An extension of an inequality for ratios of gamma functions

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Abstract

In this paper, we prove that for $x + y > 0$ and $y + 1 > 0$ the inequality

$$\frac{[\Gamma(x + y + 1) / \Gamma(y + 1)]^{1/x}}{[\Gamma(x + y + 2) / \Gamma(y + 1)]^{1/(x+1)}} < \left( \frac{x + y}{x + y + 1} \right)^{1/2}$$

is valid if $x > 1$ and reversed if $x < 1$ and that the power $\frac{1}{2}$ is the best possible, where $\Gamma(x)$ is the Euler gamma function. This extends the result of [Y. Yu, An inequality for ratios of gamma functions, J. Math. Anal. Appl. 352 (2) (2009) 967–970] and resolves an open problem posed in [B.-N. Guo, F. Qi, Inequalities and monotonicity for the ratio of gamma functions, Taiwanese J. Math. 7 (2) (2003) 239–247].

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1. Introduction

It is common knowledge that the classical Euler gamma function $\Gamma(x)$ may be defined for a real argument $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.$$  \hfill (1)
The logarithmic derivative of \( \Gamma(x) \), denoted by \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \), is called the psi or digamma function, and the \( \psi^{(k)}(x) \) for \( k \in \mathbb{N} \) are called the polygamma functions. It is general knowledge that these functions are basic and that they have very extensive applications in mathematical sciences.

In [9, Theorem 2], the function
\[
\frac{[\Gamma(x + y + 1)/\Gamma(y + 1)]^{1/x}}{x + y + 1}
\]
was proved to be decreasing with respect to \( x \geq 1 \) for fixed \( y \geq 0 \). Consequently, the inequality
\[
\frac{x + y + 1}{x + y + 2} \leq \frac{[\Gamma(x + y + 1)/\Gamma(y + 1)]^{1/x}}{[\Gamma(x + y + 2)/\Gamma(y + 1)]^{1/(x+1)}}
\]
holds for positive real numbers \( x \geq 1 \) and \( y \geq 0 \). Meanwhile, an open problem was posed in [9, p. 245], to ask for an upper bound \( \sqrt{\frac{x+y}{x+y+1}} \) for the function in the right-hand side of the inequality (3).

In [22], the above-mentioned open problem was partially resolved as follows: If \( y > 0 \) and \( x > 1 \), then
\[
\frac{[\Gamma(x + y + 1)/\Gamma(y + 1)]^{1/x}}{[\Gamma(x + y + 2)/\Gamma(y + 1)]^{1/(x+1)}} < \left( \frac{x + y}{x + y + 1} \right)^{1/2};
\]
if \( y > 0 \) and \( 0 < x < 1 \), then the inequality (4) is reversed.

For more information on the origin, history, background, motivations and recent developments of this topic, please refer to [1,4,5,3,6,13,20,18] and closely related references cited therein.

The aim of this paper is to extend the one-side inequality (4) and to resolve the above-mentioned open problem.

Our results may be stated as the following theorem.

**Theorem 1.1.** For \( y + 1 > 0 \) and \( x + y > 0 \), the inequality (4) holds if \( x > 1 \) and reverses if \( x < 1 \). The cases \( x = 0, -1 \) are understood to be the limits as \( x \to 0, -1 \) on both sides of the inequality (4), that is,
\[
e^{\psi(y+1)} > (y + 1) \left( \frac{y}{y + 1} \right)^{1/2}, \quad y > 0
\]
and
\[
e^{-\psi(y+1)} > \frac{1}{y} \left( \frac{y - 1}{y} \right)^{1/2}, \quad y > 1.
\]

Moreover, the powers \( \frac{1}{2} \) in (4)–(6) are the best possible in the sense that the power \( \frac{1}{2} \) in the inequality (4) cannot be replaced by a larger number and that the powers \( \frac{1}{2} \) in the reversed inequality of (4)–(6) cannot be replaced by a smaller number.

As a ready consequence of the proof of Theorem 1.1, the following inequality is concluded.
Corollary 1.1. For \( x + y > 0 \) and \( y + 1 > 0 \), if \( 0 < |x| < 1 \), then
\[
\left[ \frac{\Gamma(x + y + 1)}{\Gamma(y + 1)} \right]^{1/x} > \left[ \frac{(x + y)^{x+1}}{(x + y + 1)^{x+1}} \right]^{1/2};
\] (7)
if \( |x| > 1 \), then the inequality (7) is reversed. In particular, the inequality
\[
\Gamma(x + 1) > \left[ \frac{x^{x+1}}{(x + 1)^{x+1}} \right]^{x/2}
\] (8)
holds for \( 0 < x < 1 \) and reverses for \( x > 1 \).

2. Lemmas

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. For \( k \in \mathbb{N} \) and \( t > s > 0 \) with \( t - s \neq 1 \), we have
\[
\min \left\{ s, \frac{s + t - 1}{2} \right\} < \left[ \frac{\Gamma(s)}{\Gamma(t)} \right]^{1/(s-t)} < \max \left\{ s, \frac{s + t - 1}{2} \right\}
\] (9)
and
\[
\frac{(k - 1)!}{\left( \max \left\{ s, \frac{s + t - 1}{2} \right\} \right)^k} < \frac{(-1)^{k-1}\left[ \psi^{(k-1)}(t) - \psi^{(k-1)}(s) \right]}{t - s} < \frac{(k - 1)!}{\left( \min \left\{ s, \frac{s + t - 1}{2} \right\} \right)^k},
\] (10)
where \( \psi^{(0)}(x) \) stands for \( \psi(x) \). Moreover, the lower and upper bounds in (9) and (10) are the best possible constants for which the inequalities hold.

Proof. For real numbers \( a, b \) and \( c \), define \( \rho = \min\{a, b, c\} \) and
\[
H_{a,b;c}(x) = (x + c)^{b - a} \frac{\Gamma(x + a)}{\Gamma(x + b)}
\] (11)
for \( x \in (-\rho, \infty) \). In [19, p. 283, Theorem 1], it was obtained that

(1) the function \( H_{a,b;c}(x) \) is logarithmically completely monotonic, that is,
\[
0 \leq (-1)^i[\ln H_{a,b;c}(x)]^{(i)} < \infty
\]
for \( i \geq 1 \), on \( (-\rho, \infty) \) if and only if
\[
(a, b; c) \in D_1(a, b; c) \overset{\Delta}{=} \{(a, b; c) : (b - a)(1 - a - b + 2c) \geq 0 \} \cap \{(a, b; c) : (b - a)(|a - b| - a - b + 2c) \geq 0 \} \setminus \{(a, b; c) : a = c + 1 = b + 1 \} \setminus \{(a, b; c) : b = c + 1 = a + 1 \};
\] (12)
(2) and so is the function \( H_{b,a;c}(x) \) on \((-\rho, \infty)\) if and only if

\[
(a, b; c) \in D_2(a, b; c) \triangleq \{(a, b; c) : (b - a)(1 - a - b + 2c) \leq 0, \forall\}
\]
\[
\cap\{(a, b; c) : (b - a)(|a - b| - a - b + 2c) \leq 0\}
\]
\[
\setminus\{(a, b; c) : b = c + 1 = a + 1\}
\]
\[
\setminus\{(a, b; c) : a = c + 1 = b + 1\}.
\]

See also [10, pp. 1241–1242, Theorem 4.1]. It is well-known that the limit

\[
\lim_{x \to \infty} \left[ x^{b-a} \frac{\Gamma(x + a)}{\Gamma(x + b)} \right] = 1
\]

holds for real numbers \( a \) and \( b \); see [2, p. 257, 6.146] or [16, p. 3, Section 1.1.6]. This implies that

\[
\lim_{x \to \infty} H_{a,b,c}(x) = 1.
\]

From the logarithmically complete monotonicity of \( H_{a,b,c}(x) \), it is deduced that the function \( H_{a,b,c}(x) \) is decreasing if \( (a, b; c) \in D_1(a, b; c) \) and increasing if \( (a, b; c) \in D_2(a, b; c) \) on \((-\rho, \infty)\). As a result of the limit (15) and the monotonicity of the function \( H_{a,b,c}(x) \), it follows that the inequality \( H_{a,b,c}(x) > 1 \) holds if \( (a, b; c) \in D_1(a, b; c) \) and reverses if \( (a, b; c) \in D_2(a, b; c) \), that is, the inequality

\[
x + \lambda < \left[\frac{\Gamma(x + a)}{\Gamma(x + b)}\right]^{1/(a-b)} < x + \mu
\]

for \( b > a \) holds if \( \lambda \leq \min \left\{ a, \frac{a+b-1}{2} \right\} \) and \( \mu \geq \max \left\{ a, \frac{a+b-1}{2} \right\} \), which may be reduced to the inequality (9) by replacing \( x + a \) and \( x + b \) by \( s \) and \( t \) respectively.

Further, by virtue of the logarithmically complete monotonicity of \( H_{a,b,c}(x) \) on \((-\rho, \infty)\) again and the fact from [8, p. 98] that a completely monotonic function which is non-identically zero cannot vanish at any point on \((0, \infty)\), it is readily deduced that

\[
(-1)^k [\ln H_{a,b,c}(x)]^{(k)} = (-1)^k [(b - a) \ln(x + c) + \ln \Gamma(x + a) - \ln \Gamma(x + b)]^{(k)}
\]

\[
= (-1)^k \left[ \frac{(-1)^{k-1}(k - 1)!(b - a)}{(x + c)^k} + \psi^{(k-1)}(x + a) - \psi^{(k-1)}(x + b) \right]
\]

\[
> 0
\]

for \( k \in \mathbb{N} \) is valid if \( (a, b; c) \in D_1(a, b; c) \) and reversed if \( (a, b; c) \in D_2(a, b; c) \). Consequently, the double inequality

\[
-\frac{(k - 1)!(b - a)}{(x + c)^k} < (-1)^k \left[ \psi^{(k-1)}(x + b) - \psi^{(k-1)}(x + a) \right] < -\frac{(k - 1)!(b - a)}{(x + c_1)^k}
\]

holds with respect to \( x \in (-\rho, \infty) \) if \( (a, b; c_1) \in D_1(a, b; c) \) and \( (a, b; c_2) \in D_2(a, b; c) \), which may be rearranged as

\[
\frac{(k - 1)!(b - a)}{(x + c)^k} > \frac{(-1)^{k-1}[\psi^{(k-1)}(x + b) - \psi^{(k-1)}(x + a)]}{b - a} < \frac{(k - 1)!}{(x + \beta)^k}
\]

(16)
for \( x \in (-\rho, \infty) \) if \( \alpha \geq \max\left\{ a, \frac{a+b-1}{2} \right\} \) and \( \beta \leq \min\left\{ a, \frac{a+b-1}{2} \right\} \), where \( b > a \) and \( k \in \mathbb{N} \). In the end, replacing \( x + a \) and \( x + b \) by \( s \) and \( t \) respectively in (16) leads to (10). The proof of Lemma 2.1 is thus complete. □

Lemma 2.2. For \( x \in (0, \infty) \) and \( k \in \mathbb{N} \), we have

\[
\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}
\]

and

\[
\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1} \psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}.
\]

Proof. In [12, Theorem 2.1] and [15, Lemma 1.3], the function \( \psi(x) - \ln x + \frac{\alpha}{x} \) was proved to be completely monotonic on \((0, \infty)\), i.e.,

\[
(-1)^i \left[ \psi(x) - \ln x + \frac{\alpha}{x} \right]^{(i)} \geq 0
\]

for \( i \geq 0 \), if and only if \( \alpha \geq 1 \), and so is its negative, i.e., the inequality (19) is reversed, if and only if \( \alpha \leq \frac{1}{2} \). In [7, Theorem 2], [11, Theorem 2.1] and [14, Theorem 2.1], the function \( \frac{x^{x-a}}{e^{x\Gamma(x)}} \) was proved to be logarithmically completely monotonic on \((0, \infty)\), i.e.,

\[
(-1)^k \left[ \ln \frac{x^{x-a}}{e^{x\Gamma(x)}} \right]^{(k)} \geq 0
\]

for \( k \in \mathbb{N} \), if and only if \( \alpha \geq 1 \), and so is its reciprocal, i.e., the inequality (20) is reversed, if and only if \( \alpha \leq \frac{1}{2} \). Considering the fact from [8, p. 98] that a completely monotonic function which is non-identically zero cannot vanish at any point on \((0, \infty)\) and rearranging either (19) or (20) leads to the double inequalities (17) and (18). Lemma 2.2 is proved. □

Lemma 2.3 ([21]). If \( t > 0 \), then

\[
\frac{2t}{2+t} < \ln(1+t) < \frac{t(2+t)}{2(1+t)},
\]

If \(-1 < t < 0\), the inequality (21) is reversed.

3. Proofs of Theorem 1.1 and Corollary 1.1

Now we are in a position to prove Theorem 1.1 and Corollary 1.1.

Proof of Theorem 1.1. When \( 0 \geq y > -1 \) and \( x > -y \), define

\[
f_y(x) = \frac{\ln \Gamma(x+y+1) - \ln \Gamma(y+1)}{x} - \frac{1}{2} \ln(x+y).
\]

When \( y > 0 \) and \( x > -y \), define

\[
f_y(0) = \psi(y+1) - \frac{1}{2} \ln y
\]
and \( f_y(x) \) for \( x \neq 0 \) to be the same as in (22). Making use of the well-known recursion formula 
\[
\Gamma(x + 1) = x \Gamma(x)
\]
and computing straightforwardly yields
\[
f_y(x + 1) - f_y(x) = \left( \frac{1}{x + 1} - \frac{1}{x} \right) \ln \frac{\Gamma(x + y + 1)}{\Gamma(y + 1)} + \frac{\ln(x + y + 1)}{x + 1} + \frac{1}{2} \ln x + y + 1
\]
\[
= \frac{1}{x + 1} \left\{ \ln \left[ \frac{(x + y)^{(x + 1)/2}}{(x + y + 1)^{(x - 1)/2}} \right] - \ln \left[ \frac{\Gamma(x + y + 1)}{\Gamma(y + 1)} \right] \right\}^{1/x}.
\]
(23)

Substituting \( s = y + 1 > 0 \) and \( t = x + y + 1 > 1 \) into (9) in Lemma 2.1 leads to
\[
\min \left\{ y + 1, \frac{x + 2y + 1}{2} \right\} < \left[ \frac{\Gamma(x + y + 1)}{\Gamma(y + 1)} \right]^{1/x} < \max \left\{ y + 1, \frac{x + 2y + 1}{2} \right\}
\]
which is equivalent to
\[
\left[ \frac{\Gamma(x + y + 1)}{\Gamma(y + 1)} \right]^{1/x} < \begin{cases} \frac{x + 2y + 1}{2}, & x > 1 \\ y + 1, & x < 1 \end{cases}
\]
and
\[
\left[ \frac{\Gamma(x + y + 1)}{\Gamma(y + 1)} \right]^{1/x} > \begin{cases} y + 1, & x > 1 \\ \frac{x + 2y + 1}{2}, & x < 1 \end{cases}
\]
for \( y + 1 > 0 \) and \( x + y > 0 \). Consequently, it follows readily from (23) that, for \( y > -1 \) and \( x + y > 0 \),

1. if \( x > 1 \) and
\[
\frac{(x + y)^{(x + 1)/2}}{(x + y + 1)^{(x - 1)/2}} > \frac{x + 2y + 1}{2},
\]
then \( f_y(x + 1) - f_y(x) > 0; \)
2. if \(-1 < x < 1 \) and the inequality (24) reverses, then \( f_y(x + 1) - f_y(x) < 0. \)

For \( x + y > 0 \) and \( y > -1 \), define
\[
g(x, y) = \frac{(x + y)^{x+1}}{(x + 2y + 1)^2(x + y + 1)^{x-1}}.
\]
The partial derivative of \( g(x, y) \) with respect to \( y \) is
\[
\frac{\partial g(x, y)}{\partial y} = \frac{1 - x^2}{(x + 2y + 1)^3} \left( \frac{x + y}{x + y + 1} \right)^x.
\]
This shows that

1. when \( |x| > 1 \), the function \( g(x, y) \) is strictly decreasing with respect to \( y > -1; \)
2. when \( |x| < 1 \), the function \( g(x, y) \) is strictly increasing with respect to \( y > -1. \)
In addition, it is clear that \( \lim_{y \to -\infty} g(x, y) = \frac{1}{4} \). As a result, it is easy to see that \( g(x, y) \gtrless \frac{1}{4} \) when \( |x| \gtrless 1 \) for \( x + y > 0 \) and \( y > -1 \). In other words, the inequality (24) is valid when \( |x| > 1 \) and reversed when \( |x| < 1 \) for all \( x + y > 0 \) and \( y > -1 \). Consequently, the inequality \( f_\alpha(x + 1) - f_\beta(x) > 0 \) holds if \( x > 1 \) and reverses if \( |x| < 1 \), where \( x + y > 0 \) and \( y > -1 \).

For \( x < -1 \), denote the function enclosed in the braces in (23) by \( Q(x, y) \). Direct computation yields

\[
Q(x, y) = \frac{x + 1}{2} \ln(x + y) - \frac{x - 1}{2} \ln(x + y + 1) - \frac{1}{x} \int_{y+1}^{x+y+1} \psi(u) du
\]

and

\[
\frac{\partial Q(x, y)}{\partial x} = \frac{3x + 2y + 1}{2(x + y)(x + y + 1)} + \frac{1}{2} \ln \frac{x + y}{x + y + 1} - \int_0^1 u \psi'((y + 1)(1 - u) + (x + y + 1)u) du.
\]

Making use of the left-hand side inequality for \( k = 1 \) in (18) results in

\[
\frac{\partial Q(x, y)}{\partial x} = \frac{3x + 2y + 1}{2(x + y)(x + y + 1)} + \frac{1}{2} \ln \frac{x + y}{x + y + 1} - \int_0^1 u \left( \frac{1}{(y + 1)(1 - u) + (x + y + 1)u} + \frac{1}{2[(y + 1)(1 - u) + (x + y + 1)u]^2} \right) du
\]

\[
= \frac{1}{2} \left[ \frac{x^2 - 2yx - y(2y + 1)}{x(x + y)(x + y + 1)} + \ln \frac{x + y}{x + y + 1} - \frac{1 + 2y}{x^2} \ln \frac{y + 1}{x + y + 1} \right].
\]

Further employing the left-hand side inequality of (21) in Lemma 2.3 leads to

\[
\frac{\partial Q(x, y)}{\partial x} = \frac{1}{2} \left[ \frac{x^2 - 2yx - y(2y + 1)}{x(x + y)(x + y + 1)} - \frac{2}{1 + 2x + 2y} + \frac{1 + 2y}{x^2} \cdot \frac{2x}{2 + x + 2y} \right]
\]

\[
= \frac{(2y + 3)x^2 + 2(y^2 + 2y + 2)x + 3y + 2}{2(x + y)(x + y + 1)(x + 2y + 2)(2x + 2y + 1)}
\]

\[
= \frac{(2y + 3)F_1(x, y)F_2(x, y)}{2(x + y)(x + y + 1)(x + 2y + 2)(2x + 2y + 1)},
\]

where

\[
F_1(x, y) = \left( x + \frac{2 + y^2 + 2y - \sqrt{y^4 + 4y^3 + 2y^2 - 5y - 2}}{2y + 3} \right)
\]

and

\[
F_2(x, y) = \left( x + \frac{2 + y^2 + 2y + \sqrt{y^4 + 4y^3 + 2y^2 - 5y - 2}}{2y + 3} \right).
\]
For $x < -1$ and $x + y > 0$, a standard argument reveals that

$$F_1(x, y) < \frac{2 + y^2 + 2y - \sqrt{y^4 + 4y^3 + 2y^2 - 5y - 2}}{2y + 3} - 1 < 0$$

and

$$F_2(x, y) > \frac{2 + y^2 + 2y + \sqrt{y^4 + 4y^3 + 2y^2 - 5y - 2}}{2y + 3} - y > 0,$$

so $\frac{\partial Q(x, y)}{\partial x} < 0$ and the function $Q(x, y)$ is decreasing with respect to $x < -1$. From the fact that $Q(-1, y) = 0$, it follows that $Q(x, y) > 0$ for $x < -1$, which means that when $y + 1 > 0$ and $x + y > 0$ the inequality (4) is reversed for $x < -1$.

For $x \geq 1$, if

$$\frac{[\Gamma(x + y + 1)/\Gamma(y + 1)]^{1/x}}{[\Gamma(x + y + 2)/\Gamma(y + 1)]^{1/(x+1)}} \geq \left(\frac{x + y}{x + y + 1}\right)^{\alpha},$$

then

$$\alpha \leq \frac{\ln \Gamma(x + y + 1) - \ln \Gamma(y + 1)}{x[\ln(x + y) - \ln(x + y + 1)]} - \frac{\ln \Gamma(x + y + 2) - \ln \Gamma(y + 1)}{(x + 1)[\ln(x + y) - \ln(x + y + 1)]}$$

is valid for $y + 1 > 0$ and $x + y > 0$. Since

$$\lim_{x \to 1} \left\{ \frac{\ln \Gamma(x + y + 1) - \ln \Gamma(y + 1)}{x[\ln(x + y) - \ln(x + y + 1)]} - \frac{\ln \Gamma(x + y + 2) - \ln \Gamma(y + 1)}{(x + 1)[\ln(x + y) - \ln(x + y + 1)]} \right\}$$

$$= \frac{\ln \Gamma(y + 2) - \ln \Gamma(y + 1)}{\ln(y + 1) - \ln(y + 2)} - \frac{\ln \Gamma(y + 3) - \ln \Gamma(y + 1)}{2[\ln(y + 1) - \ln(y + 2)]}$$

$$= \frac{1}{2},$$

it follows that $\alpha \leq \frac{1}{2}$. So the powers $\frac{1}{2}$ in Theorem 1.1 are the best possible. Theorem 1.1 is thus proved.

**Proof of Corollary 1.1.** The inequality (7) follows from the discussion in the proof of Theorem 1.1 about the positivity and negativity of the function enclosed by braces in (23).

The inequality (8) is a special case of (7) for $y = 0$. 

**Remark 3.1.** This paper is a slightly revised version of the preprint [17].

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**References**

