcup_{g=1}^{\nu} B_g\Big) \cup \Big(\bigcup_{f=1}^{\mu} A_f \cap B\Big) \end{aligned}$$

where $A^* \cap B^*$, $A^* \cap \bigcup_{f=1}^{\nu} B_g$, $\bigcup_{g=1}^{\mu} A_f \cap B$ are mutually disjoint. Therefore

$$\begin{aligned} A - B &= A^* \cup \bigcup_{f=1}^{\mu} A_f - (A \cap B) \\ &= \Big\{A^* - (A^* \cap B^*) \cup \Big(A^* \cap \bigcup_{g=1}^{\nu} B_g\Big)\Big\} \cup \Big(\bigcup_{f=1} A_f - \bigcup_{f=1}^{\mu} A_f \cap B\Big) \\ &= \Big\{A^* - (A^* \cap B^*) \cup \Big(A^* \cap \bigcup_{g=1}^{\nu} B_g\Big)\Big\} \cup \bigcup_{f=1}^{\mu} (A_f - B) \end{aligned}$$

where $A^* - (A^* \cap B^*) \cup (A^* \cap \bigcup_{g=1}^{\nu} B_g)$ and $A_f - B$, $f = 1,..., \mu$ are mutually disjoint. At first, for each f, $A_f - B$ is a finite union of disjoint fundamental linear sets of dimension $\leqslant s - 1$, by the assumption of induction, since A_f is of dimension $\leqslant s - 1$ and B is of t-dimension. It suffices to prove that

$$A - (A^* \cap B^*) \cup \left(A^* \cap \bigcup_{g=1}^{\nu} B_g\right)$$

is a finite union of disjoint fundamental linear sets of dimension $\leqslant s$. Since

$$A^* = \{A^* \cap (\hat{A}^* \cap \hat{B}^*)\} \cup \{A^* \cap (\hat{A}^* - \hat{B}^*)\},$$

$$\begin{aligned} &A^* - (A^* \cap B^*) \cup \left(A^* \cap \bigcup_{g=1}^{\nu} B_g\right) \\ &= \{A^* \cap (\hat{A}^* \cap \hat{B}^*) - A^* \cap B^*\} \cup \left\{A^* \cap (\hat{A}^* - \hat{B}^*) - A^* \cap \bigcup_{g=1}^{\nu} B_g\right\} \\ &= \{(A^* - B^*) \cap (\hat{A}^* \cap \hat{B}^*)\} \cup \left\{A^* \cap (\hat{A}^* - \hat{B}^*) - \bigcup_{g=1}^{\nu} B_g\right\} \end{aligned}$$

where $(A^* - B^*) \cap (\hat{A}^* \cap \hat{B}^*)$ and $A^* \cap (\hat{A}^* - \hat{B}^*) - \bigcup_{g=1}^{\nu} B_g$

are disjoint. By lemma 4, $(A^* - B^*) \cap (\hat{A}^* \cap \hat{B}^*)$ is a finite union of disjoint fundamental linear sets of dimension $\leqslant \min(s, t)$. On the other hand, by lemma 5, $A^* \cap (\hat{A}^* - \hat{B}^*)$ is a finite union $\bigcup_{i=1}^{\rho} G_i$ of disjoint fundamental linear sets G_i of dimension $\leqslant s$. Therefore

$$\begin{aligned} &A^* \cap (\hat{A}^* - \hat{B}^*) - \bigcup_{g=1}^{\nu} B_g \\ &= \bigcup_{i=1}^{\rho} G_i - \bigcup_{g=1}^{\nu} B_g = \bigcup_{i=1}^{\rho} \left(G_i - \bigcup_{g=1}^{\nu} B_g\right) \end{aligned}$$

Now it remains only to prove that $G_i - \bigcup_{g=1}^{\nu} B_g$ is a finite union of disjoint fundamental linear sets of dimension $\leqslant s$ for each i. We shall prove this by another induction on ν. If $\nu = 1$, $G_i - B_1$ is a finite union $\bigcup_{j=1}^{\tau} H_j$ of disjoint fundamental linear sets of dimension $\leqslant s$ by the assumption of the induction of this theorem, since B_g is of dimension $\leqslant t - 1$. Then

$$\begin{aligned} G_i - \bigcup_{g=1}^{\nu} B_g &= (G_i - B_1) - \bigcup_{g=2}^{\nu} B_g \\ &= \bigcup_{j=1}^{\tau} H_j - \bigcup_{g=2}^{\nu} B_g = \bigcup_{j=1}^{\tau} \left(H_j - \bigcup_{g=2}^{\nu} B_g\right) \end{aligned}$$

By the assumption of the induction on ν, $H_j - \bigcup_{g=2}^{\nu} B_g$ is a finite union of disjoint fundamental linear sets of dimension $\leqslant s$ for each j. Hence $G_i - \bigcup_{g=1}^{\nu} B_g$ is also a finite union of disjoint fundamental linear sets of dimension $\leqslant s$ for each i, completing the proof of this theorem.

THEOREM 2. *Every semilinear set is a finite union of disjoint fundamental linear sets.*

Proof. Let S be a given semilinear set. By Lemma A. 1 ([2], p 212), S is a finite union $\bigcup_{i=1}^{\alpha} A_i$ of fundamental linear sets A_i, $i = 1,..., \alpha$. Let us prove the theorem by induction on α. If $\alpha = 1$, the theorem is trivial. Suppose that the theorem holds for α', $1 \leqslant \alpha' < \alpha$. Then $S = A_1 \cup \bigcup_{i=2}^{\alpha} A_i$ and $\bigcup_{i=2}^{\alpha} A_i$ is a finite union $\bigcup_{i=1}^{\beta} B_j$ of disjoint fundamental linear sets by the assumption of induction. Write

$$S = A_1 \cup \bigcup_{j=1}^{\beta} B_j = A_1 \cup \bigcup_{j=1}^{\beta} (B_j - A_1).$$

Then A_1 and all $B_j - A_1$, $j = 1,..., \beta$ are mutually disjoint. By Theorem 1, each $B_j - A_1$ is a finite union $\bigcup_{h=1}^{\delta} D_h$ of disjoint fundamental linear sets. Therefore S is also a finite union of disjoint fundamental linear sets, completing the proof

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