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Semiorthogonal functions and orthogonal polynomials on the unit circle

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Abstract

We establish a bijection between Hermitian functionals on the linear space of Laurent polynomials and functionals on $\mathscr{P} \times \mathscr{P}$ satisfying some orthogonality conditions ($\mathscr{P}$ denotes the linear space of polynomials with real coefficients). This allows us to study some topics about sequences (Φ_n) of orthogonal polynomials on the unit circle from a new point of view. Whenever the polynomials Φ_n have real coefficients, we recover a well known result by Szegő. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we are concerned with an algebraic method introduced in [1] in order to study polynomial modifications of Hermitian functionals $\mathscr{L}$ defined on the linear space of Laurent polynomials and, as consequence, properties of the corresponding orthogonal polynomials (OP) on the unit circle $\mathbb{T}$. The basic idea is the well known Szegő's result about the connection between OP on $\mathbb{T}$ and OP on the interval $[-1,1]$, (see [6, Section 11.5] and also [2, Section V.1] or [3, 9.1]); in this situation, since the orthogonality measure on $\mathbb{T}$ is symmetric, the associated OP have real coefficients. We will analyze a relation such as Szegő's one, valid for OP on $\mathbb{T}$ with complex coefficients.

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Now, we proceed with some notations. In what follows, we denote $\mathscr{P}=\mathbb{R}[x]$ the linear space of polynomials with real coefficients, $\mathscr{P}_n$ is the subspace of polynomials whose degree is less than or equal to n and $\mathscr{P}_n^{\#}$ the subset of $\mathscr{P}_n$ of polynomials whose degree is exactly n. Besides, Π denotes the vector space $\mathbb{C}[z]$ of complex polynomials, and Π_n its subspace of all polynomials of degree less than or equal to n. We let L be the class of Laurent polynomials, that is

$$L=\left\{\sum_{k=-m}^{n}\alpha_k z^k;\alpha_k\in\mathbb{C}|m,n\in\mathbb{N}\cup\{0\}\right\}.$$

The paper is organized as follows: In Section 2, we give a bijective correspondence between Hermitian functionals $\mathscr{L}$ on L and a class of functionals $\mathscr{M}$ on $\mathscr{P}\times\mathscr{P}$; also, we construct a basis of $\mathscr{P}\times\mathscr{P}$ (which we call semi-orthogonal basis) satisfying some orthogonality conditions with respect to $\mathscr{M}$. Section 3 is devoted to prove that the elements of the semi-orthogonal basis satisfy recurrence formulas connected with recurrence formulas for orthogonal matrix polynomials. In Section 4, the relation between OP on $\mathbb{T}$ and the semi-orthogonal basis is analyzed. In the last Section, we give an application of the above method to a modification of the functional $\mathscr{L}$.

Next, we show some technical results that we shall use later.

For $z\in\mathbb{C}\backslash\{0\}$, let us write $x=(z+z^{-1})/2$ and $y=(z-z^{-1})/2i$; therefore $z=x+iy$, $z^{-1}=x-iy$, and $x^2+y^2=1$. Notice that if $z=e^{i\theta}$, $(\theta\in\mathbb{R})$, then $x=\cos\theta$ and $y=\sin\theta$. In any case, we can express z in terms of x with, as usually, $\sqrt{x^2-1}>0$ when $x>1$.

Let us consider a polynomial Φ of degree $2n-1$, $n\geqslant 1$, $\Phi(z)=\alpha_{2n-1}z^{2n-1}+\cdots+\alpha_0$, where $\alpha_k\in\mathbb{C}$. We define $f,g:\mathbb{C}\backslash\{0\}\to\mathbb{C}$, by means of

$$\begin{aligned} f(x) &= 2^{-n}z^{-n}[z\Phi(z)+\Phi^*(z)],\\ g(x) &= -i2^{-n}z^{-n}[z\Phi(z)-\Phi^*(z)], \end{aligned}\tag{1}$$

where $\Phi^*(z)=z^{2n-1}\overline{\Phi}(z^{-1})$ is the so-called reversed polynomial.

Lemma 1.1. *Let Φ be a monic polynomial of degree $2n-1$ $(n\geqslant 1)$ and f, g defined in* (1). *Then, there exist unique polynomials $R_1\in\mathscr{P}_n^{\#}$, $R_2\in\mathscr{P}_{n-2}$, $S_1\in\mathscr{P}_{n-1}$ and $S_2\in\mathscr{P}_{n-1}^{\#}$, such that*

$$f(x)=R_1(x)+\sqrt{1-x^2}R_2(x),$$

$$g(x)=S_1(x)+\sqrt{1-x^2}S_2(x).$$

Conversely, given $R_1\in\mathscr{P}_n^{\#}$, $R_2\in\mathscr{P}_{n-2}$, $S_1\in\mathscr{P}_{n-1}$ and $S_2\in\mathscr{P}_{n-1}^{\#}$ with R_1 and S_2 monic polynomials, then there is a unique monic polynomial $\Phi\in\Pi$ of degree $2n-1$, such that

$$R_1(x)+\sqrt{1-x^2}R_2(x)=2^{-n}z^{-n}[z\Phi(z)+\Phi^*(z)],$$

$$S_1(x)+\sqrt{1-x^2}S_2(x)=-i2^{-n}z^{-n}[z\Phi(z)-\Phi^*(z)].$$

Proof. From (1) we have

$$f(x)=2^{-n}z^{-n}\sum_{k=0}^{2n-1}(\alpha_k z^{k+1}+\overline{\alpha}_k z^{2n-k-1})$$

$$=T_n(x)+\sum_{j=0}^{n-1}2^{j-n}\,\mathrm{Re}(\alpha_{n+j-1}+\alpha_{n-j-1})T_j(x)-\sqrt{1-x^2}$$

$$\times\sum_{j=1}^{n-1}2^{-j}\,\mathrm{Im}(\alpha_{2n-j-1}-\alpha_{j-1})U_{n-j-1}(x),$$

where T_j and U_j are respectively the jth Tchebychev polynomials of the first and second kind (see [6, p.3]). So, keeping in mind that $\mathrm{Im}\,\alpha_{2n-1}=0$, we can put

$$f(x)=\sum_{k=0}^{n}a_kT_k(x)+\sqrt{1-x^2}\sum_{k=0}^{n-2}b_kU_k(x).$$

with $a_n=\mathrm{Re}\,\alpha_{2n-1}=1$.

Then, the desired expression for f is obtained if we take $R_1(x)=\sum_{k=0}^{n}a_kT_k(x)$ and $R_2(x)=\sum_{k=0}^{n-2}b_kU_k(x)$. The uniqueness is obvious.

The proof for g is similar.

To deduce the converse, it suffices to expand the polynomials R_1 and S_1 (respectively, R_2 and S_2) in terms of Tchebychev polynomials of the first (respectively, second) kind and the proof follows straightforward. □

Note that if Φ has real coefficients, then $R_2(x)\equiv 0$ and $S_1(x)\equiv 0$. Moreover, if Φ is in addition a polynomial orthogonal on $\mathbb{T}$ with respect to a weight function w, then by Szegő's theory, R_1 and S_2 are orthogonal on $[-1,1]$ with respect to the weight functions $w(x)/\sqrt{1-x^2}$ and $w(x)\sqrt{1-x^2}$, respectively.

2. Functionals on $\mathscr{P}\times\mathscr{P}$ induced by Hermitian functionals on $\mathscr{P}$

We denote $\Sigma=\mathscr{P}\times\mathscr{P}$, which with the product:

$$(P_1,Q_1)\cdot(P_2,Q_2)=(P_1P_2+(1-x^2)Q_1Q_2,P_1Q_2+P_2Q_1)$$

is a conmutative algebra with identity element. For each $n\geqslant 0$, we write $\Sigma_n=\mathscr{P}_n\times\mathscr{P}_{n-1}$ with the convention $\mathscr{P}_{-1}=\{0\}$; Σ_n is a subspace of Σ such that $\dim\Sigma_n=2n+1$. Notice that, the mapping $(P,Q)\to P+\sqrt{1-x^2}Q$ establish a bijective correspondence between Σ and $\mathscr{P}+\sqrt{1-x^2}\mathscr{P}$ which preserves the product; so, in the sequel some times we will use the expression $P+\sqrt{1-x^2}Q$ to denote elements of Σ.

For every Hermitian functional on L, we can define a linear functional on Σ in the following way:

Definition 2.1. Let $\mathscr{L}$ be a Hermitian functional on L, then $\mathscr{M}_{\mathscr{L}}:\Sigma\to\mathbb{R}$ given by

$$\mathscr{M}_{\mathscr{L}}[(P,Q)]=\mathscr{L}\left[P\left(\frac{z+z^{-1}}{2}\right)+\left(\frac{z-z^{-1}}{2i}\right)Q\left(\frac{z+z^{-1}}{2}\right)\right]$$

is a linear functional on Σ. We call it, the functional induced by $\mathscr{L}$.

Observe that, if $\mathscr{L}$ is definite positive and μ is the corresponding orthogonality measure on $\mathbb{T}$ (see [3, 4]), we can write:

$$\mathscr{M}_{\mathscr{L}}[(P,Q)]=\int_{-1}^{1}P(x)\,\mathrm{d}(v_1+v_2)(x)+\int_{-1}^{1}\sqrt{1-x^2}Q(x)\,\mathrm{d}(v_1-v_2)(x),$$

where v_1 and v_2 are the measures on $[-1,1]$ given by

$$\begin{cases}\mathrm{d}v_1(x)=-\mathrm{d}\mu(\theta), & 0\leqslant\theta<\pi,\\ \mathrm{d}v_2(x)=\mathrm{d}\mu(\theta), & \pi\leqslant\theta<2\pi,\end{cases}\qquad x=\cos\theta.$$

Using the induction method, it is easy to deduce:

Proposition 2.2. (i) *Any set B_n inductively defined by*

$$B_0=\{1\},\quad B_n=B_{n-1}\cup\{h_n^{(1)},h_n^{(2)}\},\quad n\geqslant 1, \tag{2}$$

where $h_n^{(1)}\in\mathscr{P}_n^{\#}\times\mathscr{P}_{n-2}$, $h_n^{(2)}\in\mathscr{P}_{n-1}\times\mathscr{P}_{n-1}^{\#}$ is a basis of Σ_n.

(ii) *Any set B defined by means of*

$$B=\{1\}\cup\left(\bigcup_{n\geqslant 1}\{h_n^{(1)},h_n^{(2)}\}\right),$$

where $h_n^{(1)}$, $h_n^{(2)}$ are as in (i), *is a basis of Σ.*

It is well known that $\mathscr{L}$ defines a bilinear Hermitian functional: $\langle\cdot,\cdot\rangle_{\mathscr{L}}:\Pi\times\Pi\to\mathbb{C}$ by means of

$$\langle\Phi,\Psi\rangle_{\mathscr{L}}=\mathscr{L}(\Phi(z)\overline{\Psi}(z^{-1})).$$

This bilinear functional is called quasi-definite (positive definite) when the restriction of $\langle\cdot,\cdot\rangle_{\mathscr{L}}$ to $\Pi_n\times\Pi_n$ is quasi-definite (positive definite), for each n. This definition is equivalent to the fact that all principal minors of the moment matrix $(\langle z^k,z^j\rangle_{\mathscr{L}})_{k,j=0}^{\infty}$ are different from zero (>0). So the Gram-Schmidt procedure guarantees the existence of a sequence of monic orthogonal polynomials (SMOP) $\{\Phi_n\}_{n\in\mathbb{N}}$ such that $\langle\Phi_n,\Phi_n\rangle_{\mathscr{L}}\neq 0$ (>0).

Our main purpose is to carry over these results to the set of bilinear functionals on Σ, via an orthogonalization procedure.

Definition 2.3. Let $\mathscr{M}\in\Sigma^*$, where Σ^* denotes the algebraic dual space of Σ. We define the symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathscr{M}}:\Sigma\times\Sigma\to\mathbb{R}$ by

$$\langle F,G\rangle_{\mathscr{M}}=\mathscr{M}(F\cdot G),\quad F,G\in\Sigma.$$

We say that $\mathscr{M}$ is quasi-definite (positive definite) if the restriction of $\langle , \rangle_{\mathscr{M}}$ to $\Sigma_n \times \Sigma_n$ is quasi-definite (positive definite), for each $n \in \mathbb{N}$.

Notice that the definition of the positivity of $\mathscr{M}$ is similar to the one given for Hermitian functionals. However, the quasi-definiteness of $\mathscr{M}$ does not imply that $\langle F,F\rangle_{\mathscr{M}} \neq 0$ for all $F \in \mathscr{M}$, $F \neq 0$.

Theorem 2.4 (Orthogonalization procedure). *Let $\mathscr{M} \in \Sigma^*$ be a quasi-definite (positive definite) functional. Then, for each $n \in \mathbb{N}$, there exists a unique basis B_n of Σ_n, given by*

$$B_0 = 1, \qquad B_n = B_{n-1} \cup \{f_n, g_n\},$$

where $f_n = (R_n^{(1)}, R_n^{(2)}) \in \mathscr{P}_n^{\#} \times \mathscr{P}_{n-2}$ and $g_n = (S_n^{(1)}, S_n^{(2)}) \in \mathscr{P}_{n-1} \times \mathscr{P}_{n-1}^{\#}$, with $R_n^{(1)}$ and $S_n^{(2)}$ monic polynomials, such that

$$\begin{aligned} &\mathscr{M}(f_n \cdot x^k) = \mathscr{M}(g_n \cdot x^k) = 0, \quad 0 \leqslant k \leqslant n-1, \\ &\mathscr{M}(f_n \cdot \sqrt{1-x^2}x^k) = \mathscr{M}(g_n \cdot \sqrt{1-x^2}x^k) = 0, \quad 0 \leqslant k \leqslant n-2. \end{aligned} \tag{3}$$

Besides, the matrix

$$C_n = \begin{pmatrix} \mathscr{M}(f_n^2) & \mathscr{M}(f_n g_n) \\ \mathscr{M}(f_n g_n) & \mathscr{M}(g_n^2) \end{pmatrix}$$

is quasi-definite (positive definite.).

Proof. Let $\mathscr{M} \in \Sigma^*$ be quasi-definite (positive definite). For $n = 1$, we consider, according to (2), the following basis $\{1, f_1, g_1\}$ of Σ_1 given by

$$f_1(x) = (x + c_1, 0), \qquad g_1(x) = (\tilde{c}_1, 1).$$

Let $\mathscr{M}(f_1) = \mathscr{M}(g_1) = 0$. Then,

$$\mathscr{M}(f_1) = \mathscr{M}(x) + c_1 \mathscr{M}(1) = 0,$$

$$\mathscr{M}(g_1) = \tilde{c}_1 \mathscr{M}(1) + \mathscr{M}\left(\sqrt{1-x^2}\right) = 0$$

but $\mathscr{M}(1) \neq 0$, because of the quasi-definiteness of $\langle \cdot, \cdot \rangle_{\mathscr{M}}$ on Σ_0. Thus we have uniqueness for c_1, $\tilde{c}_1$ in the above system.

Moreover, the matrix

$$\begin{pmatrix} \mathscr{M}(1) & \mathscr{M}(f_1) & \mathscr{M}(g_1) \\ \mathscr{M}(f_1) & \mathscr{M}(f_1^2) & \mathscr{M}(f_1 g_1) \\ \mathscr{M}(g_1) & \mathscr{M}(f_1 g_1) & \mathscr{M}(g_1^2) \end{pmatrix} = \left(\begin{array}{c|c} \mathscr{M}(1) & 0 \\ \hline 0 & C_1 \end{array} \right)$$

is quasi-definite (positive definite) because of the quasi-definiteness (positivity) of $\mathscr{M}$. Thus C_1 is quasi-definite (positive definite). By applying induction, the proof follows straightforward. □

If $\mathscr{M}$ is induced by a positive definite functional, formulas (3) in Theorem 2.4 leads to four 'unusual' orthogonality conditions:

Theorem 2.5. *Let $\mathscr{M}_{\mathscr{L}}$ be a functional on Σ induced by a Hermitian positive definite functional $\mathscr{L}$. Let μ be the Borel positive measure on $\mathbb{T}$ associated with $\mathscr{L}$ and v_1 and v_2 the measures on $[-1,1]$ given by $\mathrm{d}v_1(x) = -\mathrm{d}\mu(\theta)$, $0 \leqslant \theta < \pi$, and $\mathrm{d}v_2(x) = \mathrm{d}\mu(\theta)$, $\pi \leqslant \theta < 2\pi$, with $x = \cos\theta$. Then, the polynomials $R_n^{(1)}$ and $R_n^{(2)}$, introduced in Theorem 2.4, satisfy*

$$\int_{-1}^{1} R_n^{(1)}(x)x^k \,\mathrm{d}(v_1+v_2)(x) + \int_{-1}^{1} R_n^{(2)}(x)x^k\sqrt{1-x^2}\,\mathrm{d}(v_1-v_2)(x) = 0, \quad 0 \leqslant k \leqslant n-1,$$

$$\int_{-1}^{1} R_n^{(2)}(x)x^k(1-x^2)\,\mathrm{d}(v_1+v_2)(x) + \int_{-1}^{1} R_n^{(1)}(x)x^k\sqrt{1-x^2}\,\mathrm{d}(v_1-v_2)(x) = 0, \quad 0 \leqslant k \leqslant n-2.$$

Two analogous formulas are true for the polynomials $S_n^{(1)}$ and $S_n^{(2)}$.

Corollary 2.6. *Let $\mathscr{M} \in \Sigma^*$ be quasi-definite (positive). Then there is a unique basis B of Σ such that*

$$B = \{1\} \cup \left(\bigcup_{n \geqslant 1} \{f_n, g_n\} \right),$$

where f_n and g_n are as in (3). Besides, the matrix associated with $\langle \cdot,\cdot \rangle_{\mathscr{M}}$ (with respect to B) is the diagonal-block matrix:

$$\begin{pmatrix} \mathscr{M}(1) & 0 & 0 & \cdots \\ 0 & C_1 & 0 & \cdots \\ 0 & 0 & C_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Definition 2.7. Given $\mathscr{M} \in \Sigma^*$ quasi-definite, we say that the basis B of Σ from Corollary 2.6 is a semi-orthogonal basis with respect to $\mathscr{M}$.

3. Recurrence formulas

Let us consider a quasi-definite linear functional $\mathscr{M}$ on Σ and let $B = \{1\} \cup (\bigcup_{n\geqslant 1}\{f_n, g_n\})$ be the corresponding semi-orthogonal basis.

We denote

$$\boldsymbol{F}_0 = (1,0)^{\mathrm{T}}, \qquad \boldsymbol{F}_n = (f_n, g_n)^{\mathrm{T}}, \quad (n \geqslant 1).$$

Also, for p_1, p_2, q_1, $q_2 \in \Sigma$, we write

$$\boldsymbol{P} = (p_1, p_2)^{\mathrm{T}}, \qquad \boldsymbol{Q} = (q_1, q_2)^{\mathrm{T}}, \quad (n \geqslant 1).$$

We define $\langle \boldsymbol{P},\boldsymbol{Q}\rangle : \Sigma^2\times\Sigma^2 \to \mathbb{R}^{(2,2)}$ by

$$\langle \boldsymbol{P},\boldsymbol{Q}\rangle = \begin{pmatrix} \mathcal{M}(p_1q_1) & \mathcal{M}(p_1q_2) \\ \mathcal{M}(p_2q_1) & \mathcal{M}(p_2q_2) \end{pmatrix}. \tag{4}$$

Notice that $\langle \boldsymbol{Q},\boldsymbol{P}\rangle = \langle \boldsymbol{P},\boldsymbol{Q}\rangle^{\mathrm{T}}$ and $\langle f\boldsymbol{P},\boldsymbol{Q}\rangle = \langle \boldsymbol{P},f\boldsymbol{Q}\rangle \quad \forall f\in\Sigma$.

Now, we consider $x\boldsymbol{F}_n$, $n\geqslant 1$. It is obvious that $xf_n, xg_n\in\Sigma_{n+1}$ and according to (3), we have

$$\mathcal{M}(xf_n(x)\cdot x^k) = \mathcal{M}(xg_n(x)\cdot x^k) = 0, \quad 0\leqslant k\leqslant n-2,$$

$$\mathcal{M}(xf_n(x)\cdot x^k\sqrt{1-x^2}) = \mathcal{M}(xg_n(x)\cdot x^k\sqrt{1-x^2}) = 0, \quad 0\leqslant k\leqslant n-3.$$

That is, xf_n and xg_n belong to the orthogonal complement of Σ_{n-2} with respect to Σ_{n+1}. So, there exist matrices Γ_n, Λ_n, $M_n\in\mathbb{R}^{(2,2)}$ such that

$$x\boldsymbol{F}_n = \Gamma_n\boldsymbol{F}_{n+1} + \Lambda_n\boldsymbol{F}_n + M_n\boldsymbol{F}_{n-1}, \quad n\geqslant 1$$

holds. By identification of the coefficients of x^{n+1} and $x^n\sqrt{1-x^2}$, we have that Γ_n is the 2×2 identity matrix and so we get

$$x\boldsymbol{F}_n = \boldsymbol{F}_{n+1} + \Lambda_n\boldsymbol{F}_n + M_n\boldsymbol{F}_{n-1}, \quad n\geqslant 1. \tag{5}$$

In a similar way, $\sqrt{1-x^2}f_n$, $\sqrt{1-x^2}g_n\in\Sigma_{n+1}$ and

$$\mathcal{M}(\sqrt{1-x^2}f_n(x)\cdot x^k) = \mathcal{M}(\sqrt{1-x^2}g_n(x)\cdot x^k) = 0, \quad 0\leqslant k\leqslant n-2,$$

$$\mathcal{M}(xf_n(x)\cdot x^k\sqrt{1-x^2}) = \mathcal{M}(xg_n(x)\cdot x^k\sqrt{1-x^2}) = 0, \quad 0\leqslant k\leqslant n-3.$$

Proceeding as above, we get that there exist two matrices $\widetilde{\Lambda}_n$, $\widetilde{M}_n\in\mathbb{R}^{(2,2)}$ such that

$$\sqrt{1-x^2}\boldsymbol{F}_n = J\boldsymbol{F}_{n+1} + \widetilde{\Lambda}_n\boldsymbol{F}_n + \widetilde{M}_n\boldsymbol{F}_{n-1}, \quad n\geqslant 1, \tag{6}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Taking into account that $\langle \boldsymbol{F}_n,\boldsymbol{F}_m\rangle = C_n\delta_{n,m}$ $(n,m\geqslant 1)$, we can obtain, after simple calculations:

$$\langle x\boldsymbol{F}_n,\boldsymbol{F}_{n+1}\rangle = C_{n+1}, \qquad \langle\sqrt{1-x^2}\boldsymbol{F}_n,\boldsymbol{F}_n\rangle = \widetilde{\Lambda}_nC_n\ ,$$

$$\langle\sqrt{1-x^2}\boldsymbol{F}_n,\boldsymbol{F}_{n+1}\rangle = JC_{n+1}, \quad \langle x\boldsymbol{F}_{n+1},\boldsymbol{F}_n\rangle = M_{n+1}C_n\ ,$$

$$\langle x\boldsymbol{F}_n,\boldsymbol{F}_n\rangle = \Lambda_nC_n, \qquad \langle\sqrt{1-x^2}\boldsymbol{F}_{n+1},\boldsymbol{F}_n\rangle = \widetilde{M}_{n+1}C_n\ .$$

So, the coefficients for the recurrence relations (5) and (6) are determined by $(C_n)_{n\geqslant 1}$, in the following way:

$$\begin{aligned} \Lambda_n &= \langle x\boldsymbol{F}_n,\boldsymbol{F}_n\rangle C_n^{-1}, & M_{n+1} &= C_{n+1}C_n^{-1}, \\ \widetilde{\Lambda}_n &= \langle\sqrt{1-x^2}\boldsymbol{F}_n,\boldsymbol{F}_n\rangle C_n^{-1}, & \widetilde{M}_{n+1} &= -\,C_{n+1}JC_n^{-1}, \end{aligned} \qquad (n\geqslant 1). \tag{7}$$

Formulas (5) and (6) does not determine completely M_1 and $\widetilde{M}_1$. Relations (7) remain true if the initial conditions $M_1=\{\mathscr{M}(1)\}^{-1}C_1$ and $\widetilde{M}_1=\{\mathscr{M}(1)\}^{-1}C_1J^{\mathrm{T}}$ are given.

Summing up we have shown

Theorem 3.1. *Let $\mathscr{M}\in\Sigma^*$ be quasi-definite and let B be the associated semi-orthogonal basis. Then, there are matrices Λ_n, M_n, $\widetilde{\Lambda}_n$, $\widetilde{M}_n\in\mathbb{R}^{(2,2)}$ such that (5)–(7) hold. Besides M_n and $\widetilde{M}_n$ are regular matrices. When the functional $\mathscr{M}$ is positive, then the eigenvalues of M_n are positive.*

Proposition 3.2. *Let $\mathscr{M}$, $\widetilde{\mathscr{M}}\in\Sigma^*$ be quasi-definite. If $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ have the same semi-orthogonal associated basis B, then there is $\alpha\in\mathbb{R}\backslash\{0\}$ such that*

$$\mathscr{M}(F)=\alpha\widetilde{\mathscr{M}}(F)\quad\forall F\in\Sigma.$$

Proof. Let $B=\{1\}\cup(\bigcup_{n\geqslant 1}\{f_n,g_n\})$. For each $F\in\Sigma$, there exist a_k, b_k, $c\in\mathbb{R}$ such that

$$F=c+\sum_k a_k f_k+\sum_j b_j g_j$$

holds. But, B is a semi-orthogonal basis with respect to $\mathscr{M}$ and $\widetilde{\mathscr{M}}$, and this fact implies $\mathscr{M}(F)=c\mathscr{M}(1)$, $\widetilde{\mathscr{M}}(F)=c\widetilde{\mathscr{M}}(1)$. So, it suffices to take $\alpha=\mathscr{M}(1)(\widetilde{\mathscr{M}}(1))^{-1}$. □

The above proposition determines a unique quasi-definite functional $\mathscr{M}\in\Sigma^*$ (except for a factor $\alpha\in\mathbb{R}\backslash\{0\}$).

4. Orthogonal polynomials and semi-orthogonal basis

Let $\mathscr{L}$ be a quasi-definite Hermitian functional on L. Assume that $\mathscr{L}$ is normalized, i.e., $\mathscr{L}(1)=1$. We denote by $\{\Phi_n\}_{n=0}^{\infty}$ the SMOP associated with $\mathscr{L}$.

For each $n\geqslant 1$, we can define, as in (1) the functions:

$$\begin{aligned} f_n(x)&=2^{-n}z^{-n}[z\Phi_{2n-1}(z)+\Phi_{2n-1}^*(z)],\\ g_n(x)&=-\mathrm{i}2^{-n}z^{-n}[z\Phi_{2n-1}(z)-\Phi_{2n-1}^*(z)], \end{aligned}\tag{8}$$

where $x=(z+z^{-1})/2$.

(Whenever $\mathscr{L}$ is symmetric and, hence, the coefficients of Φ_{2n-1} are real, formulas (8) should be compared with those given by Szegő [6; (11.5.2)].

Obviously, $\{1\}\cup\{f_n,g_n\}_{n\geqslant 1}$ constitutes a basis of Σ, according to Lemma 1.1 and Proposition 2.2.

Theorem 4.1. *Consider $B=\{1\}\cup(\bigcup_{n\geqslant 1}\{f_n,g_n\})$, where f_n, g_n are given by (8). Then B is a semi-orthogonal basis with respect to the functional $\mathscr{M}_{\mathscr{L}}\in\Sigma^*$ defined in Definition 2.1. Besides, $\mathscr{M}_{\mathscr{L}}$ is quasi-definite and, when $\mathscr{L}$ is positive definite, then $\mathscr{M}_{\mathscr{L}}$ is positive definite.*

Proof. We will write the matrix of the bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{M}_{\mathcal{L}}}$ with respect to the basis B. Thus we have

$$\mathcal{M}_{\mathcal{L}}(f_n)=\mathcal{L}[2^{-n}z^{-n}(z\Phi_{2n-1}(z)+\Phi^*_{2n-1}(z))]=0, \quad \forall n\geqslant 1,$$

$$\mathcal{M}_{\mathcal{L}}(g_n)=\mathcal{L}\left[2^{-n}z^{-n}\mathrm{i}^{-1}(z\Phi_{2n-1}(z)-\Phi^*_{2n-1}(z))\right]=0, \quad \forall n\geqslant 1$$

because of the orthogonality of $\{\Phi_n\}_{n=0}^{\infty}$. Also $\mathcal{M}_{\mathcal{L}}(1)=\mathcal{L}(1)=1$.

Now for $n\geqslant m\geqslant 1$, it follows that

$$\begin{aligned}\mathcal{M}_{\mathcal{L}}(f_nf_m)&=2^{-n-m}\mathcal{L}(z^{-n}[z\Phi_{2n-1}(z)+\Phi^*_{2n-1}(z)]z^{-m}[z\Phi_{2m-1}(z)+\Phi^*_{2m-1}(z)])\\&=2^{-n-m}\mathcal{L}(\Phi_{2n-1}(z)\Phi_{2m-1}(z)\cdot z^{-n-m+2}+\Phi^*_{2n-1}(z)\Phi_{2m-1}(z)\cdot z^{-n-m+1}\\&\quad+\Phi_{2n-1}(z)\Phi^*_{2m-1}(z)\cdot z^{-n-m+1}+\Phi^*_{2n-1}(z)\Phi^*_{2m-1}(z)\cdot z^{-n-m}).\end{aligned}$$

When $m<n$, since

$$\mathcal{L}(\Phi_{2n-1}(z)\cdot z^{-k})=\mathcal{L}(\Phi^*_{2n-1}(z)\cdot z^{-(k+1)})=0, \quad k=0,\dots,2n-2,$$

then $\mathcal{M}_{\mathcal{L}}(f_nf_m)=0$.

In a similar way, we can obtain that, for $m<n$

$$\mathcal{M}_{\mathcal{L}}(f_ng_m)=\mathcal{M}_{\mathcal{L}}(g_mf_m)=\mathcal{M}_{\mathcal{L}}(g_ng_m)=0.$$

When $n=m$, it results that

$$\begin{aligned}\mathcal{M}_{\mathcal{L}}(f_n^2)&=2^{-2n}\mathcal{L}(\Phi_{2n-1}(z)\Phi_{2n-1}(z)\cdot z^{-2n+2})\\&\quad+2^{-2n+1}\mathcal{L}(\Phi_{2n-1}(z)\Phi^*_{2n-1}(z)\cdot z^{-2n+1})+2^{-2n}\mathcal{L}(\Phi^*_{2n-1}(z)\Phi^*_{2n-1}(z)z^{-2n}).\end{aligned}$$

Then, we have, after straightforward calculations:

$$\mathcal{L}(\Phi_{2n-1}(z)\Phi_{2n-1}(z)\cdot z^{-2n+2})=-e_{2n-1}\Phi_{2n}(0),$$

$$\mathcal{L}(\Phi_{2n-1}(z)\Phi^*_{2n-1}(z)\cdot z^{-2n+1})=e_{2n-1},$$

$$\mathcal{L}(\Phi^*_{2n-1}(z)\Phi^*_{2n-1}(z)\cdot z^{-2n})=-e_{2n-1}\overline{\Phi_{2n}(0)},$$

where $e_n=\mathcal{L}(\Phi_n(z)\cdot z^{-n})\neq 0$, and thus

$$\begin{aligned}\mathcal{M}_{\mathcal{L}}(f_n^2)&=2^{-(2n-1)}e_{2n-1}\{1-\operatorname{Re}\Phi_{2n}(0)\},\\\mathcal{M}_{\mathcal{L}}(f_ng_n)&=-2^{-(2n-1)}e_{2n-1}\operatorname{Im}\Phi_{2n}(0),\\\mathcal{M}_{\mathcal{L}}(g_n^2)&=2^{-(2n-1)}e_{2n-1}\{1+\operatorname{Re}\Phi_{2n}(0)\}.\end{aligned} \tag{9}$$

So, B constitutes a semi-orthogonal basis, according to Definition 2.7. Besides, the matrix of $\langle\cdot,\cdot\rangle_{\mathcal{M}_\mathcal{L}}$ with respect to B is

$$\begin{pmatrix} \mathcal{M}_\mathcal{L}(1) & 0 & 0 & \cdots \\ 0 & C_1 & 0 & \cdots \\ 0 & 0 & C_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where

$$C_n = 2^{-2n+1} e_{2n-1} \begin{pmatrix} 1-\operatorname{Re}\Phi_{2n}(0) & -\operatorname{Im}\Phi_{2n}(0) \\ -\operatorname{Im}\Phi_{2n}(0) & 1+\operatorname{Re}\Phi_{2n}(0) \end{pmatrix},$$

i.e., each C_n is quasi-definite (positive definite) if and only if $\mathcal{L}$ is quasi-definite (positive). □

Proposition 4.2. *For each $n\geqslant 1$ the functions f_n and g_n can be expressed as*

$$\begin{aligned} f_n(x) &= \frac{(1-\overline{\Phi_{2n}(0)})\Phi_{2n}(z)+(1-\Phi_{2n}(0))\Phi_{2n}^*(z)}{2^n z^n(1-|\Phi_{2n}(0)|^2)}, \\ g_n(x) &= \frac{(1+\overline{\Phi_{2n}(0)})\Phi_{2n}(z)-(1+\Phi_{2n}(0))\Phi_{2n}^*(z)}{2^n z^n i(1-|\Phi_{2n}(0)|^2)}. \end{aligned} \qquad (10)$$

Proof. It is an easy consequence of formula (8) and Szegő's recurrence formulas. □

Proposition 4.3. *The SMOP related to $\mathcal{L}$, can be expressed as*

$$\Phi_{2n-1}(z) = 2^{n-1}z^{n-1}\{f_n(x)+ig_n(x)\},$$

$$\Phi_{2n}(z) = 2^{n-1}z^{n}\{(1+\Phi_{2n}(0)f_n(x)+(1-\Phi_{2n}(0))ig_n(x)\}.$$

Proof. It suffices to eliminate $\Phi_{2n-1}^*(z)$ in (8) and $\Phi_{2n}^*(z)$ in (10). □

In these conditions, Theorem 4.1 has the following converse:

Theorem 4.4. *Let $\mathcal{M}\in\Sigma^*$ be quasi-definite, such that trace $C_n\neq 0$ for each $n\geqslant 1$. Then, there exists just one quasi-definite Hermitian functional $\mathcal{L}\in L^*$, such that $\mathcal{M}$ is induced by $\mathcal{L}$, that is, $\mathcal{M}=\mathcal{M}_\mathcal{L}$. Moreover, $\mathcal{L}$ is positive definite if and only if $\mathcal{M}$ is positive definite.*

Proof. Consider a semi-orthogonal basis $B=\{1\}\cup(\bigcup_{n\geqslant 1}\{f_n,g_n\})$ related to $\mathcal{M}$. Without loss of generality, we can assume that $\mathcal{M}(1)=1$. Then the matrix of $\langle\cdot,\cdot\rangle_\mathcal{M}$ with respect to B is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & C_1 & 0 & \cdots \\ 0 & 0 & C_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

with

$$C_n = \begin{pmatrix} \mathscr{M}(f_n^2) & \mathscr{M}(f_n g_n) \\ \mathscr{M}(g_n f_n) & \mathscr{M}(g_n^2) \end{pmatrix}.$$

We will obtain a Hermitian functional $\mathscr{L}$ such that the associated SMOP satisfy formulas (9). This system provides that

$$e_{2n-1} = 2^{2n-2}\{\mathscr{M}(f_n^2) + \mathscr{M}(g_n^2)\}$$

with $e_{2n-1} \neq 0$ because of trace $C_n \neq 0$ (>0). Now, the quasi-definiteness of $\mathscr{M}$ implies that

$$0 \neq \operatorname{Det} C_n = \mathscr{M}(f_n^2)\mathscr{M}(g_n^2) - \mathscr{M}(f_n g_n)^2 = e_{2n-1}^{-1} 2^{2-4n}(1 - |\Phi_{2n}(0)|^2)$$

then $|\Phi_{2n}(0)| \neq 1$ and, when $\mathscr{M}$ is positive, $|\Phi_{2n}(0)| < 1$.

So, once e_{2n-1} is known, formulas (9) give us $\Phi_{2n}(0)$ and, because of Proposition 4.3, we have $\{\Phi_n\}_{n\in\mathbb{N}}$. Thus, this sequence $\{\Phi_n\}$ is uniquely determined by the matrix sequence (C_n). Now, Favard's theorem guarantees the existence of a unique Hermitian, quasi-definite (positive) functional (except for a non-zero factor) $\mathscr{L} \in \Sigma^*$ such that $\{\Phi_n\}_{n\in\mathbb{N}}$ is the related SMOP. Besides, $\mathscr{M}$ is induced by $\mathscr{L}$ in the sense of Definition 2.1. □

5. Application

Let $\mathscr{L} \in L^*$ be quasi-definite and $\mathscr{M} := \mathscr{M}_{\mathscr{L}}$ its induced functional. If we define $\tilde{\mathscr{L}} \in L^*$ by: $\tilde{\mathscr{L}} = (\frac{1}{2}(z + z^{-1}) - a)\mathscr{L}$ with $a \in \mathbb{R}$, let $\tilde{\mathscr{M}} := \mathscr{M}_{\tilde{\mathscr{L}}} \in \Sigma^*$ be given by

$$\tilde{\mathscr{M}}[(P,Q)] = \tilde{\mathscr{L}}[P + \sqrt{1-x^2}Q] = \mathscr{L}[(x-a)(P + \sqrt{1-x^2}Q)] = \mathscr{M}[(x-a)(P,Q)]$$

$\forall (P,Q) \in \Sigma$. So, $\tilde{\mathscr{M}} = (x-a)\mathscr{M}$ is the associated functional of $\tilde{\mathscr{L}}$, with the quasi-definiteness conditions given in Theorem 4.1.

Assume that $\tilde{\mathscr{M}}$ is quasi-definite, and let $(\boldsymbol{F}_n)$ the sequence of semi-orthogonal functions related to $\tilde{\mathscr{M}}$, $\boldsymbol{F}_n = (F_n, G_n)^{\mathrm{T}}$, $n \geqslant 1$. Then, we have

$$\tilde{\mathscr{M}}(x^k \boldsymbol{F}_n(x)) = \mathscr{M}(x^k (x-a)\boldsymbol{F}_n(x)) = 0 \quad (0 \leqslant k \leqslant n-1),$$

$$\tilde{\mathscr{M}}(x^k \sqrt{1-x^2}\boldsymbol{F}_n(x)) = \mathscr{M}(x^k \sqrt{1-x^2}(x-a)\boldsymbol{F}_n(x)) = 0 \quad (0 \leqslant k \leqslant n-2)$$

from here $(x-a)F_n$, $(x-a)G_n \in \Sigma_{n+1}$ are orthogonal to Σ_n with respect to $\langle \cdot,\cdot \rangle_{\mathscr{M}}$. Thus, there exist unique matrices A_n, $B_n \in \mathbb{R}^{(2,2)}$ such that

$$(x-a)\boldsymbol{F}_n = A_n\, \boldsymbol{f}_{n+1} + B_n\, \boldsymbol{f}_n.$$

By comparing the coefficients of x^{n+1} and $\sqrt{1-x^2}x^n$ we obtain that $A_n = I$. So, if we assume that $\mathscr{M}$ is quasi-definite, there exists a unique matrix $B_n \in \mathbb{R}^{(2,2)}$ such that

$$(x-a)\boldsymbol{F}_n = \boldsymbol{f}_{n+1} + B_n\, \boldsymbol{f}_n. \tag{11}$$

When $|a| \neq 1$, notice that $f_n(a)$ has two determinations for $\alpha = a + \sqrt{a^2-1}$ and $\alpha^{-1} = a - \sqrt{a^2-1}$, that we denote by $f_n^+(a)$ and $f_n^-(a)$, respectively. Then formula (11) becomes

$$0 = [f_{n+1}^+(a), f_{n+1}^-(a)] + B_n[f_n^+(a), f_n^-(a)].$$

Thus, the quasi-definiteness of $[f_{n+1}^+(a), f_{n+1}^-(a)]$, $(n \geqslant 1)$ implies the existence of F_n $(n \geqslant 1)$ verifiying (11) and, moreover, the quasi-definiteness of $\widetilde{\mathscr{M}}$.

On the other hand,

$$\det[f_n^+(a), f_n^-(a)] = \frac{1}{i2^{2n}} \begin{vmatrix} \alpha\Phi_{2n-1}(\alpha) + \Phi_{2n-1}^*(\alpha) & \alpha^{-1}\Phi_{2n-1}(\alpha^{-1}) + \Phi_{2n-1}^*(\alpha^{-1}) \\ \alpha\Phi_{2n-1}(\alpha) - \Phi_{2n-1}^*(\alpha) & \alpha^{-1}\Phi_{2n-1}(\alpha^{-1}) - \Phi_{2n-1}^*(\alpha^{-1}) \end{vmatrix}$$

$$= \frac{1}{i2^{2n-2}} \begin{vmatrix} \Phi_{2n-1}^*(\alpha) & \Phi_{2n-1}^*(\alpha^{-1}) \\ \alpha\Phi_{2n-1}(\alpha) & \alpha^{-1}\Phi_{2n-1}(\alpha^{-1}) \end{vmatrix} \neq 0.$$

If we claim that trace $\widetilde{\mathscr{M}}(F_n \cdot F_n^T) \neq 0$, then the same relation for the even terms is obtained. That is, according to Theorem 4.4, $\widetilde{\mathscr{L}}$ is quasi-definite when

$$D_n(\alpha) = \Phi_n^*(\alpha^{-1})\alpha\Phi_n(\alpha) - \Phi_n^*(\alpha)\alpha^{-1}\Phi_n(\alpha^{-1}) \neq 0 \tag{12}$$

$\forall n \geqslant 1$ and $\alpha \neq \pm 1$. This requirement is the same given in [5] for the quasi-definiteness of $\widetilde{\mathscr{L}}$ in the particular case $\widetilde{\mathscr{L}} = \mathrm{Re}(z-\alpha)\mathscr{L}$.

By using formulas (8) and (11), we can obtain the relationship:

$$(z^2 - 2az + 1)\widetilde{\Phi}_n(z) = \Phi_{n+2}(z) + r_n z\Phi_n(z) + s_n\Phi_n^*(z),$$

where $(\widetilde{\Phi}_n)$ is the SMOP associated with $\widetilde{\mathscr{L}}$. So, we derive the determinantal formula:

$$(z^2 - 2az + 1)\widetilde{\Phi}_n(z) = D_n(\alpha)^{-1} \begin{vmatrix} \Phi_{n+2}(z) & z\Phi_n(z) & \Phi_n^*(z) \\ \Phi_{n+2}(\alpha) & \alpha\Phi_n(\alpha) & \Phi_n^*(\alpha) \\ \Phi_{n+2}(\alpha^{-1}) & \alpha^{-1}\Phi_n(\alpha^{-1}) & \Phi_n^*(\alpha^{-1}) \end{vmatrix}$$

which gives $(\widetilde{\Phi}_n)$ in terms of (Φ_n), constrained to the condition (12).

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