On \( R \)-Homomorphisms of Power Series Rings

ROBERT GILMER*

Florida State University, Tallahassee, Florida 32306 and
University of Texas, Austin, Texas 78712

AND

MATTHEW O’MALLEY

University of Houston, Houston, Texas 77004

Let \( R \) be a commutative ring with identity and let \( \{X_i\}_{i=1}^n \) and \( \{Y_j\}_{j=1}^m \) be sets of indeterminates over \( R \). A homomorphism \( \phi : R[[X_1, \ldots, X_n]] \to R[[Y_1, \ldots, Y_m]] \) of power series rings over \( R \) is an \( R \)-homomorphism if \( \phi(r) = r \) for each \( r \) in \( R \). We also use the terms \( R \)-endomorphism of \( R[[X_1, \ldots, X_n]] \) and \( R \)-automorphism of \( R[[X_1, \ldots, X_n]] \); questions concerning \( R \)-homomorphisms of power series rings can frequently be treated within the context of \( R \)-endomorphisms of a power series ring, and we sometimes take this approach. Several papers [2, 4, 6–12] have recently dealt with the structure of the sets of \( R \)-endomorphisms and \( R \)-automorphisms of \( R[[X_1, \ldots, X_n]] \), with the case \( n = 1 \) having drawn special attention. One result from [2] plays a particularly important role in this paper. It enables us to answer some previously open questions, and makes it possible to clarify, consolidate, and extend some of the general theory. The result in question is Theorem A of [2]; we repeat its statement below.

**Theorem A.** Assume that \( \phi : R[[X_1, \ldots, X_n]] \to R[[Y_1, \ldots, Y_m]] \) is an \( R \)-homomorphism, and assume that \( \phi(X_i) \) has constant term \( c_i \). Then there exists an \( R \)-automorphism of \( R[[X_1, \ldots, X_n]] \) that maps \( X_i \) onto \( c_i + X_i \) for each \( i \).

The paper is divided into three sections. Theorem A is used primarily in Section 1, where questions regarding the existence and uniqueness of \( R \)-homomorphisms of \( R[[X_1, \ldots, X_n]] \) into \( R[[Y_1, \ldots, Y_m]] \) are considered. In particular, if \( g_1, g_2, \ldots, g_n \in R[[Y_1, \ldots, Y_m]] \), does there exist an \( R \)-homomorphism \( \phi : R[[X_1, \ldots, X_n]] \to R[[Y_1, \ldots, Y_m]] \) such that \( \phi(X_i) = g_i \) for each \( i \).

* The work of this author was partially supported by NSF Grant 76–06591.
each $i$? If such a homomorphism exists, is it unique? While necessary and sufficient conditions for the existence of such a homomorphism are not as definitive as might be desired, we show that there are some strong results in relation to the question of existence (Proposition 1.1, Theorem 1.4); moreover, if such a $\phi$ exists, it is unique (Theorem 1.2).

In [6, (5.7)], it is shown that if $\psi$ is an $R$-automorphism of $R[[X_1, \ldots, X_n]]$ and if $f_1, \ldots, f_n$ are elements of $R[[X_1, \ldots, X_n]]$ such that $(f_1, \ldots, f_n) = ((\psi(X_i))_{i=1}^n)$, then there exists an $R$-automorphism of $R[[X_1, \ldots, X_n]]$ that maps $X_i$ to $f_i$ for each $i$. In Section 3 we consider analogous questions for the more general situation of an $R$-homomorphism of $R[[X_1, \ldots, X_n]]$ into $R[[Y_1, \ldots, Y_m]]$. Specifically, if $\phi$ and $\psi$ are $R$-homomorphisms of $R[[X_1, \ldots, X_n]]$ into $R[[Y_1, \ldots, Y_m]]$ for which $(\phi(X_i))_{i=1}^n = (\psi(X_i))_{i=1}^n$, we show that $\phi$ is surjective (an isomorphism) if and only if $\psi$ is surjective (an isomorphism) (Theorems 3.4 and 3.5). Moreover, we provide an example to show that $\psi$ need not be injective if $\phi$ is injective.

Following [2], we denote by $I_c(R)$ the set of elements $b$ of the ring $R$ for which there exists an $R$-homomorphism $\phi : R[[X]] \to R$ such that $\phi(X) = b$. If $R$ is a domain and if $b_1, \ldots, b_n$ are elements of $I_c(R)$, then it is shown that $R$ is a complete Hausdorff space in its $(b_i)$-adic topology for each $i$ and that $R$ is complete in its $(b_1, \ldots, b_n)$-adic topology. Whether or not $\bigcap_{k=1}^\infty (b_1, \ldots, b_n)^k = (0)$ is an open question. Much of Section 2 is devoted to a proof that, for $n = 2$, the equality $\bigcap_{k=1}^\infty (b_1, b_2)^k = (0)$ holds if $b_1$ generates a primary ideal of $R$ for some positive integer $j$ (Theorem 2.10).

Before proceeding further, we introduce some simplifying notation. All rings considered in this paper are assumed to be commutative and to contain an identity element. We use $R^{(n)}$ to denote the formal power series ring $R[[X_1, \ldots, X_n]]$ in $n$ indeterminates over $R$. In considering $R$-homomorphisms of $R^{(n)}$ into $R^{(m)}$, if it seems necessary to distinguish between the two sets of indeterminates involved, then we normally use $\{X_i\}_{i=1}^n$ to denote the first set of indeterminates, and $\{Y_j\}_{j=1}^m$ to denote the second set. We use the symbol $\mathcal{X}$ to denote the ideal of $R[[X_1, \ldots, X_n]]$ generated by $\{X_i\}_{i=1}^n$; if two power series rings $R[[X_1, \ldots, X_n]]$ and $R[[Y_1, \ldots, Y_m]]$ are under consideration, then we use $\mathcal{X}_1$ and $\mathcal{X}_2$ to denote the ideals generated by $\{X_i\}_{i=1}^n$ and $\{Y_j\}_{j=1}^m$, respectively. If $A$ is an ideal of the ring $R$, then we refer to the topological ring $(R, A)$ if $R$ is considered as a topological ring under its $A$-adic topology. We use the symbols $\omega$, $\omega_0$, and $\mathbb{Z}$ to denote the sets of positive integers, nonnegative integers, and integers, respectively.

If $f \in R^{(n)}$, then $f$ is uniquely expressible in the form $\sum_{j \in \omega_0} f_j$, where for each $j$, $f_j \in R[[X_1, \ldots, X_n]]$ is $0$ or a homogeneous polynomial (that is, a form) of degree $j$. We call $\sum_{j \in \omega_0} f_j$ the homogeneous decomposition of $f$, and $f_j$ is called the $j$th homogeneous component of $f$. 
1. Existence and Uniqueness of $R$-Homomorphisms of $R((n))$ into $R((m))$

Assume that $g_1, \ldots, g_n \in R((m))$. In this section we consider the questions of existence and uniqueness of an $R$-homomorphism of $R((n))$ into $R((m))$ that maps $X_i$ to $g_i$ for each $i$. Results similar to those of Section 1 have also been obtained by the Stellenbosch Algebra Group. Our first result represents a technical extension of Theorem 4.1 of [6].

**Proposition 1.1.** Assume that $g_1, \ldots, g_n \in R((m)) = R[[Y_1, \ldots, Y_m]]$ and let $c_i$ be the constant term of $g_i$. Let $C$ be the ideal of $R$ generated by $\{c_i\}_{i=1}^n$, and assume that $(R, C)$ is a Hausdorff space. Then there exists an $R$-homomorphism $\phi$ of $R((n)) = R[[X_1, \ldots, X_n]]$ into $R((m))$ such that $\phi(X_i) = g_i$ for each $i$ if and only if $(R, C)$ is complete. Moreover, if such a $\phi$ exists, it is the unique $R$-homomorphism of $R((n))$ into $R((m))$ that maps $X_i$ onto $g_i$ for each $i$.

**Proof.** We remark that Theorem 4.1 of [6] is precisely the case of Proposition 1.1 where $R((n)) = R((m))$—that is, where $\phi$ is an $R$-endomorphism of $R((n))$. Assume first that $(R, C)$ is complete. Theorem 4.1 implies that there exists an $R$-endomorphism $\psi$ of $R((m+n)) = R[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]]$ such that $\psi(X_i) = g_i$ and $\psi(Y_j) = 0$ for all $i$ and $j$. If $\pi$ denotes the canonical projection of $R((m+n)) = R[[Y_1, \ldots, Y_m]][[X_1, \ldots, X_n]]$ onto $R[[Y_1, \ldots, Y_m]]$, then it follows that the restriction of $\pi \circ \psi$ to $R[[X_1, \ldots, X_n]]$ is an $R$-homomorphism into $R[[Y_1, \ldots, Y_m]]$ that maps $X_i$ onto $g_i$ for each $i$.

For the converse, assume that $\phi$ is an $R$-homomorphism of $R((n))$ into $R((m+n))$ such that $\phi(X_i) = g_i$ for each $i$, and let $\pi$ denote the canonical projection of $R((m+n))$ onto $R[[X_1, \ldots, X_n]]$. Then $\phi \circ \pi$ is an $R$-endomorphism of $R((m+n))$ such that $(\phi \circ \pi)(X_i) = g_i$ and $(\phi \circ \pi)(Y_j) = 0$ for all $i$ and $j$. Therefore $(R, C)$ is complete by Theorem 4.1 of [6]. Uniqueness of $\phi$ follows by the same reasoning: if $\phi_1$ and $\phi_2$ are $R$-homomorphisms of $R((n))$ into $R((m))$ that agree on $\{X_i\}_{i=1}^n$, then $\phi_1 \circ \pi$ and $\phi_2 \circ \pi$ are $R$-endomorphisms of $R((m+n))$ that agree on the indeterminates $X_i$ and $Y_j$. Theorem 4.1 implies that $\phi_1 \circ \pi = \phi_2 \circ \pi$, and since $\pi$ restricts to the identity mapping on $R((n))$, it follows that $\phi_1 = \phi_2$.

Theorem A allows us to extend the proof in Proposition 1.1 that an $R$-endomorphism $\phi$ of $R((n))$ into $R((m))$ is uniquely determined by $\phi(X_1), \ldots, \phi(X_n)$, without regard to the topology on $R$ induced by the ideal generated by the constant terms of $\phi(X_1), \ldots, \phi(X_n)$.

**Theorem 1.2.** Assume that $\phi_1$ and $\phi_2$ are $R$-homomorphisms of $R((n))$ into $R((m))$ such that $\phi_1(X_i) = \phi_2(X_i)$ for each $i$. Then $\phi_1 = \phi_2$.

**Proof.** Let $\phi_1(X_i) = \phi_2(X_i) = g_i$ for each $i$, let $c_i$ be the constant term of $g_i$, and let $f_i = g_i - c_i$. Theorem A implies that there exists an $R$-
automorphism \( \tau \) of \( R^{(m)} \) such that \( \tau(X_i) = X_i + c_i \) for each \( i \). Therefore
\[ \tau^{-1}(X_i) = X_i - c_i, \]
and \( (\phi_1 \circ \tau^{-1})(X_i) = f_i = (\phi_2 \circ \tau^{-1})(X_i) \) for each \( i \). Since each \( f_i \) has constant term 0, Proposition 1.1 implies that \( \phi_1 \circ \tau^{-1} = \phi_2 \circ \tau^{-1} \). Consequently, \( \phi_1 = \phi_2 \), as asserted.

The proof of Theorem 1.1 establishes the following corollary.

**Corollary 1.3.** Assume that there exists an \( R \)-homomorphism \( \phi : R^{(n)} \to R^{(m)} \) such that \( \phi(X_i) = g_i = c_i + f_i \), where \( c_i \in R \) and \( f_i \in \mathcal{X}_2 \) for each \( i \). Then \( \phi \) differs from the \( R \)-homomorphism \( \psi : R^{(n)} \to R^{(m)} \) determined by \( \psi(X_i) = f_i \) for each \( i \) by an automorphism of \( R^{(n)} \). Hence \( \phi \) and \( \psi \) are simultaneously injective.

In the case where \( n = m \), Theorem 5.5 of [6] implies that the homomorphisms \( \phi \) and \( \psi \) of Corollary 1.3 are simultaneously surjective. (Moreover, \( \phi \) surjective implies that \( \phi \) is an automorphism in this case.) On the other hand, simultaneous injectivity of \( \phi \) and \( \psi \) was not known for the case \( n = m \). Corollary 1.3 shows, of course, that both these statements are true in the more general case.

Given existence of an \( R \)-homomorphism \( \phi : R^{(n)} \to R^{(m)} \) such that \( \phi(X_i) = g_i = c_i + f_i \) for each \( i \), Theorem 1.2 implies uniqueness of \( \phi \). The next result shows that the question of existence itself depends only upon \( c_1, c_2, \ldots, c_n \).

**Theorem 1.4.** Assume that \( g_1, g_2, \ldots, g_n \in R^{(m)} \) and that \( c_i \) is the constant term of \( g_i \). The following conditions are equivalent.

1. There exists an \( R \)-homomorphism \( \phi : R^{(n)} \to R^{(m)} \) such that \( \phi(X_i) = g_i \) for each \( i \).
2. For each \( i \) between 1 and \( n \), there exists an \( R \)-homomorphism \( \phi_i : R[[X_i]] \to R \) such that \( \phi_i(X_i) = c_i \).
3. For each \( i \) between 1 and \( n \), there exists an \( R \)-automorphism of \( R[[X_i]] \) that maps \( X_i \) to \( X_i + c_i \).

**Proof.** Assume that (1) is satisfied. The mapping \( \sigma \) of \( R^{(m)} \) into \( R \) that sends each element of \( R^{(m)} \) to its constant term is an \( R \)-homomorphism, and hence the composite \( \sigma \circ \phi : R^{(n)} \to R \) is also an \( R \)-homomorphism. Moreover, the restriction \( \phi_i \) of \( \sigma \circ \phi \) to \( R[[X_i]] \) is an \( R \)-homomorphism that maps \( X_i \) to \( c_i \) for each \( i \) between 1 and \( n \).

Theorem A shows that (2) implies (3). If (3) is satisfied, then we prove first that there exists an \( R \)-automorphism of \( R[[X_1, \ldots, X_n]] \) that maps \( X_i \) to \( X_i + c_i \) for each \( i \). For \( R[[X_1]] \), this is clear. We assume the existence of an \( R \)-automorphism \( \tau \) of \( R[[X_1, \ldots, X_k]] \), where \( 1 \leq k < n \), such that \( \tau(X_i) = X_i + c_i \). Regarding \( R[[X_1, \ldots, X_{k+1}]] \) as \( R[[X_1, \ldots, X_k]][[X_{k+1}]] \), it follows that there exists an \( R \)-automorphism \( \tau^* \) of \( R[[X_1, \ldots, X_{k+1}]] \) such that
\( \tau^*(X_i) = X_i + c_i \) for each \( i \) between 1 and \( k \), while \( \tau^*(X_{k+1}) = X_{k+1} \). On the other hand, since \( R[[X_1, \ldots, X_{k+1}]] = R[[X_{k+1}]] R[[X_1, \ldots, X_k]] \), condition (3) implies the existence of an \( R \)-automorphism \( \mu \) of \( R[[X_1, \ldots, X_{k+1}]] \) such that \( \mu(X_{k+1}) = X_{k+1} + c_{k+1} \), while \( \mu(X_i) = X_i \) for \( 1 \leq i \leq k \). The composite \( \mu \circ \tau^* \) is then an \( R \)-automorphism of \( R[[X_1, \ldots, X_{k+1}]] \) such that \( (\mu \circ \tau^*)(X_i) = X_i + c_i \) for each \( i \) between 1 and \( k + 1 \). By induction it follows that there exists an \( R \)-automorphism \( \psi \) of \( R[[X_1, \ldots, X_n]] \) such that \( \psi(X_i) = X_i + c_i \) for each \( i \). Proposition 1.1 implies, moreover, that there exists an \( R \)-homomorphism \( \sigma : R((t)) \to R((m)) \) such that \( \sigma(X_i) = g_i - c_i \) for each \( i \). The composite \( \sigma \circ \psi \) is then an \( R \)-homomorphism of \( R((n)) \) into \( R((m)) \) such that \( (\sigma \circ \psi)(X_i) = g_i \) for each \( t \).

We remark that the equivalence of (2) and (3) of Theorem 1.4, as well as the implication (1) \( \Rightarrow \) (2), is established in Theorem D of [2]. The proof in Theorem 1.4 that (3) implies (1) shows that if for each \( i \) there exists an automorphism of \( R[[X_i]] \) that maps \( X_i \) to \( g_i \), then there exists an \( R \)-automorphism of \( R[[X_1, \ldots, X_n]] \) that maps \( X_i \) to \( g_i \) for each \( i \); the converse of this statement can be obtained from Corollary 1.3 and from Theorem 3.5 of [6].

2. THE IDEAL \( I_c(R) \)

Following [2], we denote by \( I_c(R) \) (or simply \( I_c \) if the ring \( R \) is clear from the context) the set of elements \( b \) of \( R \) for which there exists an \( R \)-homomorphism \( \phi : R[[X]] \to R \) such that \( \phi(X) = b \). Eakin and Sathaye prove in Theorem E of [2] that \( I_c(R) \) is an ideal of \( R \). It follows from Lemma 5.1 of [7] that \( I_c(R) \) is contained in the Jacobson radical \( J(R) \) of \( R \), and results from Section 4 of [7] show that \( I_c(R) \) contains each element \( b \) of \( R \) such that \( R \) is a complete Hausdorff space in its \((b)\)-adic topology. Conversely, if \( b \in I_c(R) \), then it is not difficult to show that \( R \) is complete in its \((b)\)-adic topology. Thus, if the ring \( R \) has the following property (\( P \)), then \( I_c(R) \) is the set of elements \( b \in R \) for which \( (R, (b)) \) is a complete Hausdorff space.

\[
(P) : \bigcap_{n=1}^{\infty} (b^n) = (0) \quad \text{for each } b \in I_c(R).
\]

O'Malley proves in Section 5 of [7] that the class of rings with property (\( P \)) includes the classes of Noetherian rings and semisimple rings. Moreover, he shows that if \( b \) is a regular element of \( R \) and if there exists an \( R \)-automorphism \( \phi \) of \( R[[X]] \) such that \( \phi(X) \) has constant term \( b \), then \( \bigcap_{n=1}^{\infty} (b^n) = 0 \). Theorem 1.4 shows that this statement is true for \( \phi \) an \( R \)-endomorphism of \( R[[X]] \), and hence, for a regular element \( b \) of \( R \), the relation \( b \in I_c(R) \) holds if and only if \( (R, (b)) \) is a complete Hausdorff space.
Thus the class of rings with property \((P)\) includes the class of integral domains. In addition, a Hilbert ring also has property \((P)\), for in such a ring, the Jacobson radical and the nilradical coincide.

Not all rings have property \((P)\), however. In [4], Gilmer gave an example of a ring \(R\) containing an element \(r\) such that \(r \in I_c(R)\), but \(\bigcap_{n=1}^{\infty} (r^n) \neq (0)\). While the condition that \(R\) is complete in its \((b)\)-adic topology is necessary in order that \(b \in I_c(R)\), it is not sufficient. For example, if \(b\) is a nonzero idempotent of \(R\), then \((R, (b))\) is complete, but \(b\) is not in \(I_c(R)\) since \(0\) is the only idempotent in the Jacobson radical of \(R\). For a different kind of example, we note that a slight modification of an example of Fields in [3, p. 433] yields an example of a ring \(R\) containing an element \(y \in J(R)\) such that \(R\) is complete in its \((y)\)-adic topology, while the element \(f(x) = y - X\) of \(R[[X]]\) is a zero divisor. Thus, since \(X\) is a regular element of \(R[[X]]\), there exists no \(R\)-automorphism of \(R[[X]]\) that maps \(X\) to \(f(X)\), and hence \(y \notin I_c(R)\). Moreover, since \(y \in J(R) \setminus \{0\}\), the ideal \((y)\) is not idempotent.

As stated in [2], it would be desirable to have an intrinsic characterization of the elements of \(I_c(R)\); the existence of a useful characterization seems doubtful. A result in this direction, however, follows from Theorem 3.2 of [4]:

The element \(b\) of \(R\) is in \(I_c(R)\) if and only if \(R[[X]]\) is the (group-theoretic) direct sum of \(R\) and \((b + X)\), the ideal of \(R[[X]]\) generated by \(b + X\).

The preceding result generalizes to the case of several variables [6, (5.8)]: there exists an \(R\)-automorphism of \(R^{(n)}\) that maps \(X_i\) to \(g_i\) for each \(i\), if and only if \(R^{(n)} = R \oplus B\), where \(B\) is the ideal of \(R^{(n)}\) generated by \(\{g_i\}_{i=1}^{n}\).

Of the classes of rings with property \((P)\) already mentioned, the classes of Noetherian rings, semisimple rings, and Hilbert rings have the property that \(\bigcap_{k=1}^{\infty} (b_1, b_2, ..., b_n)^k = (0)\) for each finite subset \(\{b_i\}_{i=1}^{n}\) of \(I_c\); whether the class of integral domains has the same property is an open question. We note that this same question arises in [11] and, as observed there, an affirmative answer to the question would have significantly reduced the work required in [11]. A related question—that of whether \(\bigcap_{k=1}^{\infty} (b_1, b_2, ..., b_n)^k = (0)\) for each finite set \(\{b_i\}_{i=1}^{n}\) of regular elements of \(I_c\)—has a negative answer. This statement follows from the next result.

**Proposition 2.1.** Assume that \(\{X_i\}_{i=1}^{n}\) is a finite set of indeterminates over \(R\) and that \(A = (f_1, ..., f_m)\) is a finitely generated ideal of \(R[[X_1, ..., X_n]]\).

(1) \(I_c(R[X_1, ..., X_n])\) is the nilradical of \(R[X_1, ..., X_n]\).

(2) \(I_c(R^{(n)}) = I_c(R) + X\).

(3) \(I_c(R^{(n)}/A) \cong (I_c(R), \{X_i\}_{i=1}^{n}, A)/A\).

**Proof.** The first statement follows because the Jacobson radical of
The inclusion $\mathcal{I} \subseteq I_c(R((n)))$ follows from the fact that $(R((n)), (X_i))$ is a complete Hausdorff space for each $i$. If $r \in I_c(R)$, then there exists an $R$-automorphism $\phi$ of $R[[X]]$ such that $\phi(X) = r + X$, and $\phi$ can be extended to an automorphism $\phi^*$ of $R[[X]][[X_1, \ldots, X_n]] = R((n))[[X]]$ that maps $X_i$ to $X_i$ for each $i$. Thus $\phi^*$ is an $R((n))$-automorphism of $R((n))[[X]]$ that maps $X$ to $r + X$, and Theorem 1.4 implies that $r \in I_c(R((n)))$. Therefore $I_c(R) \subset I_c(R((n)))$. To prove the reverse containment, we need only prove that each element $b$ of $I_c(R((n))) \cap R$ is in $I_c(R)$. This follows at once from Theorem 1.4: since $b \in I_c(R((n)))$, there exists an $R((n))$-homomorphism $\psi$ of $R((n))[[X]] \to R((n))$ such that $\psi(X) = b$; thus $\psi$ is an $R$-homomorphism of $R[[X_1, \ldots, X_n]]$ into $R[[X_1, \ldots, X_n]]$ that maps $X$ to $b$, and Theorem 1.4 implies that $b \in I_c(R)$.

To prove (3), consider first an element $r$ of $I_c(R)$. As shown in the proof of (2), there exists an $R((n))$-automorphism $\phi^*$ of $R((n))[[X]]$ such that $\phi^*(X) = X + r$. Since $\phi^*(f_i) = f_i$ for each $i$, $\phi^*$ maps the ideal $A[[X]] = A \cdot R((n))[[X]]$ of $R((n))[[X]]$ generated by $\{f_i\}_{i=1}^m$ onto itself. Hence $\phi^*$ induces an automorphism of $(R((n))/A)[[X]] \cong R((n))/[[X]]/A[[X]]$ that is the identity on $R((n))/A$ and maps $X$ to $(r + A) + X$ (see [4]). Consequently, $(I_c(R) + A)/A \subseteq I_c(R((n))/A)$. To prove that each $X_i + A$ is in $I_c(R((n))/A)$, let $\{Y_i\}_{i=1}^n$ be a set of indeterminates over $R((n))$. Proposition 1.1 implies that there exists an $R((n))$-endomorphism $\psi$ of $R((n))[[Y_1, \ldots, Y_n]]$ such that $\psi(Y_i) = Y_i - X_i$ for each $i$, and it follows from Corollary 1.3 that $\psi$ is an automorphism. As above, $\psi$ maps $A[[Y_1, \ldots, Y_n]]$ onto itself, and hence $\psi$ induces an automorphism of $(R((n))/A)[[Y_1, \ldots, Y_n]]$ that maps $Y_i$ to $Y_i - (X_i + A)$ for each $i$. Therefore each $X_i + A$ is in $I_c(R((n))/A)$, and this establishes the containment relation $(I_c(R), A, \{X_i\}_{i=1}^n)/A \subseteq I_c(R((n))/A)$.

Let $S$ be a ring such that $\cap_{n=1}^\infty(s^n) \neq (0)$ for some element $s \in I_c(S)$. By Proposition 2.1, $S$ and $s + X$ are in $I_c(S[[X]])$. Moreover, $s + X$ is regular in $S[[X]]$ since $X$ is regular in $S[[X]]$ and there exists an $S$-automorphism of $S[[X]]$ that maps $X$ onto $s + X$. On the other hand, $s \in (X, s + X)$ so that $\cap_{n=1}^\infty(X, s + X)^n \neq (0)$. This is the same type of example that Eakin and Sathaye use in [2] to show that a topological ring $(R, (c))$ and $(R, (d))$ are complete Hausdorff spaces (see [5, 11]). The connection between the two questions is quite natural, because as observed previously, for a regular element $b$ of a ring $R$, the relation $b \in I_c(R)$ holds if and only if $(R, (b))$ is a complete Hausdorff space. Thus, an integral domain $D$ has the property that $\cap_{k=1}^\infty(b_1, \ldots, b_n)^k = (0)$ for each finite subset $\{b_i\}_{i=1}^n$ of $I_c(D)$ if and only if $(D, (b_i))$ a complete Hausdorff space for each $i$ implies that $(D, (b_1, \ldots, b_n))$ is a Hausdorff space.
difficult to determine. Although we conjecture that the question has a negative answer in general, even for the case \( n = 2 \), it is true that if either \( b_i^k \) or \( b_i^k \) generates a primary ideal of \( D \) for some \( k \in \omega \), then \( \bigcap_{n=1}^{\infty} (b_1, b_2)^n = (0) \). In order to prove this statement, we require several preliminary results.

**Lemma 2.2.** Let \( b(X) = \sum_{i=0}^{\infty} b_i X^i \in R[[X]] \). If \( b_0 \) is regular in \( R \), then \( (b(X)) \) is closed in the \((X)-adic topology on \( R[[X]] \).

Proof. Let \( c(X) = \sum_{i=0}^{\infty} c_i X^i \in \bigcap_{n=1}^{\infty} (b(X), X^n) \). Then, for each \( n \geq 1 \), there exists a polynomial \( f_n(X) = f_{n,0} + f_{n,1}X + \cdots + f_{n,n-1}X^{n-1} \in R[X] \) and a power series \( g_n(X) \in R[[X]] \) such that \( c(X) = f_n(X) b(X) + X^n g_n(X) \). If \( S = R[1/b_0] \), then \( b(X) \) is a unit of \( S[[X]] \) and

\[
c(X) b^{-1}(X) = f_n(X) + X^n g_n(X) b^{-1}(X). \tag{1}
\]

Since the coefficients of \( 1, X, \ldots, X^{n-1} \) on the right hand side of (1) are determined by \( f_{n,0}, f_{n,1}, \ldots, f_{n,n-1} \), it follows that for \( i, j \in \omega, i \leq j, f_{i,0} = f_{j,0}, f_{i,1} = f_{j,1}, \ldots, f_{i,i-1} = f_{j,i-1} \). Let \( f(X) = f_{1,0} + \sum_{n=1}^{\infty} f_{n,n+1,n} X^n \); we prove that \( c(X) = f(X) b(X) \). It suffices to prove that \( c(X) - f(X) b(X) \in (X) \) for each \( m \in \omega \). Clearly, \( c(X) - f_m(X) b(X) \equiv 0 \pmod{X^m} \), and moreover, since \( f_{m,0} = f_{1,0}, f_{m,1} = f_{2,1}, \ldots, f_{m,m-2} = f_{m-1,m-2} \), we have \( f(X) \equiv f_m(X) \pmod{X^m} \). Consequently, \( c(X) - f(X) b(X) \equiv 0 \pmod{X^m} \), which is what we wished to prove.

**Lemma 2.3.** If \( b \) is a regular element of the ring \( R \), then \( (X-b) R[[X]] \cap R = \bigcap_{n=1}^{\infty} (b^n R) \).

Proof. Let \( S = R[1/b] \); then \( X-b \) is invertible in \( S[[X]] \) and \( (X-b)^{-1} = -b^{-1}[1 + X/b + X^2/b^2 + \cdots] \). Thus, if \( d \in R \), then \( d \in (X-b) R[[X]] \) if and only if \( d(X-b)^{-1} \in R[[X]] \)—that is, if and only if \( db^{-n} \in R \) for all \( n \in \omega \), and hence if and only if \( d \in \bigcap_{n=1}^{\infty} (b^n R) \).

The next proposition provides, in theory, a method of constructing an integral domain \( J \) containing elements \( s \) and \( t \) such that \( (J, (s)) \) and \( (J, (t)) \) are complete Hausdorff spaces, but \( (J, (s, t)) \) is not Hausdorff.

**Proposition 2.4.** Let \( b \) and \( d \) be elements of the ring \( R \) with the following properties:

1. \( d \) and \( b-d \) are regular elements of \( R \).
2. \( X-d \) generates a prime ideal of \( R[[X]] \).
3. \( (R, (b)) \) is a complete Hausdorff space.
4. \( \bigcap_{n=1}^{\infty} [(b, d) R]^n \not\subseteq (X-d) R[[X]] \).

Then if \( J = R[[X]]/(X-d) \), \( J \) is an integral domain that is a complete
Hausdorff space in its \((X,X-d)/(X-d)\)- and \((X-b,X-d)/(X-d)\)-adic topologies, but not Hausdorff in the \((X,X-d)/(X-d) + (X-b,X-d)/(X-d) = (X,b,X-d)/(X-d)\)-adic topology.

**Proof.** Since \(R[[X]]\) is complete in the \((X)\)-adic topology, it follows that \(J\) is complete in the \((X,X-d)/(X-d)\)-adic topology, and since \((X-d)\) is closed in the \((X)\)-adic topology (Lemma 2.2), it follows that \(J\) is a Hausdorff space in the \((X,X-d)/(X-d)\)-adic topology. Since \((R, (b))\) is a complete Hausdorff space, Theorems 4.12 and 4.18 of [7] show that there exists an \(R\)-automorphism \(\phi\) of \(R[[X]]\) that maps \(X\) to \(X-b\). Clearly, \(\phi\) is a homeomorphism of \((R[[X]], (X)) \) onto \((R[[X]], (X-b))\), and hence \(R[[X]]\) is a complete Hausdorff space in its \((X-b)\)-adic topology. Moreover, since \((X+b-d)\) is closed in the \((X)\)-adic topology (Lemma 2.2), the image \((X-d)\) of \((X+b-d)\) under \(\phi\) is closed in the \((X-b)\)-adic topology, and therefore, as before, \(J\) is a complete Hausdorff space in its \((X,b,X-d)/(X-d)\)-adic topology. Finally, since \(((b,d)R[[X]])^n + (X-d) = (X,b,X-d)^n + (X-d)\) for each \(n \in \omega\), we have

\[
\bigcap_{n=1}^{\infty} \left( (b,d)R[[X]] \right)^n + (X-d)/(X-d) = \bigcap_{n=1}^{\infty} \left( (X,b,X-d)/(X-d) \right)^n.
\]

Therefore, since \(\bigcap_{n=1}^{\infty} ((b,d)R)^n \not\subseteq (X-d)R[[X]]\), it follows that \((0) \neq \bigcap_{n=1}^{\infty} ((b,d)R)^n + (X-d)/(X-d) \subseteq \bigcap_{n=1}^{\infty} ((X,b,X-d)/(X-d))^n\). Thus \(J\) is not Hausdorff in the \((X,b,X-d)/(X-d)\)-adic topology.

**Lemma 2.5.** Let \(b\) be an element of the ring \(R\), let \(h(X) = \sum_{i=0}^{\infty} h_iX^i \in R[[X]]\), and for each \(n \in \omega\), let \(h_n(X) = h_0 + h_1X + \cdots + h_{n-1}X^{n-1}\). If \(h(X) \in (X-b)\), then \(h_n(b) \in (b^n)\) for each \(n \in \omega\); if \(b\) is regular in \(R\), then the converse also holds.

**Proof.** Assume that \(h(X) = (X-b) \sum_{i=0}^{\infty} g_iX^i \in (X-b)\). Then, for each \(n \in \omega\), \(h_n(X) = (X-b)(g_0 \mid g_1X \mid \cdots \mid g_{n-1}X^{n-1} - g_{n-1}X^n)\), so that \(h_n(b) = -g_{n-1}b^n \in (b^n)\). Conversely, if \(b\) is regular in \(R\) and if, for each \(n \in \omega\), \(h_n(b) = -s_{n-1}b^n \in (b^n)\), where \(s_{n-1} \in R\), then we prove that \(h(X) = (X-b) \sum_{i=0}^{\infty} s_iX^i\). Thus, \(h(X) = -bs_0\), and \(h_n = s_{n-1} - bs_n\) for each \(n \in \omega\). It is clear that \(h_0 = -bs_0\), and if we assume that \(h_i = s_{i-1} - bs_i\) for \(i = 1, \ldots, n-1\), then \(h_{n+1}(b) = h_0 + h_1b + \cdots + h_nb^n = -s_nb^{n+1}\). Substituting, we have \(-s_0b^{n+1} = -bs_0 + (s_0 - bs_1)b + \cdots + (s_{n-2} - bs_{n-1})b^{n-1} + h_nb^n = -b^n s_{n-1} + h_nb^n\). Since \(b^n\) is regular in \(R\), it follows that \(h_n = s_{n-1} - bs_n\).

The proof of the next result is straightforward and will be omitted.
LEMMA 2.6. If $b_1, \ldots, b_n$ are regular elements of the ring $R$ and if the element $d$ of $R$ is such that $(b_i):(d) = (b_i)$ for each $i$, then $(b_1b_2 \cdots b_n):(d) = (b_1b_2 \cdots b_n)$.

In the terminology of [13, p. 223] Lemma 2.6 states that if $d$ is prime to $b_1b_2 \cdots b_n$ if $d$ is prime to each $b_i$. We use the following special case of Lemma 2.6: if $b$ is a regular element of $R$ and if $d$ is prime to $b^j$ for some $j \in \omega$, then $d$ is prime to $b^k$ for each $k \in \omega$.

LEMMA 2.7. Let $b, d \in R$ and suppose that $\bigcap_{n=1}^{\infty} (b^n) = (0)$, while $\bigcap_{n=1}^{\infty} (b, d)^n \neq (0)$. If $(b')$ is primary, then $((b')):(d) = (b')$.

Proof. Since $\bigcap_{n=1}^{\infty} (b^n) = (0)$ and $\bigcap_{n=1}^{\infty} (b, d)^n \neq (0)$, it follows that $d^k \notin (b')$ for any $k, p \in \omega$. In particular, $d^k \notin (b')$ for all $k \in \omega$. Thus, if $t \in ((b')):(d)$, then $td \in (b')$, and since $(b')$ is primary, it follows that $t \in (b')$.

PROPOSITION 2.8. Let $b$ and $d$ be regular elements of the ring $R$ and suppose that $(R, (b))$ is a complete Hausdorff space. If $((b')):(d)$ for a fixed $j \in \omega$, then $(X - d)$ is closed in $(R[[X]], (X, b))$.

Proof. Since the $(b)$-adic topology on $R$ is equivalent to the $(b')$-adic topology, and since the ideals $(X, b)$ and $(X, b')$ induce equivalent topologies on $R[[X]]$, we assume without loss of generality that $j = 1$, that is, we assume $d$ is prime to $b$. Let $t(X) = \sum_{i=0}^{\infty} t_i X^i \in \bigcap_{n=1}^{\infty} (X - d, (X, b)^n) = \bigcap_{n=1}^{\infty} (X - d, X^n, b^n)$. Then, for each $n \in \omega$, $t(X) = (X - d) \sum_{i=0}^{\infty} r_{n,i}X^i + X^n \sum_{i=0}^{\infty} s_{n,i}X^i + b^n \sum_{i=0}^{\infty} u_{n,i}X^i$. Regrouping, it follows that, for each $n \in \omega$,

$$t(X) = (-dr_{n,0} + b^nu_{n,0}) + \sum_{i=1}^{n-1} (r_{n,i-1} - dr_{n,i} + b^nu_{n,i})X^i + X^n h_n(X),$$

where $h_n(X) \in R[[X]]$. Thus, $t_0 = -dr_{n,0} + b^nu_{n,0}$ for $n \geq 1$ and, for $k \in \omega$, $t_k = r_{n,k-1} - dr_{n,k} + b^nu_{n,k}$ for $n \geq k + 1$. We show by induction that 

$$\{r_{p,i}\}_{p=i+1}^{\infty}$$

is a Cauchy sequence of $(R, (b))$ for each $i \in \omega$. Let $(b^m)$ be a neighborhood of $(0)$. Then $t_0 = -dr_{q,0} + b^mu_{q,0} = -dr_{m,0} + b^mu_{m,0}$ for $q \geq m$. Therefore, $d(r_{q,0} - r_{m,0}) \in (b^m)$ and from Lemma 2.6, we have that $r_{q,0} - r_{m,0} \in (b^m)$. Thus, 

$$\{r_{p,0}\}_{p=1}^{\infty}$$

is a Cauchy sequence of $(R, (b))$ and $r_{q,0} - r_{m,0} \in (b^m)$ for $q \geq m, m \in \omega$. Fix $k \geq 1$ and suppose that 

$$\{r_{p,k-1}\}_{p=k}^{\infty}$$

is a Cauchy sequence of $(R, (b))$ and that $r_{q,k-1} - r_{m,k-1} \in (b^m)$ for $q \geq m \geq k$. Fix $n \geq k + 1$ and let $(b^m)$ be a neighborhood of $(0)$. Then $t_k = r_{q,k-1} - dr_{q,k} + b^nu_{q,k} = r_{n,k-1} - dr_{n,k} + b^nu_{n,k}$ for $q \geq n$, and therefore 

$$d(r_{n,k} - r_{q,k}) = b^q(b^q - b^nu_{q,k} - u_{n,k}) + (r_{q,k-1} - r_{m,k-1}) \in (b^m).$$

Again Lemma 2.6 implies that $r_{q,k} - r_{n,k} \in (b^m)$ and therefore 

$$\{r_{p,k}\}_{p=k+1}^{\infty}$$

is a Cauchy sequence of $(R, (b))$.

As $(R, (b))$ is complete, we let $\lim_{p} r_{p,i} = u_i$ for each $i \in \omega$.,
and we consider the sequences \( \{t_0\} = \{-dr_{p,0} + b^pu_{p,0}\}_{p=1}^\infty \) and \( \{t_k\} = \{r_{n,k-1} - dr_{n,k} + b^nu_{n,k}\}_{n=1}^\infty, k \geq 1 \). Observe that \( t_0 = \lim_{p} t_0 = \lim_{p}(dr_{p,0} + b^pu_{p,0}) = -du_0 + 0 \) and hence \( t_0 = -du_0 \in (d) \). Similarly, \( t_1 = u_0 - du_1 \), so that \( dt_1 = du_0 - d^2u_1 \) or \( t_0 + dt_1 = -d^2u_1 \in (d^2) \), and by induction, it follows that \( t_0 + t_1d + \cdots + t_md^m = -d^{m+1}u_m \in (d^{m+1}) \) for each \( m \in \omega_0 \). Thus, by Lemma 2.5, \( t(X) \in (X - d) \), which is what we wished to prove.

**Proposition 2.9.** Let \( b \) and \( d \) be elements of the integral domain \( D \) and suppose that \( (D, (b)) \) and \( (D, (d)) \) are complete Hausdorff spaces. If \( (X - d) \) is closed in \( (D[[X]], (X, b)) \), then \( (D, (b, d)) \) is a complete Hausdorff space.

**Proof.** It is straightforward to show that \( (D, (b, d)) \) is complete if \( (D, (b)) \) and \( (D, (d)) \) are complete. Moreover, if \( d = 0 \) or \( b - d = 0 \), then clearly \( \bigcap_{n=1}^\infty (b, d)^n = (0) \). Thus, we may assume that \( d \neq 0 \) and \( b - d \neq 0 \). Since \( (D, (d)) \) is a complete Hausdorff space, there exists a \( D \)-automorphism of \( D[[X]] \) that maps \( X \) onto \( X - d \) ([7, (4.12) and (4.18)]), and therefore, since \( D \) is a domain, \( (X - d) \) is a prime ideal of \( D[[X]] \). By Lemma 2.3, \( (X - d) D[[X]] \cap D = \bigcap_{n=1}^\infty (d^nD) = (0) \), and hence, if \( \lim_{n=1}^{\infty} (b, d)^n D \neq (0) \), then \( \bigcap_{n=1}^\infty (b, d)^n D \not\subseteq (X - d) D[[X]] \). Thus, if \( J = D[[X]]/(X - d) \), then Proposition 2.4 implies that \( J \) is a complete Hausdorff space in its \( (X, X - d)/(X - d) \)- and \( (X - b, X - d)/(X - d) \)-adic topologies, but \( J \) is not Hausdorff in the \( (X, X - b, X - d)/(X - d) \)-adic topology. This last statement is equivalent to the statement that \( (X - d) \) is closed in the \( (X) \) - and \( (X - b) \)-adic topologies, but \( (X - d) \) is not closed in the \( (X, b) \)-adic topology on \( D[[X]] \).

We are now ready to prove the result mentioned before Lemma 2.2. It is an easy consequence of Propositions 2.8 and 2.9.

**Theorem 2.10.** Let \( b \) and \( d \) be elements of the integral domain \( D \) and suppose that \( (D, (b)) \) and \( (D, (d)) \) are complete Hausdorff spaces. If either

1. \( ((b^j) : (d)) = (b^j) \) for some \( j \in \omega \), or
2. \( (b^j) \) is a primary ideal of \( D \) for some \( j \in \omega \),

then \( (D, (b, d)) \) is a complete Hausdorff space. In particular, if \( b \) or \( d \) is a prime element of \( D \), then \( \bigcap_{n=1}^\infty (b, d)^n = (0) \).

**Proof.** It is clear that if either \( b \) or \( d \) is 0, then \( \bigcap_{n=1}^\infty (b, d)^n = (0) \). We assume that \( b \) and \( d \) are nonzero elements of \( D \). If condition (1) is satisfied, then the conclusion follows immediately from Propositions 2.8 and 2.9. On the other hand, if condition (2) is satisfied and \( \bigcap_{n=1}^\infty (b, d)^n \neq (0) \), then by Lemma 2.7, \( ((b^j) : (d)) = (b^j) \). Since condition (1) implies that \( \bigcap_{n=1}^\infty (b, d)^n = (0) \), we have a contradiction.
Remark. As noted previously, Proposition 2.4 provides sufficient conditions for there to exist an integral domain \( J \) containing elements \( s \) and \( t \) such that \( (J, (s)) \) and \( (J, (t)) \) are complete Hausdorff spaces, but \( (J, (s, t)) \) is not a Hausdorff space. In Proposition 2.4, the condition that \( X - d \) generates a prime ideal of \( R[[X]] \) is used to make \( J = R[[X]]/(X - d) \) an integral domain. A sufficient condition in order that \( (X - d) \) be a prime ideal of \( R[[X]] \) is that \( d \) is a prime element of \( R \). (More generally, if \( c \) is a regular, prime element of the commutative ring \( T \) with identity and if \( f(X) \in T[[X]] \) has constant term \( c \), then \( f(X) \) generates a prime ideal of \( T[[X]] \).)

If, however, condition (2) of Proposition 2.4 is replaced by the condition (2') that \( d \) should be a prime element of \( R \), then conditions (1), (2'), and (3) of Proposition 2.4 imply that \( \bigcap_{n=1}^{\infty} (d^n) = \bigcap_{n=1}^{\infty} (b, d)^n \). Thus by Lemma 2.3, \( \bigcap_{n=1}^{\infty} [(b, d) R^n] \subseteq (X - d) R[[X]] \), so that condition (4) of Proposition 2.4 is not satisfied. Therefore, if Proposition 2.4 is to be used to construct an example of a domain with the required properties, then it must be that \( d \) is not a prime element of \( R \).

3. Relations on the Image sets \( \{\phi(X_i)\}_{i=1}^{n} \)

Assume that \( \phi \) and \( \psi \) are \( R \)-homomorphisms of \( R^{((n))} \) into \( R^{((m))} \). In this section we consider how properties of \( \phi \) and \( \psi \) are determined by the image sets \( \{\phi(X_i)\}_{i=1}^{n} \) and \( \{\psi(X_i)\}_{i=1}^{n} \) and by relations between the ideals of \( R^{((m))} \) generated by these image sets. We begin with a result related to Proposition 5.7 of [6] that follows from Theorem 1.4 and the fact that \( I_c(R) \) is an ideal of \( R \).

**Proposition 3.1.** Assume that there exists an \( R \)-homomorphism of \( R^{((n))} \) into \( R^{((m))} \) that maps \( X_i \) to \( g_i \) for each \( i \). If \( f_1, \ldots, f_n \) are elements of the ideal of \( R^{((m))} \) generated by \( \{g_i\}_{i=1}^{n} \), then there exists an \( R \)-homomorphism of \( R^{((n))} \) into \( R^{((m))} \) that maps \( X_i \) to \( f_i \) for each \( i \).

**Proof.** Let \( c_i \) be the constant term of \( g_i \) and \( d_i \) the constant term of \( f_i \) for each \( i \). Each \( c_i \) is in \( I_c(R) \), and hence \( (c_1, \ldots, c_n) \subseteq I_c(R) \). Because \( f_i \) belongs to the ideal of \( R^{((m))} \) generated by \( \{g_i\}_{i=1}^{n} \), it follows that \( d_i \) belongs to \( (c_1, \ldots, c_n) \). Therefore each \( d_i \) is in \( I_c(R) \), and Proposition 3.1 then follows from Theorem 1.4.

Several questions arise in relation to Proposition 3.1. In [6, (5.7)], it is shown that if \( \phi \) is an \( R \)-automorphism of \( R^{((n))} \) and if \( f_1, \ldots, f_n \in R^{((n))} \) are such that \( f_1, \ldots, f_n = (\{\phi(X_i)\}_{i=1}^{n}) \), then there exists an \( R \)-automorphism of \( R^{((n))} \) that maps \( X_i \) to \( f_i \) for each \( i \). We observe that a partial converse of this result is valid, and we consider the more general situation of an \( R \)-homomorphism of \( R^{((m))} \) into \( R^{((m))} \).
PROPOSITION 3.2. Assume that \( \phi \) and \( \psi \) are \( R \)-automorphisms of \( R^{(n)} \) such that \( (\{ \psi(X_i) \}_i^{n}_{i=1}) \subseteq (\{ \phi(X_i) \}_i^{n}_{i=1}) \). Then these two ideals are equal.

Proof. The hypothesis implies that \( \phi^{-1} \psi \) is an \( R \)-automorphism of \( R^{(n)} \) such that \( (\{ \phi^{-1} \psi(X_i) \}_i^{n}_{i=1}) \subseteq \mathcal{J}' \), and in this special case, Theorem 3.5 of [6] implies that \( \mathcal{J}' = (\{ \phi^{-1} \psi(X_i) \}_i^{n}_{i=1}) \). Therefore \( (\{ \phi(X_i) \}_i^{n}_{i=1}) = (\{ \psi(X_i) \}_i^{n}_{i=1}) \), as asserted.

We prove in Theorem 3.4 that if \( \phi \) and \( \psi \) are \( R \)-homomorphisms of \( R^{(n)} \) into \( R^{(m)} \) such that \( (\{ \phi(X_i) \}_i^{n}_{i=1}) = (\{ \psi(X_i) \}_i^{n}_{i=1}) \), then \( \phi \) is surjective if and only if \( \psi \) is surjective. The proof of Theorem 3.4 uses an extension of Theorem 3.5 of [6].

THEOREM 3.3. Let \( \{ g_i \}_i^{n}_{i=1} \) be a subset of \( R^{(m)} \) such that each \( g_i \) has constant term 0, and let \( f_i \) be the first homogeneous component of \( g_i \). Denote by \( \phi \) the \( R \)-homomorphism \( h(X_1, ..., X_n) \to h(g_1, ..., g_n) \) of \( R^{(n)} \) into \( R^{(m)} \). The following conditions are equivalent.

1. \( \phi \) is surjective.
2. \( Rf_1 + ... + Rf_n = RY_1 + ... + RY_m \).
3. \( R[f_1, ..., f_n] = R[Y_1, ..., Y_m] \).
4. The ideal \( \mathcal{J}' \) of \( R^{(m)} \) is generated by \( \{ g_i \}_i^{n}_{i=1} \).

If \( \phi \) is surjective, then \( n \geq m \).

The proof of the equivalence of conditions (1)–(4) is analogous to the proof of Theorem 3.5 of [6], and is therefore omitted. If \( \phi \) is surjective, then the inequality \( n \geq m \) follows from either (2) (see part (4) of [6, (3.1)]) or from (3) [13, p. 39].

THEOREM 3.4. Assume that \( \phi \) and \( \psi \) are \( R \)-homomorphisms of \( R^{(n)} \) into \( R^{(m)} \) such that \( (\{ \phi(X_i) \}_i^{n}_{i=1}) = (\{ \psi(X_i) \}_i^{n}_{i=1}) \). Then \( \phi \) is surjective if and only if \( \psi \) is surjective. Moreover, \( n \geq m \) if \( \phi \) is surjective.

Proof. Let \( \phi(X_i) = g_i \), let \( c_i \) be the constant term of \( g_i \), let \( f_i = g_i - c_i \), and let \( \sigma \) be the \( R \)-homomorphism of \( R^{(n)} \) into \( R^{(m)} \) that maps \( X_i \) to \( f_i \) for each \( i \). By Corollary 1.3, \( \phi \) is surjective if and only if \( \sigma \) is surjective.

Let \( g_i^{(1)} \) be the first homogeneous component of \( g_i \) for each \( i \). If \( \phi \) and hence \( \sigma \) is surjective, then Theorem 3.3 implies that \( RY_1 + ... + RY_m = M = Rg_1^{(1)} + ... + Rg_n^{(1)} \). Hence if \( h_i^{(1)} \) is the first homogeneous component of \( \psi(X_i) = h_i \) for each \( i \), then the equality \( (g_1, ..., g_n) = (h_1, ..., h_n) \) implies that \( Rg_1^{(1)} + ... + Rg_n^{(1)} = M \subseteq Rh_1^{(1)} + ... + Rh_n^{(1)} + (d_1, ..., d_n)M \), where \( d_i \) is the constant term of \( h_i \) for each \( i \). Because each \( d_i \) is in \( J(R) \), it follows that \( M = Rh_1^{(1)} + ... + Rh_n^{(1)} + J(R)M \), and Nakayama's Lemma then implies that \( M = Rh_1^{(1)} + ... + Rh_n^{(1)} \). Applying Corollary 1.3 and Theorem 3.3 again, we conclude that \( \psi \) is surjective if \( \phi \) is surjective.
Since $\sigma$ is surjective if $\phi$ is surjective, the inequality $n \geq m$ follows from Theorem 3.3 in the case where $\phi$ is surjective.

While a surjective $R$-endomorphism of $R^{(n)}$ is necessarily an automorphism [6, (5.5)], it is clear that a surjective $R$-homomorphism of $R^{(n)}$ onto $R^{(m)}$ need not be an isomorphism. On the other hand, Abhyankar in [1] shows that if $L$ is a field and if $n \geq 2$, then there exists an injective $L$-homomorphism of $L^{(n)}$ into $L^{(2)}$, while there does not exist an injective $L$ homomorphism of $L^{(n)}$ into $L^{(1)}$. We can use the method of proof of [1] to show that the analogue of Theorem 3.4 fails for injectivity; thus, for example, we can show that there exist $L$-homomorphisms $\phi$ and $\psi$ of $L[[X_1, X_2, X_3]]$ into $L[[Y_1, Y_2]]$ such that $\phi$ is injective, $((\phi(X_i))_{i=1}^3) = ((\psi(X_i))_{i=1}^3)$, and $\psi$ is not injective. To obtain an example, we use the fact that the quotient field $L((Y_2))$ of $L[[Y_2]]$ has infinite transcendence degree over $L$ and the fact that $L(Y_2L[[Y_2]]) = L((Y_2))$. Thus the algebraically independent subset $\{Y_2\}$ of $Y_2L[[Y_2]]$ can be extended to a transcendence basis for $L((Y_2))$ over $L$. In particular, there exist $f, g \in L[[Y_2]]$ such that the set $\{Y_2, Y_2f, Y_2g\}$ is algebraically independent over $L$. It then follows from [1] that the $L$-homomorphism $\phi : L[[X_1, X_2, X_3]] \to L[[Y_1, Y_2]]$ determined by $\phi(X_1) = Y_1, Y_2, \phi(X_2) = Y_1, Y_2f, \phi(X_3) = Y_1, Y_2g$ is injective. Also $((\phi(X_i))_{i=1}^3) = (Y_1, Y_2)$. It is clear, however, that there exists an $L$-homomorphism $\psi : L[[X_1, X_2, X_3]] \to L[[Y_1, Y_2]]$ such that $((\psi(X_i))_{i=1}^3) = (Y_1, Y_2)$ and $\psi$ is not injective.

We prove next that Theorem 3.4 extends in the natural way to the case where $\phi$ or $\psi$ is an isomorphism.

**Theorem 3.5.** Assume that $\phi$ and $\psi$ are $R$-homomorphisms of $R^{(m)}$ into $R^{(n)}$ such that $((\phi(X_i))_{i=1}^n) = ((\psi(X_i))_{i=1}^n)$. Then $\phi$ is an isomorphism if and only if $\psi$ is an isomorphism. Moreover, $n = m$ if $\phi$ is an isomorphism.

**Proof.** Assume that $\phi$ is an isomorphism. Theorem 3.4 applied to $\phi$ yields $n \geq m$, and applied to $\phi^{-1}$ yields $m \geq n$. Therefore, $n = m$ and we can assume without loss of generality that $\phi$ is an $R$-automorphism and $\psi$ is an $R$-endomorphism of $R^{(n)}$. Since $\phi$ is surjective, Theorem 3.4 implies that $\psi$ is surjective, and hence $\psi$ is an automorphism of $R^{(n)}$ by Theorem 5.5 of [6].

**REFERENCES**