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Stability of BSDEs with Random Terminal Time and Homogenization of Semilinear Elliptic PDEs

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In this paper, we extend the probabilistic method for homogenization of semilinear parabolic PDEs, developed by Buckdahn, Hu, and Peng to the case of elliptic PDEs. First, we give a stability result for BSDEs with random terminal time which are related to elliptic PDEs as shown in Peng (*Stochastics Stochastics Rep.* **37** (1991), 61–74). In the one dimensional case, we also partially relax the monotonicity assumption on the coefficient. Then, we use these stability results for BSDEs with random terminal time to study homogenization of systems of semilinear elliptic PDEs.

1. INTRODUCTION

In [3], Buckdahn, Hu, and Peng study some homogenization properties of semilinear parabolic PDEs by a probabilistic method based upon the nonlinear Feynman–Kac formula developed by Pardoux and Peng in [8]. The key point, in this approach, is to obtain a powerful stability result for backward stochastic differential equations (BSDEs) with fixed terminal time, say T. The proof of the result mentioned above depends heavily on the idea of subdividing the time interval $[0, T]$ into a finite number of small time intervals.

On the other hand, Peng [9], and recently Darling and Pardoux [4], have shown that semilinear elliptic PDEs are related to BSDEs with random terminal time.

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The aim of this paper is to obtain stability results for BSDEs with random terminal time in order to study homogenization of systems of semilinear elliptic PDEs. However, the unboundedness of the terminal time does not allow us to use the same method as in [3]. This is why our proof is based upon an approximation procedure.

The paper is organized as follows. In Section 2, we study the stability of BSDEs in a general framework. In Section 3, we prove some results for BSDEs in the one-dimensional case, for which we can partially relax the monotonicity condition on the coefficient. The last section is devoted to the application to elliptic PDEs.

2. STABILITY OF BSDES WITH RANDOM TERMINAL TIME

2.1. *Preliminaries*

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space carrying a standard d-dimensional Brownian motion $(W_t)_{t \in \mathbb{R}_+}$, and let $(\mathscr{F}_t)$ be the filtration generated by W. We make the usual $\mathbb{P}$-augmentation to $\mathscr{F}_t$ so that $\mathscr{F}_t$ contains all $\mathbb{P}$-null sets. Then the filtration $(\mathscr{F}_t)$ satisfies the usual hypothesis.

Let τ be an $(\mathscr{F}_t)$-stopping time and let α be some real number. $\mathscr{M}^{2,\alpha}(0, \tau; E)$ denotes the Hilbert space consisting of all progressively measurable processes X, with values in the Euclidean space E such that:

$$\|X\|_\alpha^2 := \mathbb{E}\left[\int_O^\tau e^{\alpha s} \|X_s\|^2 \, ds\right] < \infty.$$

Suppose that τ is a finite stopping time and let ξ be an $\mathscr{F}_\tau$-adapted random variable and $f\colon \Omega \times \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k$ such that $f(\cdot, y, z)$ is a progressively measurable process for each $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$. Consider the BSDE with random terminal time

$$-dY(t) = \mathbb{1}_{t \leqslant \tau}(f(t, Y(t), Z(t)) \, dt - Z(t) \, dW_t), \qquad Y(\tau) = \xi,$$

or equivalently,

$$Y(t \wedge \tau) = \xi + \int_{t \wedge \tau}^\tau f(s, Y(s), Z(s)) \, ds - \int_{t \wedge \tau}^\tau Z(s) \, dW_s. \tag{1}$$

BSDE (1) was first introduced by Peng in [9]. We recall the existence and uniqueness result established by Darling and Pardoux in a more general context [4].

Suppose that (ξ, f) satisfy the following assumption:

(A1). There exist constants $C \geqslant 0$, $\gamma \geqslant 0$ and μ such that, $d\mathbb{P} \otimes ds$ *a.e.*,

1. f is uniformly Lipschitz; i.e., $\forall (y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$|f(s, y_1, z_1) - f(s, y_2, z_2)| \leqslant C\,|y_1 - y_2| + \gamma\,\|z_1 - z_2\|;$$

2. f is monotone in y: $\forall z \in \mathbb{R}^{k \times d}$, $\forall (y, y') \in \mathbb{R}^d$,

$$\langle y - y', f(s, y, z) - f(s, y', z) \rangle \leqslant -\mu\,|y - y'|^2;$$

3. $\exists \rho \in \mathbb{R}$, such that $\rho > \gamma^2 - 2\mu$, and

$$\mathbb{E}[e^{\rho\tau}\,|\xi|^2] \leqslant C; \qquad \mathbb{E}\left[\int_0^\tau e^{\rho s}\,|f(s, 0, 0)|^2\,ds\right] \leqslant C.$$

Set $\lambda = (\gamma^2/2) - \mu$.

LEMMA 2.1. *If* (A1) *holds, then the BSDE* (1) *has a unique solution* (Y, Z) *in the Hilbert space* $\mathscr{M}^{2, 2\lambda}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d})$. *The solution belongs actually to* $\mathscr{M}^{2, \rho}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d})$ *and satisfies*

$$\mathbb{E}[\sup_{0 \leqslant s \leqslant \tau} e^{\rho s}\,|Y(s)|^2] < \infty,$$

and $(M(t))_{t \geqslant 0}$ *is a uniformly integrable martingale, where*

$$M(t) = \int_0^{t \wedge \tau} e^{\rho s}\,Y(s).Z(s)\,dW_s.$$

We end this subsection by a proposition based upon the approximation method introduced in [4].

First we fix some notations.

Let $(\bar{Y}_n, \bar{Z}_n)$ be the unique solution in $\mathscr{M}^2(0, n; \mathbb{R}^k \times \mathbb{R}^{k \times d})$ of the classical (the terminal time is deterministic) BSDE on $[0, n]$ (see [5] for a survey)

$$\bar{Y}_n(t) = \mathbb{E}(e^{\lambda\tau}\xi \mid \mathscr{F}_n) + \int_{t \wedge \tau}^{n \wedge \tau} [e^{\lambda s} f(s, e^{-\lambda s}\bar{Y}_n(s), e^{-\lambda s}\bar{Z}_n(s)) - \lambda\,\bar{Y}_n(s)]\,ds$$

$$- \int_t^n \bar{Z}_n(s)\,dW_s.$$

Remark that we have

$$\bar{Y}_n(t \wedge \tau) = \bar{Y}_n(t) \qquad \text{and} \qquad \bar{Z}_n(t) = 0 \qquad \text{on} \quad \{t > \tau\}.$$

Indeed,

$$\bar{Y}_n(t \wedge \tau) = \mathbb{E}(e^{\lambda\tau}\xi \mid \mathscr{F}_{n \wedge \tau}) + \int_{t \wedge \tau}^{n \wedge \tau} [e^{\lambda s} f(s, e^{-\lambda s}\bar{Y}_n(s), e^{-\lambda s}\bar{Z}_n(s)) - \lambda \bar{Y}_n(s)]\, ds$$

$$- \int_{t \wedge \tau}^{n} \bar{Z}_n(s)\, dW_s,$$

and then

$$\bar{Y}_n(n \wedge \tau) = \mathbb{E}(e^{\lambda\tau}\xi \mid \mathscr{F}_{n \wedge \tau}) - \int_{n \wedge \tau}^{n} \bar{Z}_n(s)\, dW_s.$$

It follows that $\int_{n \wedge \tau}^{n} \bar{Z}_n(s)\, dW_s$ belongs to $\mathscr{F}_{n \wedge \tau}$, and thus $\mathbb{1}_{\{n \geqslant s > \tau\}} \bar{Z}_n(s) = 0$ which gives the result.

Since $e^{\lambda\tau}\xi$ belongs to $L^2(\mathscr{F}_\tau)$ there exists a process (η) in $\mathscr{M}^2(0, \tau; \mathbb{R}^{k \times d})$ such that

$$e^{\lambda\tau}\xi = \mathbb{E}[e^{\lambda\tau}\xi] + \int_0^\tau \eta(s)\, dW_s.$$

We define, for each $t > n$,

$$\bar{Y}_n(t) = \mathbb{E}(e^{\lambda\tau}\xi \mid \mathscr{F}_t) = \zeta(t) \qquad \text{and} \qquad \bar{Z}_n(t) = \eta(t),$$

and finally set

$$Y_n(t) = e^{-\lambda(t \wedge \tau)} \bar{Y}_n(t), \qquad Z_n(t) = e^{-\lambda(t \wedge \tau)} \bar{Z}_n(t).$$

Then we have the following result:

Proposition 2.2. *Let* (A1) *hold and fix δ such that $\rho > \delta > 2\lambda$. Then there exists a constant K which depends only on C, γ, μ, and ρ, such that, $\forall t \in \mathbb{R}_+$,*

$$\mathbb{E}\left[e^{\delta(t \wedge \tau)} \,|\, \widetilde{Y}_n(t \wedge \tau)|^2 + \int_0^\tau e^{\delta s}(|\widetilde{Y}_n(s)|^2 + \|\widetilde{Z}_n(s)\|^2)\, ds \right] \leqslant K e^{(\delta - \rho)n},$$

where

$$\widetilde{Y}_n(t) = Y(t) - Y_n(t) \qquad \textit{and} \qquad \widetilde{Z}_n(t) = Z(t) - Z_n(t).$$

Proof. First we remark that, according to Lemma 4.1 of [4], (Y_n, Z_n) belongs to the space $\mathscr{M}^{2,\rho}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k\times d})$ and solves the BSDE

$$Y_n(t\wedge\tau)=\xi+\int_{t\wedge\tau}^{\tau} f_n(s, Y_n(s), Z_n(s))\,ds-\int_{t\wedge\tau}^{\tau} Z_n(s)\,dW_s,$$

where $f_n(s, y, z)=\mathbb{1}_{s\leqslant n} f(s, y, z)+\mathbb{1}_{s>n}\lambda y$.

Using Itô's formula to calculate $\mathbb{E}[e^{\delta(t\wedge\tau)}\,|\widetilde{Y}_n(t\wedge\tau)|^2]$ and noting that the expectation of the stochastic integral $\int_{t\wedge\tau}^{\tau} e^{\delta s}\,\widetilde{Y}_n(s).\widetilde{Z}_n(s)\,dW_s$ vanishes in view of Lemma 2.1, we get

$$\begin{aligned}\mathbb{E}&\left[e^{\delta(t\wedge\tau)}\,|\widetilde{Y}_n(t\wedge\tau)|^2+\int_{t\wedge\tau}^{\tau} e^{\delta s}\,\|\widetilde{Z}_n(s)\|^2\,ds\right]\\&=\mathbb{E}\left[\int_{t\wedge\tau}^{\tau} e^{\delta s}(-\delta\,|\widetilde{Y}_n(s)|^2+2\,\widetilde{Y}_n(s).[f(s, Y(s), Z(s))\right.\\&\qquad\left.-f_n(s, Y_n(s), Z_n(s))])\,ds\right].\end{aligned}$$

Writing now

$$\begin{aligned}f(s, y, z)-f_n(s, y', z')=&f(s, y, z)-f(s, y', z)+f(s, y', z)-f(s, y', z')\\&+f(s, y', z')-f_n(s, y', z'),\end{aligned}$$

and using the fact that f is Lipschitz and monotone, we obtain

$$\begin{aligned}\mathbb{E}&\left[e^{\delta(t\wedge\tau)}\,|\widetilde{Y}_n(t\wedge\tau)|^2+\int_{t\wedge\tau}^{\tau} e^{\delta s}\,\|\widetilde{Z}_n(s)\|^2\,ds\right]\\&\leqslant\mathbb{E}\left[\int_{t\wedge\tau}^{\tau} e^{\delta s}((-\delta-2\mu)\,|\widetilde{Y}_n(s)|^2+2\gamma\,|\widetilde{Y}_n(s)|\cdot\|\widetilde{Z}_n(s)\|)\,ds\right]\\&\quad+\mathbb{E}\left[\int_{(t\vee n)\wedge\tau}^{\tau} 2e^{\delta s}\,|\widetilde{Y}_n(s)|\cdot|f(s, Y_n(s), Z_n(s))-\lambda Y_n(s)|\,ds\right].\end{aligned}$$

Since $\delta>\gamma^2-2\mu$, there exist positive numbers α and β such that $1-\alpha>0$ and, moreover, $\nu=\delta-(\gamma^2/\alpha)+2\mu-\beta>0$. Thus, from $2ab\leqslant(a^2/\kappa^2)+\kappa^2b^2$, we have

$$\begin{aligned}\mathbb{E}&\left[e^{\delta(t\wedge\tau)}\,|\widetilde{Y}_n(t\wedge\tau)|^2+(1-\alpha)\int_{t\wedge\tau}^{\tau} e^{\delta s}\,\|\widetilde{Z}_n(s)\|^2\,ds+\nu\int_{t\wedge\tau}^{\tau} e^{\delta s}\,|\widetilde{Y}_n(s)|^2\,ds\right]\\&\leqslant\frac{3}{\beta}\,\mathbb{E}\left[\int_{n\wedge\tau}^{\tau} e^{\delta s}(|f(s,0,0)|^2+(C+|\lambda|)^2\,|Y_n(s)|^2+\gamma^2\,\|Z_n(s)\|^2)\,ds\right].\end{aligned}$$

Since $\mathbb{E}[\mathbb{1}_{\tau\leqslant n}\int_{n\wedge\tau}^{\tau}\cdots]=0$, we obtain

$$\mathbb{E}\left[\int_{n\wedge\tau}^{\tau} e^{\delta s}\,|f(s,0,0)|^2\,ds\right]\leqslant e^{(\delta-\rho)n}\,\mathbb{E}\left[\int_{n\wedge\tau}^{\tau} e^{\rho s}\,|f(s,0,0)|^2\,ds\right],$$

and the same is true for the other two terms.

Now coming back to the definition of $\bar{Y}_n$ and $\bar{Z}_n$, we get, setting the constant K' equal to $(3/\beta)\max(1,(C+|\lambda|)^2,\gamma^2)$,

$$\begin{aligned}&\mathbb{E}\left[e^{\delta(t\wedge\tau)}\,|\widetilde{Y}_n(t\wedge\tau)|^2+(1-\alpha)\int_{t\wedge\tau}^{\tau} e^{\delta s}\,\|\widetilde{Z}_n(s)\|^2\,ds+v\int_{t\wedge\tau}^{\tau} e^{\delta s}\,|\widetilde{Y}_n(s)|^2\,ds\right]\\&\quad\leqslant K'e^{(\delta-\rho)n}\,\mathbb{E}\left[\int_{n\wedge\tau}^{\tau} e^{\rho s}(|f(s,0,0)|^2+e^{-2\lambda s}(|\zeta(s)|^2+\|\eta(s)^2))\,ds\right]\\&\quad\leqslant K'e^{(\delta-\rho)n}\left(\mathbb{E}\left[\int_{n\wedge\tau}^{\tau} e^{\rho s}\,|f(s,0,0)|^2\,ds\right]+\|\zeta\|^2_{\rho-2\lambda}+\|\eta\|^2_{\rho-2\lambda}\right).\end{aligned}$$

In view of Lemma 4.1 of [4], we have moreover

$$\|\zeta\|^2_{\rho-2\lambda}+\|\eta\|^2_{\rho-2\lambda}\leqslant\left(1+\frac{1}{\rho-2\lambda}\right)\mathbb{E}[e^{\rho\tau}\,|\xi|^2].$$

Finally, from the integrability assumption on (ξ,f), we obtain

$$\mathbb{E}\left[e^{\delta(t\wedge\tau)}\,|\widetilde{Y}_n(t\wedge\tau)|^2+\int_{t\wedge\tau}^{\tau} e^{\delta s}(\|\widetilde{Z}_n(s)\|^2+|\widetilde{Y}_n(s)|^2)\,ds\right]\leqslant Ke^{(\delta-\rho)n},$$

where $K=K'C(2+1/(\rho-2\lambda))/\min(1-\alpha,v)$, which is the desired result. ∎

2.2. *Stability of BSDEs*

In this subsection, we prove our main result concerning the stability of BSDEs with random terminal time. As was shown in [3] for standard BSDEs, it is possible to obtain a stability property in L^2 even if the coefficients do not converge themselves strongly in L^2. We first recall this result.

Consider the BSDEs depending on the parameter $(\varepsilon\geqslant 0)$

$$Y^\varepsilon(t)=\xi^\varepsilon+V^\varepsilon_T-V^\varepsilon_t+\int_t^T f^\varepsilon(s,Y^\varepsilon(s),Z^\varepsilon(s))\,ds-\int_t^T Z^\varepsilon(s)\,dW_s,$$

where $f^\varepsilon\colon\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k$ is such that $f^\varepsilon(\cdot,y,z)$ is progressively measurable for each (y,z) and $\xi^\varepsilon\in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^k)$. Assume that:

(A2). There exists a constant $C \geqslant 0$, such that $\forall \varepsilon \geqslant 0$, $d\mathbb{P} \otimes ds$ *a.e.*,

1. $\forall (y_1, z_1), (y_2, z_2), \quad |f^\varepsilon(s, y_1, z_1) - f^\varepsilon(s, y_2, z_2)| \leqslant C(|y_1 - y_2| + \|z_1 - z_2\|)$;

2. $(V^\varepsilon_\cdot)$ is a continuous progressively measurable process with values in $\mathbb{R}^k$ and

$$\mathbb{E}\left[\int_0^T |f^\varepsilon(s, 0, 0)|^2\, ds\right] + \sup_{0 \leqslant t \leqslant T} \mathbb{E}[|V^\varepsilon_t|^2] \leqslant C.$$

(A3). $\forall t \in [0, T]$, we have

1. $\mathbb{E}[|\int_t^T (f^\varepsilon(s, Y^0(s), Z^0(s)) - f^0(s, Y^0(s), Z^0(s)))\, ds|^2] \to 0$ as $\varepsilon \to 0$;

2. $\forall t \in [0, T]$, $\mathbb{E}[|V^\varepsilon_t - V^0_t|^2] \to 0$ as $\varepsilon \to 0$ and $\mathbb{E}[|\xi^\varepsilon - \xi^0|^2] \to 0$ as $\varepsilon \to 0$.

LEMMA 2.3. *Let* (A2) *and* (A3) *hold. Then, for each t in* $[0, T]$ *we have*

$$\mathbb{E}\left[|Y^\varepsilon(t) - Y^0(t)|^2 + \int_0^T \|Z^\varepsilon(s) - Z^0(s)\|^2\, ds\right] \to 0, \qquad \textit{as } \varepsilon \textit{ goes to } 0.$$

Now we can state our main result of the section. Let $(f^\varepsilon)_{\varepsilon \geqslant 0}$ be a family of functions defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$ with values in $\mathbb{R}^k$ such that $f^\varepsilon(\cdot, y, z)$ is a progressively measurable process for each (y, z) and each ε. Let τ be a finite $(\mathcal{F}_t)$-stopping time and (ξ^ε) a family of $\mathcal{F}_\tau$-adapted random variables. (V^ε) is a family of continuous semimartingales.

Suppose that we have

(A4). For each $\varepsilon \geqslant 0$, assumption (A1) holds for $(\xi^\varepsilon, f^\varepsilon)$ with constant C, γ, ρ and μ not depending on ε and

$$\mathbb{E}[e^{\rho\tau} |V^\varepsilon_\tau|^2] \leqslant C, \qquad \mathbb{E}\left[\int_0^\tau e^{\rho s} |V^\varepsilon_s|^2\, ds\right] \leqslant C$$

Consider the solution $(Y^\varepsilon, Z^\varepsilon)$ of the BSDE

$$Y^\varepsilon(t \wedge \tau) = \xi^\varepsilon + V^\varepsilon_\tau - V^\varepsilon_{t \wedge \tau} + \int_{t \wedge \tau}^\tau f^\varepsilon(s, Y^\varepsilon(s), Z^\varepsilon(s))\, ds - \int_{t \wedge \tau}^\tau Z^\varepsilon(s)\, dW_s. \tag{2}$$

In addition, we assume that

(A5). $\forall n \in \mathbb{N}$, $\forall t \in [0, n]$,

1. $\mathbb{E}[|\int_{t \wedge \tau}^{n \wedge \tau} (f^\varepsilon(s, Y^0(s), Z^0(s)) - f^0(s, Y^0(s), Z^0(s)))\, ds|^2] \to 0$ as $\varepsilon \to 0$;

2. the random variables $e^{\lambda\tau}\xi^{\varepsilon}$ and $e^{\lambda\tau}V^{\varepsilon}_{\tau}$ converge respectively towards $e^{\lambda\tau}\xi^{0}$ and $e^{\lambda\tau}V^{0}_{\tau}$ in $L^2(\Omega, \mathscr{F}_{\tau}, \mathbb{P})$;

3. $\forall t \in \mathbb{R}_+$, $V^{\varepsilon}_{t\wedge\tau} \to V^{0}_{t\wedge\tau}$ in L^2 and for each integer n there exists C_n such that

$$\sup_{0\leqslant t\leqslant n}\ \sup_{\varepsilon\geqslant 0} \mathbb{E}[|V^{\varepsilon}_{t\wedge\tau}|^2] \leqslant C_n.$$

THEOREM 2.4. *Let* (A4) *and* (A5) *hold, and fix* δ *such that* $\rho > \delta > \gamma^2 - 2\mu$. *Then, for each* t *in* $\mathbb{R}_+$, *we have*

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[|Y^{\varepsilon}(t\wedge\tau) - Y^{0}(t\wedge\tau)|^2 + \int_0^{\tau} e^{\delta s} \, \|Z^{\varepsilon}(s) - Z^{0}(s)\|^2 \, ds \right] = 0.$$

Proof. We split the proof into two steps.

Step 1. First, suppose that for each $\varepsilon \geqslant 0$, $V^{\varepsilon} \equiv 0$.

Remark that, in view of assumption (A4), Lemma 2.1 supplies a unique solution to BSDE (2) in $\mathscr{M}^{2,2\lambda}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k\times d})$ which belongs in fact to $\mathscr{M}^{2,\rho}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k\times d})$.

We set $\widetilde{Y^{\varepsilon}}(s) = Y^{\varepsilon}(s) - Y^{0}(s)$ and $\widetilde{Z^{\varepsilon}}(s) = Z^{\varepsilon}(s) - Z^{0}(s)$. Thus, we have

$$\widetilde{Y^{\varepsilon}}(t\wedge\tau) = \alpha^{\varepsilon} + \int_{t\wedge\tau}^{\tau} g^{\varepsilon}(s, \widetilde{Y^{\varepsilon}}(s), \widetilde{Z^{\varepsilon}}(s))\, ds - \int_{t\wedge\tau}^{\tau} \widetilde{Z^{\varepsilon}}(s)\, dW_s,$$

where $\alpha^{\varepsilon} = \xi^{\varepsilon} - \xi^{0}$ and $g^{\varepsilon}(s, y, z) = f^{\varepsilon}(s, y + Y^{0}(s), z + Z^{0}(s)) - f^{0}(s, Y^{0}(s), Z^{0}(s))$.

Since (Y^0, Z^0) belongs to $\mathscr{M}^{2,\rho}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k\times d})$ and f^{ε} satisfies assumption (A1) uniformly with respect to ε, we deduce that (A1) also hold for $(\alpha^{\varepsilon}, g^{\varepsilon})$ with the same ρ, μ and γ but for another constant C which does not depend on ε.

According to Proposition 2.2, let us introduce $(\overline{Y^{\varepsilon}_n}, \overline{Z^{\varepsilon}_n})$ the solution of the BSDE on $[0, n]$,

$$\overline{Y^{\varepsilon}_n}(t) = \mathbb{E}(e^{\lambda\tau}\alpha^{\varepsilon} \,|\, \mathscr{F}_n) + \int_{t\wedge\tau}^{n\wedge\tau} [e^{\lambda s} g^{\varepsilon}(s, e^{-\lambda s}\overline{Y^{\varepsilon}_n}(s), e^{-\lambda s}\overline{Z^{\varepsilon}_n}(s)) - \lambda \overline{Y^{\varepsilon}_n}(s)]\, ds$$
$$- \int_t^n \overline{Z^{\varepsilon}_n}(s)\, dW_s,$$

and, for $t > n$, we set

$$\overline{Y^{\varepsilon}_n}(t) = \mathbb{E}(e^{\lambda\tau}\alpha^{\varepsilon} \,|\, \mathscr{F}_t), \qquad \overline{Z^{\varepsilon}_n}(t) = \widetilde{\eta^{\varepsilon}}(t),$$

where $(\widetilde{\eta^\varepsilon})$ is given by

$$e^{\lambda\tau}\alpha^\varepsilon = \mathbb{E}\,[e^{\lambda\tau}\alpha^\varepsilon] + \int_0^\tau \widetilde{\eta^\varepsilon}(s)\,dW_s.$$

Now, we define, for each t in $\mathbb{R}_+$,

$$\widetilde{Y_n^\varepsilon}(t) = e^{-\lambda(t\wedge\tau)}\overline{Y_n^\varepsilon}(t) \qquad \text{and} \qquad \widetilde{Z_n^\varepsilon}(t) = e^{-\lambda(t\wedge\tau)}\overline{Z_n^\varepsilon}(t).$$

Hence, by Proposition 2.2, there exists a constant K, independent of ε, such that for each ε and each t,

$$\mathbb{E}\left[e^{\delta(t\wedge\tau)}\,|\widetilde{Y^\varepsilon}(t\wedge\tau) - \widetilde{Y_n^\varepsilon}(t\wedge\tau)|^2 + \int_0^\tau e^{\delta s}\,\|\widetilde{Z^\varepsilon}(s) - \widetilde{Z_n^\varepsilon}(s)\|^2\,ds\right] \leqslant Ke^{(\delta-\rho)n}.$$

Thus, we get

$$\mathbb{E}[e^{\delta(t\wedge\tau)}\,|\widetilde{Y^\varepsilon}(t\wedge\tau)|^2] \leqslant 2\mathbb{E}[e^{\delta(t\wedge\tau)}\,|\widetilde{Y_n^\varepsilon}(t\wedge\tau)|^2] + 2Ke^{(\delta-\rho)n}. \qquad (3)$$

Now, recall that as we see in the previous subsection $\widetilde{Y_n^\varepsilon}(t\wedge\tau) = \widetilde{Y_n^\varepsilon}(t)$. Moreover, $(\widetilde{Y_n^\varepsilon}, \widetilde{Z_n^\varepsilon})$ solves the BSDE on $[0,n]$,

$$\widetilde{Y_n^\varepsilon}(t) = e^{-\lambda(n\wedge\tau)}\mathbb{E}(e^{\lambda\tau}\alpha^\varepsilon\,|\,\mathcal{F}_n) + \int_{t\wedge\tau}^{n\wedge\tau} g^\varepsilon(s, \widetilde{Y_n^\varepsilon}(s), \widetilde{Z_n^\varepsilon}(s))\,ds - \int_t^n \widetilde{Z_n^\varepsilon}(s)\,dW_s.$$

But, in view of (A5), we have

$$\mathbb{E}\left[\left|\int_{t\wedge\tau}^{n\wedge\tau} g^\varepsilon(s,0,0)\,ds\right|^2\right] \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

and

$$e^{-\lambda(n\wedge\tau)}\mathbb{E}(e^{\lambda\tau}\alpha^\varepsilon\,|\,\mathcal{F}_n) \to 0 \qquad \text{in } L^2, \qquad \text{as} \quad \varepsilon \to 0.$$

So, Lemma 2.3 implies that, for all integers n,

$$\mathbb{E}\left[|\widetilde{Y_n^\varepsilon}(t)|^2 + \int_0^n \|\widetilde{Z_n^\varepsilon}(s)\|^2\,ds\right] \to 0, \qquad \text{as} \quad \varepsilon \to 0. \qquad (4)$$

For n larger than t, the inequality (3) yields

$$\mathbb{E}[\,|\widetilde{Y^\varepsilon}(t\wedge\tau)|^2] \leqslant 2e^{2|\delta|\,t}\,\mathbb{E}[\,|\widetilde{Y_n^\varepsilon}(t)|^2] + 2Ke^{|\delta|\,t}e^{(\delta-\rho)n}.$$

Since $\delta<\rho$, the second term of the right hand side can be done arbitrary small uniformly with respect to ε by choosing n large enough. Thus, from Eq. (4), we derive

$$\mathbb{E}[|\widetilde{Y}^{\varepsilon}(t\wedge\tau)|^2]\to 0 \qquad \text{as} \quad \varepsilon\to 0. \tag{5}$$

Moreover, we have

$$\mathbb{E}\left[\int_0^{\tau} e^{\delta s}\,\|\widetilde{Z}^{\varepsilon}(s)\|^2\,ds\right]\leqslant 2Ke^{(\delta-\rho)n}+2\mathbb{E}\left[\int_0^{\tau} e^{\delta s}\,\|\widetilde{Z}^{\varepsilon}_n(s)\|^2\,ds\right],$$

and

$$\mathbb{E}\left[\int_0^{\tau} e^{\delta s}\,\|\widetilde{Z}^{\varepsilon}_n(s)\|^2\,ds\right]$$

$$\leqslant e^{|\delta|\,n}\mathbb{E}\left[\int_0^{n\wedge\tau}\|\widetilde{Z}^{\varepsilon}_n(s)\|^2\,ds\right]+\mathbb{E}\left[\int_{n\wedge\tau}^{\tau} e^{\delta s}\,\|\widetilde{Z}^{\varepsilon}_n(s)\|^2\,ds\right].$$

It is worth noting that, for n fixed, Lemma 2.3 shows that, if $\varepsilon\to 0$, then

$$\mathbb{E}\left[\int_0^{n\wedge\tau}\|\widetilde{Z}^{\varepsilon}_n(s)\|^2\,ds\right]\to 0.$$

In addition, for $n<t<\tau$, we have $\widetilde{Z}^{\varepsilon}_n(t)=e^{-\lambda t}\,\widetilde{\eta}^{\varepsilon}(t)$ and then

$$\mathbb{E}\left[\int_{n\wedge\tau}^{\tau} e^{\delta s}\,\|\widetilde{Z}^{\varepsilon}_n(s)\|^2\,ds\right]=\mathbb{E}\left[\mathbb{1}_{\tau>n}\int_0^{\tau} e^{\delta s}e^{-2\lambda s}\,\|\,\widetilde{\eta}^{\varepsilon}(s)\|^2\,ds\right]$$

$$\leqslant\mathbb{E}\left[\mathbb{1}_{\tau>n}e^{(\delta-\rho)n}\int_n^{\tau} e^{(\rho-2\lambda)s}\,\|\,\widetilde{\eta}^{\varepsilon}(s)\|^2\,ds\right]$$

$$\leqslant e^{(\delta-\rho)n}\,\|\,\widetilde{\eta}^{\varepsilon}\,\|^2_{\rho-2\lambda}.$$

Moreover, by Lemma 4.1 in [4], we have, since the random variable $e^{\rho\tau/2}\alpha^{\varepsilon}$ is in L^2,

$$\|\,\widetilde{\eta}^{\varepsilon}\,\|^2_{\rho-2\lambda}\leqslant\mathbb{E}[e^{\rho\tau}\,|\alpha^{\varepsilon}|^2].$$

Finally, we get, using the assumption (A4),

$$\mathbb{E}\left[\int_0^{\tau} e^{\delta s}\,\|\widetilde{Z}^{\varepsilon}(s)\|^2\,ds\right]\leqslant K'e^{(\delta-\rho)n}+2e^{|\delta|\,n}\mathbb{E}\left[\int_0^{n\wedge\tau}\|\widetilde{Z}^{\varepsilon}_n(s)\|^2\,ds\right].$$

By choosing n large enough, since $\rho > \delta$, the first term of the right hand side of the previous inequality can be done arbitrary small, uniformly with respect to ε. Hence, taking into account Eq. (4), we deduce that

$$\mathbb{E}\left[\int_0^\tau e^{\delta s}\, \|\widetilde{Z^\varepsilon}(s)\|^2\, ds\right] \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Coming back to the definition of $\widetilde{Y^\varepsilon}$ and $\widetilde{Z^\varepsilon}$, we get, from Eq. (5),

$$\mathbb{E}\left[\,|Y^\varepsilon(t \wedge \tau) - Y^0(t \wedge \tau)|^2 + \int_0^\tau e^{\delta s}\, \|Z^\varepsilon(s) - Z^0(s)\|^2\, ds\right] \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

which complete the proof of Step 1.

Step 2. First, we notice that $(Y^\varepsilon, Z^\varepsilon)$ solves the BSDE (2) if and only if $(\widehat{Y^\varepsilon}, \widehat{Z^\varepsilon})$ solves the following BSDE

$$\begin{cases} -d\,\widehat{Y^\varepsilon}(t) = \mathbb{1}_{t \leqslant \tau}(g^\varepsilon(t, \widehat{Y^\varepsilon}(t), \widehat{Z^\varepsilon}(t))\, dt - \widehat{Z^\varepsilon}(t)\, dW_t), \\ \widehat{Y^\varepsilon}(\tau) = \xi^\varepsilon + V^\varepsilon_\tau, \end{cases} \tag{6}$$

where we have set, for t in $\mathbb{R}_+$, $\widehat{Y^\varepsilon}(t) = Y^\varepsilon(t) + V^\varepsilon_{t \wedge \tau}$, $\widehat{Z^\varepsilon}(t) = Z^\varepsilon(t)$ and, for each (s, y, z) in $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$, $g^\varepsilon(s, y, z) = f^\varepsilon(s, y - V^\varepsilon_s, z)$.

Since (V^ε) is bounded in $\mathscr{M}^{2,\rho}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d})$ and $\mathbb{E}[e^{\rho\tau}\, |V^\varepsilon_\tau|^2] \leqslant C$, Lemma 2.1 provides a unique solution $(\widehat{Y^\varepsilon}, \widehat{Z^\varepsilon})$ to (6) in $\mathscr{M}^{2,2\lambda}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d})$ which belongs to $\mathscr{M}^{2,\rho}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d})$.

It follows that the BSDE (2) has a unique solution $(Y^\varepsilon, Z^\varepsilon)$ in $\mathscr{M}^{2,2\lambda}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d})$ which belongs to $\mathscr{M}^{2,\rho}(0, \tau; \mathbb{R}^k \times \mathbb{R}^{k \times d})$.

Moreover, we can apply Step 1 to $(\widehat{Y^\varepsilon}, \widehat{Z^\varepsilon})$. Recall that

$$g^\varepsilon(s, y, z) - g^0(s, y, z) = f^\varepsilon(s, y - V^\varepsilon_s, z) - f^0(s, y - V^0_s, z),$$

and, since f^ε is Lipschitz, from Hölder's inequality, we get

$$\begin{aligned} &\mathbb{E}\left[\left|\int_{t \wedge \tau}^{n \wedge \tau} (g^\varepsilon(s, \widehat{Y^0}(s), \widehat{Z^0}(s)) - g^0(s, \widehat{Y^0}(s), \widehat{Z^0}(s)))\, ds\right|^2\right] \\ &\quad \leqslant 2\mathbb{E}\left[\left|\int_{t \wedge \tau}^{n \wedge \tau} (f^\varepsilon(s, Y^0(s), Z^0(s)) - f^0(s, Y^0(s), Z^0(s)))\, ds\right|^2\right] \\ &\qquad + 2C^2 n \mathbb{E}\left[\int_0^n |V^0_{s \wedge \tau} - V^\varepsilon_{s \wedge \tau}|^2\, ds\right]. \end{aligned}$$

By assumption (A5), the first term tends to 0. But, on the other hand, in view of assumption (A5).3, Lebesgue's dominated convergence theorem ensures that

$$\mathbb{E}\left[\int_0^n |V^0_{s\wedge\tau} - V^\varepsilon_{s\wedge\tau}|^2\, ds\right] \to 0, \qquad \text{as} \quad \varepsilon \to 0.$$

Thus, from Step 1, we obtain, for each t in $\mathbb{R}_+$,

$$\mathbb{E}[|\widehat{Y^\varepsilon}(t\wedge\tau) - \widehat{Y^0}(t\wedge\tau)|^2] + \mathbb{E}\left[\int_0^\tau e^{\delta s}\,\|\widehat{Z^\varepsilon}(s) - \widehat{Z^0}(s)\|^2\, ds\right] \to 0.$$

Since, for each t, $V^\varepsilon_{t\wedge\tau}$ tends to $V^0_{t\wedge\tau}$ in L^2, we finally obtain, as $\widehat{Z^\varepsilon} = Z^\varepsilon$,

$$\lim_{\varepsilon\to 0}\left(\mathbb{E}[|Y^\varepsilon(t\wedge\tau) - Y^0(t\wedge\tau)|^2] + \mathbb{E}\left[\int_0^\tau e^{\delta s}\,\|Z^\varepsilon(s) - Z^0(s)\|^2\, ds\right]\right) = 0,$$

which is the desired result.

The proof is complete. ∎

We end this section with another stability result.

Let (τ^ε) be a family of $(\mathcal{F}_t)$-stopping times and assume that

(A4'). For each ε, assumption (A1) holds for $(\xi^\varepsilon, f^\varepsilon, \tau^\varepsilon)$ uniformly with respect to ε and

$$\mathbb{E}[e^{\rho\tau^\varepsilon}|V^\varepsilon_{\tau^\varepsilon}|^2]\Big] \leqslant C, \qquad \mathbb{E}\left[\int_0^{\tau^\varepsilon} e^{\rho s}\,|V^\varepsilon_s|^2\, ds\right] \leqslant C.$$

(A5'). $\forall n\in\mathbb{N}$, $\forall t\in[0,n]$,

1. $\mathbb{E}[|\int_{t\wedge\tau^\varepsilon}^{n\wedge\tau^\varepsilon} f^\varepsilon(s,0,0)\, ds|^2] \to 0$ as $\varepsilon\to 0$;
2. the random variables $e^{\lambda\tau^\varepsilon}\xi^\varepsilon$ and $e^{\lambda\tau^\varepsilon}V^\varepsilon_{\tau^\varepsilon}$ converge to 0 in L^2;
3. $\forall t\in\mathbb{R}_+$, $V^\varepsilon_{t\wedge\tau^\varepsilon}\to 0$ in L^2 and for each integer n there exists C_n such that

$$\sup_{0\leqslant t\leqslant n}\ \sup_{\varepsilon\geqslant 0} \mathbb{E}[|V^\varepsilon_{t\wedge\tau^\varepsilon}|^2] \leqslant C_n.$$

Consider $(Y^\varepsilon, Z^\varepsilon)$, the solution of the BSDE

$$\begin{cases} -dY^\varepsilon(t) = \mathbb{1}_{t\leqslant\tau^\varepsilon}(f^\varepsilon(t, Y^\varepsilon(t), Z^\varepsilon(t))\, dt + dV^\varepsilon_t - Z^\varepsilon(t)\, dW_t), \\ Y^\varepsilon(\tau^\varepsilon) = \xi^\varepsilon. \end{cases}$$

We can state

PROPOSITION 2.5. *Under the assumptions* (A4') *and* (A5'), *for each t in* $\mathbb{R}_+$, *we have*

$$\mathbb{E}\left[\,|Y^\varepsilon(t\wedge\tau^\varepsilon)|^2+\int_0^{\tau\varepsilon} e^{\delta s}\,\|Z^\varepsilon(s)\|^2\,ds\right]\to 0 \qquad as \quad \varepsilon\to 0,$$

where δ *is some fixed number such that* $\rho>\delta>2\lambda$.

Proof. We adopt the same strategy as in the proof of the previous result. So we just outline the proof.

Suppose, first, that $V^\varepsilon\equiv 0$; fix t and pick n larger than t. From Proposition 2.2, there exists a constant K, which does not depend on ε since all assumptions are fulfilled uniformly with respect to ε, such that

$$\mathbb{E}\left[\,e^{\delta(t\wedge\tau^\varepsilon)}\,|Y^\varepsilon(t\wedge\tau^\varepsilon)-Y_n^\varepsilon(t\wedge\tau^\varepsilon)|^2+\int_0^{\tau^\varepsilon} e^{\delta s}\,\|Z^\varepsilon(s)-Z_n^\varepsilon(s)\|^2\,ds\right]$$
$$\leqslant Ke^{(\delta-\rho)n}.$$

But, n being fixed, Lemma 2.3 gives

$$\mathbb{E}\left[\,|Y_n^\varepsilon(t\wedge\tau^\varepsilon)|^2+\int_0^{n\wedge\tau^\varepsilon} \|Z_n^\varepsilon(s)\|^2\,ds\right]\to 0 \qquad \text{as} \quad \varepsilon\to 0,$$

from which we easily deduce the result.

For the general case, we do the change of variables $\widehat{Y^\varepsilon}(t)=Y^\varepsilon(t)+V^\varepsilon_{t\wedge\tau^\varepsilon}$ and $\widehat{Z^\varepsilon}(t)=Z^\varepsilon(t)$. We can apply the previous step to $(\widehat{Y^\varepsilon},\widehat{Z^\varepsilon})$. Since, $\mathbb{E}[|V^\varepsilon_{t\wedge\tau^\varepsilon}|^2]$ goes to 0, we easily obtain the result. ∎

5. ONE-DIMENSIONAL CASE

As we have seen in the previous section, to solve BSDEs with random terminal time requires a "structural" condition on the coefficient f which links the constant of monotonicity, μ, and the Lipschitz constant of f in z, γ that is "$\rho>\gamma^2-2\mu$." For applications we have in mind, homogenization of PDEs, this condition is not natural. The aim of this section is to solve BSDEs with random terminal time in dimension 1 without this assumption and to obtain also some stability results.

3.1. *Solutions of BSDEs*

Let τ be an $(\mathcal{F}_t)$-stopping time and ξ an $\mathcal{F}_\tau$-adapted random variable. As in the preceding section, we work with a function f defined on $\Omega\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^d$ which takes values in $\mathbb{R}$ and such that $f(\cdot,y,z)$ is a progressively measurable process for each (y,z) in $\mathbb{R}\times\mathbb{R}^d$.

We want to construct an adapted process $(Y(t), Z(t))_{t \in \mathbb{R}_+}$, which solves the BSDE

$$\begin{cases} -dY(t) = \mathbb{1}_{t \leqslant \tau}(f(t, Y(t), Z(t))\,dt - Z(t)\,dW_t), \\ Y(\tau) = \xi \qquad \text{on} \quad \{\tau < \infty\}. \end{cases} \tag{7}$$

We begin with a lemma. Assume that:

(A6). There exist two constants $K \geqslant 0$ and $\mu > 0$ such that, $d\mathbb{P} \otimes dt$ *a.e.*,

1. f is uniformly Lipschitz, i.e., $\forall (y_1, z_1), (y_2, z_2)$ in $\mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leqslant K(|y_1 - y_2| + |z_1 - z_2|);$$

2. f is monotone in y, i.e., $\forall y_1, y_2 \in \mathbb{R}$, $\forall z \in \mathbb{R}^d$,

$$(y_1 - y_2) \cdot (f(t, y_1, z) - f(t, y_2, z)) \leqslant -\mu(y_1 - y_2)^2;$$

3. $|f(t, 0, 0)| \leqslant K$.

LEMMA 3.1. *Let assumption* (A6) *hold. Then, the BSDE* (7) *with* $\xi = 0$ *has a solution* (Y, Z) *which belongs to* $\mathcal{M}^{2, -2\mu}(0, \tau; \mathbb{R} \times \mathbb{R}^d)$ *and such that* Y *is a bounded process. This solution is unique in the class of processes* (Y, Z) *such that* Y *is continuous and bounded and* Z *belongs to* $\mathcal{M}^2_{\mathrm{loc}}(0, \tau; \mathbb{R}^d)$.

Proof. First we prove uniqueness. Suppose that (Y^1, Z^1) and (Y^2, Z^2) are both solutions of the BSDE (7) such that (Y^1, Z^1) (respectively (Y^2, Z^2)) satisfies that Y^1 (respectively Y^2) is continuous and bounded and that Z^1 (respectively Z^2) belongs to $\mathcal{M}^2_{\mathrm{loc}}(0, \tau; \mathbb{R}^d)$. Set, as usual, $\tilde{Y} = Y^1 - Y^2$ and $\tilde{Z} = Z^1 - Z^2$. Since both Y^1 and Y^2 are continuous and bounded, we can assert that $\tilde{Y}$ is continuous and for some $M > 0$, we have

$$\sup_{t \in \mathbb{R}_+} |\tilde{Y}(t)| \leqslant M, \qquad a.s.$$

We introduce a notation. For (z, z') in $\mathbb{R}^d$, define, for $k = 1, ..., d+1$,

$$(z, z')_k = (z'_1, ..., z'_{k-1}, z_k, ..., z_d),$$

and remark that $(z, z')_1 = z$, $(z, z')_{d+1} = z'$.

We define a new process (α, β), with values in $\mathbb{R} \times \mathbb{R}^d$, by setting

$$\alpha(s) = \begin{cases} \dfrac{f(s, Y^1(s), Z^1(s)) - f(s, Y^2(s), Z^1(s))}{Y^1(s) - Y^2(s)}, & \text{if} \quad Y^1(s) - Y^2(s) \neq 0, \\ -\mu, & \text{otherwise,} \end{cases}$$

and, for $i=1, ..., d$,

$$\beta_i(s)=\begin{cases} \dfrac{f(s, Y^2(s), U_i(s))-f(s, Y^2(s), U_{i+1}(s))}{Z_i^1(s)-Z_i^2(s)}, & \text{if} \quad Z_i^1(s)-Z_i^2(s)\neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where, for $i=1, ..., d$, $U_i(s)=(Z^1(s), Z^2(s))_i$.

Note that, since f is Lipschitz, α and β are bounded processes. Moreover, we have, in view of Eq. (7),

$$-d\tilde{Y}(t)=\mathbb{1}_{t\leqslant\tau}([f(t, Y^1(t), Z^1(t))-f(t, Y^2(t), Z^2(t))]\,dt-\tilde{Z}(t)\,dW_t),$$

which can be rewritten in the following way,

$$-d\tilde{Y}(t)=\mathbb{1}_{t\leqslant\tau}([\alpha(t)\,\tilde{Y}(t)+\beta(t)\cdot\tilde{Z}(t)]\,dt-\tilde{Z}(t)\,dW_t).$$

Fix $\theta\in\mathbb{R}_+$ and set, for $t\geqslant\theta$,

$$R(t)=\exp\left(\int_{\theta\wedge\tau}^{t}\alpha(u)\,du\right).$$

Pick $n\in\mathbb{N}$ such that $n\geqslant\theta$; Itô's formula yields

$$R(n\wedge\tau)\,\tilde{Y}(n\wedge\tau)-R(\theta\wedge\tau)\,\tilde{Y}(\theta\wedge\tau)=\int_{\theta\wedge\tau}^{n\wedge\tau}R(s)\,\tilde{Z}(s)(dW_s-\beta(s)\,ds).$$

Since on the set $\{\tau\leqslant n\}$ we have $\tilde{Y}(n\wedge\tau)=\tilde{Y}(\tau)=0$ $\mathbb{P}$ *a.s.*, we deduce from the previous inequality that

$$\tilde{Y}(\theta\wedge\tau)=R(n\wedge\tau)\,\tilde{Y}(n\wedge\tau)\,\mathbb{1}_{\tau>n}-\int_{\theta\wedge\tau}^{n\wedge\tau}R(s)\,\tilde{Z}(s)\,d\tilde{W}_s, \tag{8}$$

where we have set $\tilde{W}_t=W_t-\int_0^t\beta(s)\,ds$.

Let $\mathbb{Q}_n$ be the probability measure on $(\Omega, \mathscr{F}_n)$ whose density with respect to $\mathbb{P}_{|\mathscr{F}_n}$ is

$$\exp\left(\int_0^n\beta(s)\,dW_s-\tfrac{1}{2}\int_0^n|\beta|^2(s)\,ds\right).$$

Since β is a bounded process, the probability measures $\mathbb{Q}_n$ and $\mathbb{P}_{|\mathscr{F}_n}$ are mutually absolutely continuous and $(\tilde{W}(t))_{0\leqslant t\leqslant n}$ is a Brownian motion under $\mathbb{Q}_n$.

Taking the conditional expectation with respect to $\mathscr{F}_{\theta \wedge \tau}$ of the equation (8), we obtain

$$|\tilde{Y}(\theta \wedge \tau)| \leqslant \mathbb{E}^{\mathbb{Q}_n}(R(n)\,|\tilde{Y}(n)|\,\mathbb{1}_{\tau > n}|\mathscr{F}_{\theta \wedge \tau}), \qquad \mathbb{Q}_n \text{ a.s.} \tag{9}$$

Since f satisfies the assumption (A6).2, we have $\alpha(t) \leqslant -\mu \, d\mathbb{P} \otimes dt$ *a.e.* and then we get $R(t) \leqslant e^{-\mu(t-\theta)}\,\mathbb{P}$ *a.s.* Moreover the variable $R(n)\,|\tilde{Y}(n)|\,\mathbb{1}_{\tau > n}$ is $\mathscr{F}_n$-measurable and, since the processes Y^1 and Y^2 are bounded, we have

$$R(n)\,|\tilde{Y}(n)|\,\mathbb{1}_{\tau > n} \leqslant 2Me^{-\mu(n-\theta)} \qquad \mathbb{P}_{|\mathscr{F}_n} \text{ a.s.}$$

and, as $\mathbb{Q}_n \sim \mathbb{P}_{|\mathscr{F}_n}$, we get

$$R(n)\,|\tilde{Y}(n)|\,\mathbb{1}_{\tau > n} \leqslant 2Me^{-\mu(n-\theta)} \qquad \mathbb{Q}_n \text{ a.s.}$$

Coming back to inequality (9), we obtain

$$|\tilde{Y}(\theta \wedge \tau)| \leqslant 2Me^{-\mu(n-\theta)} \qquad \mathbb{Q}_n \text{ a.s.}$$

Using once again the fact that $\mathbb{Q}_n$ and the restriction of $\mathbb{P}$ to $\mathscr{F}_n$ are equivalent, we deduce that

$$\mathbb{P} \text{ a.s. } \forall n \in \mathbb{N} \text{ s.t. } n \geqslant \theta, \qquad |\tilde{Y}(\theta \wedge \tau)| \leqslant 2Me^{-\mu(n-\theta)}.$$

Sending n to infinity, we deduce from the previous estimate

$$\tilde{Y}(\theta \wedge \tau) = 0 \qquad \mathbb{P} \text{ a.s.}$$

and then, by continuity, $Y^1 = Y^2$.

Moreover, for $t \in \mathbb{R}_+$, Itô's formula yields

$$\mathbb{E}[|\tilde{Y}(t \wedge \tau)|^2] + 2\mathbb{E}\left[\int_0^{t \wedge \tau} \tilde{Y}(s)(f(s, Y_1(s), Z_1(s)) - f(s, Y_2(s), Z_2(s)))\,ds\right]$$
$$= \mathbb{E}[|\tilde{Y}(0)|^2] + \mathbb{E}\left[\int_0^{t \wedge \tau} |\tilde{Z}(s)|^2\,ds\right],$$

which gives

$$\mathbb{E}\left[\int_0^{t \wedge \tau} |\tilde{Z}(s)|^2\,ds\right] = 0.$$

Now, we turn to existence. Denote by (Y_n, Z_n) the unique solution of the BSDE

$$Y_n(t) = \int_{t \wedge \tau}^{n \wedge \tau} f(s, Y_n(s), Z_n(s))\,ds - \int_t^n Z_n(s)\,dW_s,$$

and recall that in fact $Y_n(t) = Y_n(t \wedge \tau)$, $Z_n(t) = \mathbb{1}_{t \leqslant \tau} Z_n(t)$, in other words,

$$Y_n(t \wedge \tau) = \int_{t \wedge \tau}^{n \wedge \tau} f(s, Y_n(s), Z_n(s))\, ds - \int_{t \wedge \tau}^{n \wedge \tau} Z_n(s)\, dW_s.$$

First we give an a priori estimate. Using the same notations as in the proof of uniqueness, we have

$$Y_n(t \wedge \tau) = \int_{t \wedge \tau}^{n \wedge \tau} (\alpha_n(s)\, Y_n(s) + \beta_n(s)\, Z_n(s) + f(s, 0, 0))\, ds - \int_{t \wedge \tau}^{n \wedge \tau} Z_n(s)\, dW_s,$$

where

$$\alpha_n(s) = \begin{cases} \dfrac{f(s, Y_n(s), Z_n(s)) - f(s, 0, Z_n(s))}{Y_n(s)}, & \text{if} \quad Y_n(s) \neq 0, \\ -\mu, & \text{otherwise,} \end{cases}$$

and, for $i = 1, ..., d$,

$$(\beta_n)_i(s) = \begin{cases} \dfrac{f(s, 0, (Z_n(s), 0)_i) - f(s, 0, (Z_n(s), 0)_{i+1})}{(Z_n)_i(s)}, & \text{if} \quad (Z_n)_i(s) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We define $R_n(t) = \exp(\int_{\theta \wedge \tau}^t \alpha_n(s)\, ds)$ and $W_n(t) = W(t) - \int_0^t \beta_n(s)\, ds$. Thus, applying Itô's formula, we obtain

$$\begin{aligned} Y_n(\theta \wedge \tau) = R_n(n \wedge \tau)\, Y_n(n \wedge \tau) &+ \int_{\theta \wedge \tau}^{n \wedge \tau} R_n(s)\, f(s, 0, 0)\, ds \\ &- \int_{\theta \wedge \tau}^{n \wedge \tau} R_n(s)\, Z_n(s)\, dW_n(s), \end{aligned}$$

and taking into account that $Y_n(n \wedge \tau) = Y_n(n) = 0$, we deduce from the previous equality that

$$Y_n(\theta \wedge \tau) = \int_{\theta \wedge \tau}^{n \wedge \tau} R_n(s)\, f(s, 0, 0)\, ds - \int_{\theta \wedge \tau}^{n \wedge \tau} R_n(s)\, Z_n(s)\, dW_n(s).$$

We introduce the probability measure $\mathbb{Q}_n$ on $(\Omega, \mathscr{F}_n)$ whose density with respect to the restriction of $\mathbb{P}$ on $\mathscr{F}_n$ is given by

$$\exp\left(\int_0^n \beta_n(s)\, dW(s) - \tfrac{1}{2} \int_0^n |\beta_n|^2(s)\, ds\right).$$

From Girsanov's theorem, $\mathbb{Q}_n$ and $\mathbb{P}_{|\mathscr{F}_n}$ are mutually absolutely continuous and moreover the process $(W_n(t))_{0 \leqslant t \leqslant n}$ is a Brownian motion under $\mathbb{Q}_n$. Hence, we deduce that

$$|Y_n(\theta \wedge \tau)| \leqslant \mathbb{E}^{\mathbb{Q}_n}\left(\int_{\theta \wedge \tau}^{n \wedge \tau} |f(s, 0, 0)|\, R_n(s)\, ds \mid \mathscr{F}_{\theta \wedge \tau}\right) \qquad \mathbb{Q}_n \text{ a.s.}$$

Remark that $|f(s, 0, 0)| \leqslant K$ and $R_n(s) \leqslant e^{-\mu(s-\theta)}$. Thus, we deduce from the previous inequality that

$$|Y_n(\theta \wedge \tau)| \leqslant K e^{\mu\theta}\, \mathbb{E}^{\mathbb{Q}_n}\left(\int_{\theta \wedge \tau}^{n \wedge \tau} e^{-\mu s}\, ds \,\middle|\, \mathscr{F}_{\theta \wedge \tau}\right) \qquad \mathbb{Q}_n \text{ a.s.},$$

and then

$$|Y_n(\theta \wedge \tau)| \leqslant K e^{\mu\theta}\, \mathbb{E}^{\mathbb{Q}_n}\left(\int_{\theta}^{n} e^{-\mu s}\, ds \,\middle|\, \mathscr{F}_{\theta \wedge \tau}\right) \qquad \mathbb{Q}_n \text{ a.s.}$$

From the previous inequality, we easily derive that

$$\forall \theta \in [0, n], \qquad |Y_n(\theta \wedge \tau)| \leqslant \frac{K}{\mu} \qquad \mathbb{Q}_n \text{ a.s.}$$

Since Y_n is a continuous process and the measures $\mathbb{P}$ (in fact its restriction to $\mathscr{F}_n$) and $\mathbb{Q}_n$ are equivalent on $\mathscr{F}_n$, we get

$$\mathbb{P} \text{ a.s. } \forall n \in \mathbb{N}, \qquad \forall \theta \in [0, n], \qquad |Y_n(\theta \wedge \tau)| \leqslant \frac{K}{\mu}. \tag{10}$$

Now, we study the convergence of the sequence of processes $((Y_n, Z_n))_{\mathbb{N}}$ in the Hilbert space $\mathscr{M}^{2, -2\mu}(0, \tau; \mathbb{R} \times \mathbb{R}^d)$. We define Y_n and Z_n on the whole time axis by setting

$$Y_n(t) = 0 \qquad \text{and} \qquad Z_n(t) = 0, \qquad \text{if} \quad t > n.$$

Fix $\theta \leqslant n \leqslant m$ and set $\tilde{Y} = Y_m - Y_n$, $\tilde{Z} = Z_m - Z_n$, and $\tilde{f}(s, y, z) = \mathbb{1}_{s \leqslant n} f(s, y, z)$. We get, from Itô's formula, since $\tilde{Y}(m) = 0$,

$$\tilde{Y}(t \wedge \tau) = \int_{t \wedge \tau}^{m \wedge \tau} (f(s, Y_m(s), Z_m(s)) - \tilde{f}(s, Y_n(s), Z_n(s)))\, ds$$
$$- \int_{t \wedge \tau}^{m \wedge \tau} \tilde{Z}(s)\, dW_s.$$

As in the proof of uniqueness, we use the same kind of linearization together with Girsanov's transformation. So we write

$$\begin{aligned} f(s,\, Y_m(s),\, Z_m(s)) - \tilde{f}(s,\, Y_n(s),\, Z_n(s)) \\ = \alpha_{n,m}(s)\ \tilde{Y}(s) + \beta_{n,m}(s)\ \tilde{Z}(s) + \mathbb{1}_{s>n}\, f(s,0,0), \end{aligned}$$

where

$$\alpha_{n,m}(s) = \begin{cases} \dfrac{f(s,\, Y_m(s),\, Z_m(s)) - f(s,\, Y_n(s),\, Z_m(s))}{Y_m(s) - Y_n(s)}, & \text{if } Y_m(s) - Y_n(s) \neq 0, \\ -\mu, & \text{otherwise,} \end{cases}$$

and, for $i = 1, ..., d$,

$$\begin{aligned} &(\beta_{n,m})_i\,(s) \\ &\quad = \begin{cases} \dfrac{f(s,\, Y_n(s),\, U_i(s)) - f(s,\, Y_n(s),\, U_{i+1}(s))}{(Z_m)_i\,(s) - (Z_n)_i\,(s)}, & \text{if } \ (Z_m - Z_n)_i\,(s) \neq 0, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where, for $i = 1, ..., d$, $U_i(s) = (Z_m(s), Z_n(s))_i$.

We set $R_{n,m}(t) = \exp(\int_{\theta \wedge \tau}^{t} \alpha_{n,m}(s)\, ds)$ and $W_{n,m}(t) = W_t - \int_0^t \beta_{n,m}(s)\, ds$. We define a new probability measure on $(\Omega, \mathscr{F}_m)$, say $\mathbb{Q}_{n,m}$, whose density with respect to the restriction of $\mathbb{P}$ to $\mathscr{F}_m$ is

$$\exp\left(\int_0^m \beta_{n,m}(s)\, dW_s - \tfrac{1}{2} \int_0^m |\beta_{n,m}|^2(s)\, ds \right).$$

The process $(\beta_{n,m})$ being bounded, $\mathbb{Q}_{n,m}$ is equivalent to $\mathbb{P}_{|\mathscr{F}_m}$ and $(W_{n,m}(t))_{0 \leqslant t \leqslant m}$ is a Brownian motion under $\mathbb{Q}_{n,m}$.

We use Itô's formula to compute $R_{n,m}(\theta \wedge \tau)\ \tilde{Y}(\theta \wedge \tau)$ and obtain, noting that $\tilde{Y}(m \wedge \tau) = 0$,

$$\tilde{Y}(\theta \wedge \tau) = \int_{n \wedge \tau}^{m \wedge \tau} R_{n,m}(s)\, f(s,0,0)\, ds - \int_{\theta \wedge \tau}^{m \wedge \tau} R_{n,m}(s)\, dW_{n,m}(s).$$

As $f(s, 0, 0)$ is bounded by K, we get

$$|\tilde{Y}(\theta \wedge \tau)| \leqslant K \mathbb{E}^{\mathbb{Q}_{n,m}} \left(\int_{n \wedge \tau}^{m \wedge \tau} R_{n,m}(s)\, ds \,\middle|\, \mathscr{F}_{\theta \wedge \tau} \right), \qquad \mathbb{Q}_{n,m}\ a.s.$$

and, from $\alpha_{n,m}(t) \leqslant -\mu$, we finally obtain

$$|Y_m(\theta \wedge \tau) - Y_n(\theta \wedge \tau)| \leqslant \frac{K}{\mu} e^{\mu\theta}(e^{-\mu n} - e^{-\mu m}), \qquad \mathbb{Q}_{n,m}\ a.s.$$

Since $\mathbb{Q}_{n,m}$ and $\mathbb{P}$ are equivalent, we deduce from the continuity of the process $\tilde{Y}$ that

$$\mathbb{P}\ a.s.\ \forall 0 \leqslant \theta \leqslant n \leqslant m, \qquad |Y_m(\theta \wedge \tau) - Y_n(\theta \wedge \tau)| \leqslant \frac{K}{\mu} e^{\mu\theta}(e^{-\mu n} - e^{-\mu m}). \tag{11}$$

Taking into consideration that by construction $Y_n(\theta \wedge \tau) = Y_n(\theta)$, the previous inequality implies that, for each $\theta \geqslant 0$, the sequence of random variables $(Y_n(\theta))_{\mathbb{N}}$ is a Cauchy sequence in L^∞ and then converges to $Y(\theta)$, uniformly with respect to θ on compact sets.

Moreover, $Y(\theta) = Y(\theta \wedge \tau)$ and letting m go to infinity in (11), it comes that $\mathbb{P}$ *a.s.*,

$$\forall 0 \leqslant \theta \leqslant n, \qquad |Y(\theta \wedge \tau) - Y_n(\theta \wedge \tau)| \leqslant \frac{K}{\mu} e^{-\mu(n-\theta)}. \tag{12}$$

Now, we show that the sequence $(Y_n)_{\mathbb{N}}$ is a Cauchy sequence in the space $\mathcal{M}^{2,-2\mu}(0, \tau; \mathbb{R})$. Indeed, we have

$$\mathbb{E}\left[\int_0^\tau e^{-2\mu s}\, |\tilde{Y}(s)|^2\, ds\right]$$
$$= \mathbb{E}\left[\int_0^{n \wedge \tau} e^{-2\mu s}\, |\tilde{Y}(s)|^2\, ds\right] + \mathbb{E}\left[\int_{n \wedge \tau}^{m \wedge \tau} e^{-2\mu s}\, |\tilde{Y}(s)|^2\, ds\right],$$

and then, we derive, from the inequality (11) and the a priori estimate (10), that

$$\mathbb{E}\left[\int_0^\tau e^{-2\mu s}\, |\tilde{Y}(s)|^2\, ds\right] \leqslant \mathbb{E}\left[\int_0^{n \wedge \tau} \frac{K^2}{\mu^2} e^{-2\mu n}\, ds\right] + \mathbb{E}\left[\int_{n \wedge \tau}^{m \wedge \tau} \frac{K^2}{\mu^2} e^{-2\mu s}\, ds\right].$$

Finally, we have

$$\mathbb{E}\left[\int_0^\tau e^{-2\mu s}\, |Y_m(s) - Y_n(s)|^2\, ds\right] \leqslant \frac{K^2}{\mu^2} e^{-2\mu n}\left(n + \frac{1}{2\mu}\right), \tag{13}$$

which shows that $(Y_n)_{\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}^{2,-2\mu}(0, \tau; \mathbb{R})$.

We show that the same is true for the sequence $(Z_n)_{\mathbb{N}}$. From Itô's formula, we get

$$\mathbb{E}\left[\,|\tilde{Y}(0)|^2+\int_0^{m\wedge\tau} e^{-2\mu s}\,|\tilde{Z}(s)|^2\,ds\right]$$

$$=\mathbb{E}\left[\int_0^{m\wedge\tau} e^{-2\mu s}(2\mu\,|\tilde{Y}(s)|^2+2\,\tilde{Y}(s)\cdot[\,f(s,\,Y_m(s),\,Z_m(s))\right.$$

$$\left.-f(s,\,Y_n(s),\,Z_n(s))])\,ds\right]+\mathbb{E}\left[\int_{n\wedge\tau}^{m\wedge\tau} 2e^{-2\mu s}\,Y_m(s)\,f(s,0,0)\,ds\right].$$

Taking into account that f is Lipschitz and satisfies the assumption (A6), we have the estimate

$$\mathbb{E}\left[\int_0^{m\wedge\tau} e^{-2\mu s}\,|\tilde{Z}(s)|^2\,ds\right]\leqslant\mathbb{E}\left[\int_0^{m\wedge\tau} 2Ke^{-2\mu s}\,|\tilde{Y}(s)|\,|\tilde{Z}(s)|\,ds\right]+\frac{K^2}{\mu^2}\,e^{-2\mu n}.$$

Using the fact that $2K\,|\tilde{Y}(s)|\,|\tilde{Z}(s)|\leqslant 2K^2\,|\tilde{Y}(s)|^2+\frac{1}{2}|\tilde{Z}(s)|^2$, we obtain finally

$$\mathbb{E}\left[\int_0^{m\wedge\tau} e^{-2\mu s}\,|\tilde{Z}(s)|^2\,ds\right]\leqslant 4K^2\,\mathbb{E}\left[\int_0^{m\wedge\tau} e^{-2\mu s}\,|\tilde{Y}(s)|^2\,ds\right]+2\,\frac{K^2}{\mu^2}\,e^{-2\mu n},$$

from which we deduce, using the inequality (13),

$$\mathbb{E}\left[\int_0^{\tau} e^{-2\mu s}(|\tilde{Y}(s)|^2+|\tilde{Z}(s)|^2)\,ds\right]\leqslant\frac{K^2}{\mu^2}\,e^{-2\mu n}\left[(1+4K^2)\left(n+\frac{1}{2\mu}\right)+2\right].$$

As a byproduct of the previous inequality, the sequence of processes $((Y_n, Z_n))_{\mathbb{N}}$ is a Cauchy sequence in $\mathscr{M}^{2,-2\mu}(0,\tau;\mathbb{R}\times\mathbb{R}^d)$ and thus converges in this Hilbert space towards a process (Y, Z).

It remains to show that (Y, Z) satisfies the BSDE (7). First, remark that in view of the a priori estimate (10), we have

$$\forall t\in\mathbb{R}_+,\qquad |Y(t\wedge\tau)|\leqslant\frac{K}{\mu}.$$

Secondly, for $0\leqslant\theta\leqslant t\leqslant n$, we have

$$Y_n(\theta\wedge\tau)-Y_n(t\wedge\tau)=\int_{\theta\wedge\tau}^{t\wedge\tau} f(s,\,Y_n(s),\,Z_n(s))\,ds-\int_{\theta\wedge\tau}^{t\wedge\tau} Z_n(s)\,dW_s,$$

and, then passing to the limit in L^2, we obtain

$$Y(\theta \wedge \tau) - Y(t \wedge \tau) = \int_{\theta \wedge \tau}^{t \wedge \tau} f(s, Y(s), Z(s))\, ds - \int_{\theta \wedge \tau}^{t \wedge \tau} Z(s)\, dW_s,$$

that is

$$-dY(t) = \mathbb{1}_{t \leqslant \tau}(f(t, Y(t), Z(t))\, dt - Z(t)\, dW_t).$$

Moreover, obviously,

$$|Y(n \wedge \tau)| \leqslant |Y(n \wedge \tau) - Y_{2n}(n \wedge \tau)| + |Y_{2n}(n \wedge \tau)|,$$

and we deduce from (12) that

$$|Y(n \wedge \tau)| \leqslant \frac{K}{\mu} e^{-\mu n} + |Y_{2n}(n \wedge \tau)|.$$

On the set $\{\tau < \infty\}$, we have, for n large enough ($n \geqslant \tau(\omega)$), since $Y_{2n}(n \wedge \tau) = Y_{2n}(2n \wedge \tau) = 0$,

$$|Y(\tau)| \leqslant \frac{K}{\mu} e^{-\mu n},$$

and thus, sending n to infinity, we get

$$Y(\tau) = 0 \qquad \text{on} \quad \{\tau < \infty\}.$$

This completes the proof of Lemma 3.1. ∎

We prove the following estimate:

PROPOSITION 3.2. *Let* (A6) *hold and let* $\xi = 0$. *Then there exists a constant* C *depending only on* K *and* μ, *such that for each* $0 \leqslant t \leqslant n$,

$$\mathbb{E}[|Y(t \wedge \tau) - Y_n(t \wedge \tau)|^2] + \mathbb{E}\left[\int_0^{t \wedge \tau} |Z(s) - Z_n(s)|^2\, ds\right] \leqslant Ce^{-2\mu(n-t)}.$$

Proof. The key point in this proof is the inequality (12). Indeed, fix $0 \leqslant t \leqslant n$, and, using Itô's formula, we get

$$\mathbb{E}[\,|Y(t\wedge\tau)-Y_n(t\wedge\tau)|^2]+2\mathbb{E}\left[\int_0^{t\wedge\tau}(Y(s)-Y_n(s))(f(s,\,Y(s),\,Z(s))\right.$$
$$\left.-f(s,\,Y_n(s),\,Z_n(s)))\,ds\right]$$
$$=\mathbb{E}[\,|Y(0)-Y_n(0)|^2]+\mathbb{E}\left[\int_0^{t\wedge\tau}|Z(s)-Z_n(0)|^2\,ds\right],$$

from which we derive, using the fact that f is Lipschitz and satisfies (A6).2.,

$$\mathbb{E}\left[\int_0^{t\wedge\tau}|Z(s)-Z_n(s)|^2\,ds\right]$$
$$\leqslant\mathbb{E}[\,|Y(t\wedge\tau)-Y_n(t\wedge\tau)|^2]$$
$$+2K\mathbb{E}\left[\int_0^{t\wedge\tau}|Y(s)-Y_n(s)|\cdot|Z(s)-Z_n(s)|\,ds\right].$$

Since $2K\,|Y(s)-Y_n(s)|\cdot|Z(s)-Z_n(s)|\leqslant 2K^2\,|Y(s)-Y_n(s)|^2+\frac{1}{2}|Z(s)-Z_n(s)|^2$, we finally obtain

$$\tfrac{1}{2}\mathbb{E}\left[\int_0^{t\wedge\tau}|Z(s)-Z_n(s)|^2\,ds\right]$$
$$\leqslant\mathbb{E}\left[\,|Y(t\wedge\tau)-Y_n(t\wedge\tau)|^2\right]+2K^2\mathbb{E}\left[\int_0^{t\wedge\tau}|Y(s)-Y_n(s)|^2\,ds\right].$$

Thus, in view of (12),

$$\mathbb{E}\left[\int_0^{t\wedge\tau}|Z(s)-Z_n(s)|^2\,ds\right]\leqslant 2\,\frac{K^2}{\mu^2}\,e^{2\mu t}\left(1+\frac{K^2}{\mu}\right)e^{-2\mu n}.$$

It remains only to choose $C=(K^2/\mu^2)(3+2(K^2/\mu))$. ∎

We can state our main result concerning the existence and uniqueness of solutions of BSDE (7).

THEOREM 3.3. *Let* (A6) *hold and in addition suppose that* ξ *belongs to* $L^\infty(\mathscr{F}_\tau)$ *and, moreover,* $d\mathbb{P}\otimes dt$ *a.e.*,

$$\forall z\in\mathbb{R}^d,\qquad |f(t,\,0,\,z)|\leqslant K.$$

Then there exists a solution (Y,Z) *to BSDE* (7), *such that* Y *is a bounded and continuous process and* Z *belongs to* $\mathscr{M}^{2,\,-2\mu}(0,\,\tau;\,\mathbb{R}^d)$. *This solution is unique in the class of processes* (Y,Z) *such that* Y *is continuous and uniformly bounded and* Z *belongs to* $\mathscr{M}^2_{\text{loc}}(0,\,\tau;\,\mathbb{R}^d)$.

Proof. Since ξ belongs to $L^\infty(\mathscr{F}_\tau)$, there exists a process (η) in $\mathscr{M}^2(0, \tau; \mathbb{R}^d)$ such that

$$\xi = \mathbb{E}[\xi] + \int_0^\tau \eta(s)\, dW_s.$$

It is worth noting that (Y, Z) solves the BSDE (7) if and only if the process (y, z), defined by $(y(t), z(t)) = (Y(t) - \mathbb{E}(\xi \mid \mathscr{F}_t), Z(t) - \eta(t))$, solves the BSDE

$$\begin{cases} -dy(t) = \mathbb{1}_{t \leqslant \tau}(g(t, y(t), z(t))\, dt - z(t)\, dW_t), \\ y(\tau) = 0 \qquad \text{on} \quad \{\tau < \infty\}, \end{cases} \tag{14}$$

where we have set $g(t, y, z) = f(t, y + \mathbb{E}(\xi \mid \mathscr{F}_t), z + \eta(t))$. Moreover, we remark that the function g satisfies $|g(t, 0, 0)| \leqslant K \, \|\xi_\infty + K$. So from Lemma 3.1, the BSDE (14) has a unique solution (y, z) such that y is continuous and bounded and z belongs to $\mathscr{M}^2_{\mathrm{loc}}(0, \tau; \mathbb{R}^d)$. Since $(\mathbb{E}(\xi \mid \mathscr{F}_t))_t$ is bounded and continuous and (η) belongs to $\mathscr{M}^2(0, \tau; \mathbb{R}^d)$, we easily obtain the result. ∎

Remark. When the random variable ξ has a Malliavin derivative which is uniformly bounded, the assumption that $|f(t, 0, z)| \leqslant K$ can be weaken to $|f(t, 0, 0)| \leqslant K$.

3.2. *Stability Results*

In the context of the previous section, BSDEs with random terminal time in dimension one, we can establish some stability results in the spirit of Section 2.2.

Assume that

(A7). Assumption (A6) hold for a family of functions $(f^\varepsilon)_{\varepsilon \geqslant 0}$, with constants K and μ independent of ε and for each ε, $d\mathbb{P} \otimes dt$ *a.e.*,

$$\forall z \in \mathbb{R}^d, \qquad |f^\varepsilon(t, 0, z)| \leqslant K,$$

(ξ^ε) is a family of $\mathscr{F}_\tau$-adapted random variables such that $\|\xi^\varepsilon\|_\infty \leqslant K$.

Introduce $(Y^\varepsilon, Z^\varepsilon)$, the solution of the BSDE depending on ε,

$$\begin{cases} -dY^\varepsilon(t) = \mathbb{1}_{t \leqslant \tau}(f^\varepsilon(t, Y^\varepsilon(t), Z^\varepsilon(t))\, dt - Z^\varepsilon(t)\, dW_t), \\ Y^\varepsilon(\tau) = \xi^\varepsilon \qquad \text{on} \quad \{\tau < \infty\}. \end{cases} \tag{15}$$

Suppose that

(A8). For each integer n, and for each t in $[0, n]$,

$$\mathbb{E}\left[\left|\int_{t\wedge\tau}^{n\wedge\tau} (f^\varepsilon(s, Y^0(s), Z^0(s)) - f^0(s, Y^0(s), Z^0(s)))\, ds\right|^2\right] \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

and ξ^ε tends to ξ^0 in $L^2(\mathscr{F}_\tau)$ as $\varepsilon \to 0$.

THEOREM 3.4. *If* (A7) *and* (A8) *hold, then* $\forall t \in \mathbb{R}$, *we have*

$$\mathbb{E}[|Y^\varepsilon(t\wedge\tau) - Y^0(t\wedge\tau)|^2] + \mathbb{E}\left[\int_0^{t\wedge\tau} |Z^\varepsilon(s) - Z^0(s)|^2\, ds\right] \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. We split the proof into two parts.

Step 1. Suppose, first, that for each ε, $\xi^\varepsilon = 0$. Fix $t \leqslant n$, and set $\widetilde{Y^\varepsilon} = Y^\varepsilon - Y^0$, $\widetilde{Z^\varepsilon} = Z^\varepsilon - Z^0$. Then $(\widetilde{Y^\varepsilon}, \widetilde{Z^\varepsilon})$ solves the BSDE

$$-d\widetilde{Y^\varepsilon}(t) = \mathbb{1}_{t\leqslant\tau}(g^\varepsilon(s, \widetilde{Y^\varepsilon}(s), \widetilde{Z^\varepsilon}(s))\, ds - \widetilde{Z^\varepsilon}(s)\, dW_s,$$

with the terminal condition $\widetilde{Y^\varepsilon}(\tau) = 0$ on $\{\tau < \infty\}$ and where we have set

$$g^\varepsilon(s, y, z) = f^\varepsilon(s, y + Y^0(s), z + Z^0(s)) - f^0(s, Y^0(s), Z^0(s)).$$

Remark that g^ε is K-Lipschitz and μ-monotone in y uniformly with respect to ε. Moreover, since $|Y^0(t)| \leqslant K/\mu$, we have

$$|g^\varepsilon(t, 0, 0)| \leqslant 2K(1 + |Y^0(t)|) \leqslant 2K\left(1 + \frac{K}{\mu}\right).$$

For each integer n, let us introduce $(\widetilde{Y^\varepsilon_n}, \widetilde{Z^\varepsilon_n})$, the solution on $[0, n]$ of the BSDE

$$\widetilde{Y^\varepsilon_n}(t) = \int_{t\wedge\tau}^{n\wedge\tau} g^\varepsilon(s, \widetilde{Y^\varepsilon_n}(s), \widetilde{Z^\varepsilon_n}(s))\, ds - \int_{t\wedge\tau}^{n\wedge\tau} \widetilde{Z^\varepsilon_n}(s)\, dW_s. \tag{16}$$

Then, by Proposition 3.2, there exists a constant C which does not depend on ε such that, for $n \geqslant t$,

$$\mathbb{E}[|\widetilde{Y^\varepsilon}(t\wedge\tau) - \widetilde{Y^\varepsilon_n}(t\wedge\tau)|^2] + \mathbb{E}\left[\int_0^{t\wedge\tau} |\widetilde{Z^\varepsilon}(s) - \widetilde{Z^\varepsilon_n}(s)|^2\, ds\right] \leqslant Ce^{-2\mu(n-t)}.$$

It follows that

$$\mathbb{E}[|\widetilde{Y^\varepsilon}(t\wedge\tau)|^2]+\mathbb{E}\left[\int_0^{t\wedge\tau}|\widetilde{Z^\varepsilon}(s)|^2\,ds\right]$$
$$\leqslant 2Ce^{-2\mu(n-t)}+2\mathbb{E}\left[|\widetilde{Y^\varepsilon_n}(t\wedge\tau)|^2+\int_0^{t\wedge\tau}|\widetilde{Z^\varepsilon_n}(s)|^2\,ds\right].$$

Moreover, from assumption (A8), we have

$$\mathbb{E}\left[\left|\int_{t\wedge\tau}^{n\wedge\tau}g^\varepsilon(s,0,0)\,ds\right|^2\right]\to 0 \qquad \text{as} \quad \varepsilon\to 0.$$

Since $(\widetilde{Y^\varepsilon_n},\widetilde{Z^\varepsilon_n})$ solves the BSDE (16), Lemma 2.3 shows that $\forall n\in\mathbb{N}$, if ε tends to 0,

$$\mathbb{E}[|\widetilde{Y^\varepsilon_n}(t\wedge\tau)|^2]+\mathbb{E}\left[\int_0^{t\wedge\tau}|\widetilde{Z^\varepsilon_n}(s)|^2\,ds\right]\to 0,$$

from which we deduce the result of Step 1.

Step 2. As in the proof of existence, we do the change of variables $y^\varepsilon(t)=Y^\varepsilon(t)-\mathbb{E}(\xi^\varepsilon\,|\,\mathcal{F}_t)$ and $z^\varepsilon(t)=Z^\varepsilon(t)-\eta^\varepsilon(t)$ where, for each ε, the process (η^ε) is given by

$$\xi^\varepsilon=\mathbb{E}[\xi^\varepsilon]+\int_0^\tau\eta^\varepsilon(s)\,dW_s.$$

Remark that, setting $g^\varepsilon(t,y,z)=f^\varepsilon(t,y+\mathbb{E}(\xi^\varepsilon\,|\,\mathcal{F}_t),z+\eta^\varepsilon(t))$, $(y^\varepsilon,z^\varepsilon)$ solves the BSDE

$$-dy^\varepsilon(t)=\mathbb{1}_{t\leqslant\tau}(g^\varepsilon(t,y^\varepsilon(t),z^\varepsilon(t))\,dt-z^\varepsilon(t)\,dW_t),$$

and satisfies $y^\varepsilon(\tau)=0$ on the set $\{\tau<\infty\}$.

In order to apply Step 1, it remains to prove that, for ε going to 0,

$$\varphi^\varepsilon:=\mathbb{E}\left[\left|\int_{t\wedge\tau}^{n\wedge\tau}[g^\varepsilon(s,y^0(s),z^0(s))-g^0(s,y^0(s),z^0(s))]\,ds\right|^2\right]\to 0,$$

for each $0\leqslant t\leqslant n$.

Coming back to the definition of g^ε and using the fact that f^ε is Lipschitz, uniformly with respect to ε, we get

$$\varphi^\varepsilon \leqslant 2K\mathbb{E}\left[\left(\int_{t\wedge\tau}^{n\wedge\tau} (|\mathbb{E}(\xi^\varepsilon - \xi^0 \,|\, \mathscr{F}_s)| + |\eta^\varepsilon(s) - \eta^0(s)|)\, ds\right)^2\right]$$
$$+2\mathbb{E}\left[\left|\int_{t\wedge\tau}^{n\wedge\tau} [f^\varepsilon(s, Y^0(s), Z^0(s)) - f^0(s, Y^0(s), Z^0(s))]\, ds\right|^2\right].$$

By assumption (A8), the second term tends to 0 as ε goes to 0, and using Hölder's inequality we derive

$$\mathbb{E}\left[\left(\int_{t\wedge\tau}^{n\wedge\tau} (|\mathbb{E}(\xi^\varepsilon - \xi^0 \mid \mathscr{F}_s)| + |\eta^\varepsilon(s) - \eta^0(s)|)\, ds|\right)^2\right]$$
$$\leqslant 2n^2\,\mathbb{E}[|\xi^\varepsilon - \xi^0|^2] + 2n\mathbb{E}\left[\int_0^{n\wedge\tau} |\eta^\varepsilon(s) - \eta^0(s)|^2\, ds\right].$$

To complete the proof, we remark that

$$\mathbb{E}\left[\int_0^{n\wedge\tau} |\eta^\varepsilon(s) - \eta^0(s)|\,^2 ds = \mathbb{E}[|\mathbb{E}(\xi^\varepsilon - \xi^0 \,|\, \mathscr{F}_n)|^2] - (\mathbb{E}[\xi^\varepsilon - \xi^0])^2\right.$$
$$\leqslant \mathbb{E}[|\xi^\varepsilon - \xi^0|^2].$$

Since, by assumption (A8), $\xi^\varepsilon \to \xi^0$ in $L^2(\mathscr{F}_\tau)$, φ^ε tends to 0 as $\varepsilon \to 0$.

Hence, applying Step 1 to $(y^\varepsilon, z^\varepsilon)$, we easily conclude the proof, since we have already seen that $\mathbb{E}(\xi^\varepsilon \,|\, \mathscr{F}_t) \to \mathbb{E}(\xi^0 \,|\, \mathscr{F}_t)$ and $\mathbb{E}[\int_0^{t\wedge\tau} |\eta^\varepsilon(s) - \eta^0(s)|^2\, ds] \to 0$. ∎

We end this subsection by the analogy of Proposition 2.5. Let (τ^ε) be a family of $(\mathscr{F}_t)$-stopping times and (ξ^ε) be $\mathscr{F}_{\tau^\varepsilon}$-adapted random variables. Assume that:

(A8') $\mathbb{E}[|\xi^\varepsilon|^2] \to 0$, and $\forall 0 \leqslant t \leqslant n$,

$$\mathbb{E}\left[\left|\int_{t\wedge\tau^\varepsilon}^{n\wedge\tau^\varepsilon} f^\varepsilon(s, 0, 0)\, ds\right|^2\right] \to 0, \qquad \text{as} \quad \varepsilon \to 0.$$

Consider $(Y^\varepsilon, Z^\varepsilon)$ which is the solution of the BSDE

$$\begin{cases} -dY^\varepsilon(t) = \mathbb{1}_{t\leqslant\tau^\varepsilon}(f^\varepsilon(t, Y^\varepsilon(t), Z^\varepsilon(t))\, dt - Z^\varepsilon(t)\, dW_t), \\ Y^\varepsilon(\tau^\varepsilon) = \xi^\varepsilon \qquad \text{on} \quad \{\tau^\varepsilon < \infty\}. \end{cases}$$

We can state the following result:

Proposition 3.5. *If* (A7) *and* (A8') *hold, then* $\forall t \in \mathbb{R}_+$, *if* $\varepsilon \to 0$, *we have*

$$\mathbb{E}[|Y^\varepsilon(t \wedge \tau^\varepsilon)|^2] + \mathbb{E}\left[\int_0^{t \wedge \tau^\varepsilon} |Z^\varepsilon(s)|^2 \, ds\right] \to 0.$$

Proof. The proof is similar to the proof of Theorem 3.4, since the upper bound in Proposition 3.2 does not depend on τ but only on K and μ. ∎

4. HOMOGENIZATION OF ELLIPTIC PDES

4.1. *Standing Assumptions*

In Chapter 3 of [2], the authors proposed a probabilistic method for studying homogenization properties of elliptic PDEs in the linear case. The purpose of this section is to give a probabilistic method for studying the homogenization properties of systems of semilinear elliptic PDEs. The approach developed here is based upon the nonlinear Feynman–Kac formula (see [8]) and the stability properties of BSDEs studied in the previous sections. The case of parabolic PDEs is studied in [3].

Consider a system of semilinear elliptic PDEs of the following form, we use convention of summation over repeated indices,

$$\begin{cases} \frac{1}{2} a_{i,j}\left(\frac{x}{\varepsilon}\right) \partial^2_{i,j} u^\varepsilon_m(x) + \frac{1}{\varepsilon} b_i\left(\frac{x}{\varepsilon}\right) \partial_i u^\varepsilon_m(x) + \frac{1}{\varepsilon} g_m\left(\frac{x}{\varepsilon}\right) \\ \qquad + f_m\left(x, \frac{x}{\varepsilon}, u^\varepsilon(x), Du^\varepsilon(x)\, \sigma\left(\frac{x}{\varepsilon}\right)\right) = 0, \qquad \text{in } \mathcal{O}, \qquad m = 1, ..., k, \\ u^\varepsilon_{|\partial\mathcal{O}} = h_{|\partial\mathcal{O}}, \end{cases} \tag{17}$$

where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^n$ and b: $\mathbb{R}^n \to \mathbb{R}^n$, σ: $\mathbb{R}^n \to \mathbb{R}^{n \times d}$, g: $\mathbb{R}^n \to \mathbb{R}^k$, f: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k$ and h: $\mathbb{R}^n \to \mathbb{R}^k$ are smooth functions which are periodic in the variable x/ε (we denote η this variable in the following). We have also $a = \sigma\sigma^t$ where the superscript t means transpose.

We are interested in the asymptotic behavior of (u^ε) which is the solution of (17). We first recall the deep connection between solutions of PDEs and BSDEs.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space carrying a standard d-dimensional Brownian motion (W_t). $(\mathcal{F}_t)$ denotes the usual right continuous and complete filtration associated to (W_t).

Assume that:

(A9). $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^n$ which is moreover of class $\mathscr{C}^5$ and the functions b, σ, g, f, h are smooth and periodic; more precisely

1. b, σ, g, f are $\mathscr{C}^3_{\mathrm{b}}$, h belongs to $\mathscr{C}^5_{\mathrm{b}}(\mathbb{R}^n)$;
2. $a=\sigma\sigma^t$ is strictly elliptic, i.e., there exists $\beta>0$ such that

$$\forall x \in \mathbb{R}^n, \qquad a(x) \geqslant \beta \mathrm{I}_n;$$

3. all functions considered are $[0, 1]^n$ periodic in the variable η $(=x/\varepsilon)$; in the following we denote $[0, 1]^n$ by Ξ.

(A10). f is monotone, i.e., there exists a constant μ strictly positive such that, for each (x, η, z) and for each (y, y'),

$$\langle y-y', f(x, \eta, y, z)-f(x, \eta, y', z)\rangle \leqslant -\mu\, |y-y'|^2.$$

In addition, if $k \geqslant 2$, suppose also

(A11). $2\mu > \gamma^2$ where $\gamma = \sup |D_z f(x, \eta, y, z)|$.

We recall first some standard results from elliptic PDEs theory and homogenization. We refer to [1, 2, and 7] for details and proofs.

Let L be the differential operator

$$L=\tfrac{1}{2}a_{i,j}(\eta)\,\partial^2_{\eta_i, \eta_j}+b_i(\eta)\partial_{\eta_i},$$

and let L^* denote its formal adjoint.

According to [2, p. 431], we can state

Proposition 4.1. *Let* (A9) *hold. Then there exists a unique continuous function m such that m is positive and Ξ-periodic and satisfies*

$$L^* m = 0 \qquad \textit{and} \qquad \int_\Xi m(\eta)\, d\eta = 1.$$

Moreover, $0<\underline{m}\leqslant m(\eta)\leqslant M$ *for each η in Ξ.*

Introduce, on the other hand, (X^ε_x), the solution of the SDE depending on $\varepsilon>0$,

$$\begin{cases} dX^\varepsilon_x(t) = \dfrac{1}{\varepsilon}\, b\left(\dfrac{X^\varepsilon_x(t)}{\varepsilon}\right) dt + \sigma\left(\dfrac{X^\varepsilon_x(t)}{\varepsilon}\right) dW_t, \\ X^\varepsilon_x(0)=x, \end{cases} \tag{18}$$

and denote by τ^ε_x the hitting time for the closed set $\mathcal{O}^c$ of the process (X^ε_x), $x \in \mathcal{O}$. Since a is strictly elliptic, we have, for each $\varepsilon>0$ and each x in $\mathbb{R}^n$, $\tau^\varepsilon_x < \infty$ *a.s.* (see [6, p. 144]).

Consider $(Y_x^\varepsilon, Z_x^\varepsilon)$ the solution of the BSDE with random terminal time, writing X_x^ε in place of $X_x^\varepsilon(t)$,

$$\begin{cases} -dY_x^\varepsilon(t) = \mathbb{1}_{t \leqslant \tau_x^\varepsilon}\left(\left(\frac{1}{\varepsilon}\, g\left(\frac{X_x^\varepsilon}{\varepsilon}\right) + f\left(X_x^\varepsilon, \frac{X_x^\varepsilon}{\varepsilon}, Y_x^\varepsilon(t), Z_x^\varepsilon(t)\right)\right) dt - Z_x^\varepsilon(t)\, dW_t\right), \\ Y_x^\varepsilon(\tau_x^\varepsilon) = h(X_x^\varepsilon(\tau_x^\varepsilon)). \end{cases} \tag{19}$$

We now precise connections between $(Y_x^\varepsilon, Z_x^\varepsilon)$ and u^ε solution of (17).

Proposition 4.2. *Let* (A9) *and* (A10) *hold and suppose also, if* $k \geqslant 2$, *that* (A11) *holds. Then the system of* PDEs (17) *has a unique classical solution* u^ε, *which belongs to* $\mathscr{C}^4(\overline{\mathcal{O}})$ *and for each* x *in* $\overline{\mathcal{O}}$, *we have*

$$Y_x^\varepsilon(t) = u^\varepsilon(X_x^\varepsilon(t \wedge \tau_x^\varepsilon)) \qquad \textit{and} \qquad Z_x^\varepsilon(t) = \mathbb{1}_{t \leqslant \tau_x^\varepsilon}\left(Du^\varepsilon(X_x^\varepsilon(t))\, \sigma\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\right),$$

where $(Y_x^\varepsilon, Z_x^\varepsilon)$ *is the solution of the BSDE* (19).

Proof. Existence of classical solution for (17) may be found in [7, p. 387]. The probabilistic interpretation of u^ε from Itô's formula, see [9].

Moreover, since u^ε and its gradient are continuous and bounded, $(Y_x^\varepsilon, Z_x^\varepsilon)$ is uniformly bounded, and then, if $k \geqslant 2$, belongs to $\mathscr{M}^{2, 2\lambda}(0, \tau_x^\varepsilon; \mathbb{R}^k \times \mathbb{R}^{k \times d})$ due to the fact that $2\lambda = \gamma^2 - 2\mu < 0$. Thus, uniqueness of solutions for BSDEs (see Lemma 2.1 and Theorem 3.3) gives uniqueness of smooth solutions for (17). Remark that in view of [1, p. 787], for $k = 1$, u^ε is the unique solution in $W^{2,p}(\mathcal{O})$. ∎

We do a further assumption on the vector fields b and g; that is

(A12). $\displaystyle\int_\Xi b(\eta)\, m(\eta)\, d\eta = 0$ and $\displaystyle\int_\Xi g(\eta)\, m(\eta)\, d\eta = 0.$

The asymptotic properties of (u^ε) remains an open question without this assumption even in the linear case.

We can state the result (see [1, p. 780] or [2, p. 432]).

Lemma 4.3. *Under assumptions* (A9) *and* (A12), *there exist two functions* χ^1 *and* χ^2, *called correctors, which belong to* $\mathscr{C}^4(\mathbb{R}^n)$ *and such that*

1. χ^1 *and* χ^2 *are* Ξ*-periodic*;
2. $\int_\Xi \chi^1(\eta)\, m(\eta)\, d\eta = 0$ *and* $\int_\Xi \chi^2(\eta)\, m(\eta)\, d\eta = 0$;
3. $L\chi_l^1 = b_l$, $l = 1, ..., n$ *and* $L\chi_m^2 = g_m$, $m = 1, ..., k$.

4.2. *Homogenization*

We start this subsection with a technical lemma, rather similar to Lemma 10.2 on p. 499 of [2].

LEMMA 4.4. *Let* (A9) *hold. Let* $\Phi\colon \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be a function of class* $\mathscr{C}^{1,2,2}$, *such that, for each* (t, x) *in* $\mathbb{R}_+ \times \mathbb{R}^n$, $\eta \to \Phi(t, x, \eta)$ *is* Ξ*-periodic and* $\int_\Xi \Phi(t, x, \eta)\, m(\eta)\, d\eta = 0$.

Then, $\forall n \in \mathbb{N}$, $\forall t \in [0, n]$,

$$\mathbb{E}\left[\left|\int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} \Phi\left(s, X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}\right) ds\right|^2\right] \to 0 \qquad as \quad \varepsilon \to 0.$$

Proof. We solve, for (t, x) fixed, $L\Psi = -\Phi$. Following [2, p. 499], Ψ is Ξ-periodic and belongs actually to $\mathscr{C}^{1,2,2}$.

Itô's formula yields

$$\begin{aligned}
&\left[\Psi\left(s, X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}\right)\right]_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} \\
&\quad = \frac{1}{\varepsilon^2} \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} (L\Psi)\left(s, X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}\right) ds \\
&\qquad + \frac{1}{\varepsilon} \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} (D_\eta \Psi + \varepsilon D_x \Psi)\, \sigma\left(s, X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}\right) dW_s \\
&\qquad + \frac{1}{\varepsilon} \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} \Big(\varepsilon \partial_s \Psi + b_i \partial_{x_i} \Psi + a_{i,j} \partial^2_{x_i, \eta_j} \Psi \\
&\qquad + \frac{\varepsilon}{2} a_{i,j} \partial^2_{x_i, x_j} \Psi\Big)\left(s, X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}\right) ds.
\end{aligned}$$

It follows, since $L\Psi = -\Phi$ that

$$\begin{aligned}
&\int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} \Phi\left(s, X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}\right) ds \\
&\quad = -\varepsilon^2 \left[\Psi\left(s, X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}\right)\right]_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} + \varepsilon \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} \alpha^\varepsilon\left(s, X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}\right) ds \\
&\qquad + \varepsilon \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} \beta^\varepsilon\left(s, X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}\right) dW_s,
\end{aligned}$$

where we have set

$$\alpha^\varepsilon(t, x, \eta) = \left(\varepsilon \partial_t \Psi + b_i \partial_{x_i} \Psi + a_{i,j} \partial^2_{x_i, \eta_j} \Psi + \frac{\varepsilon}{2} a_{i,j} \partial^2_{x_i, x_j} \Psi\right)(t, x, \eta),$$

$$\beta^\varepsilon(t, x, \eta) = (D_\eta \Psi + \varepsilon D_x \Psi)\, \sigma(t, x, \eta).$$

Since all functions are continuous and Ξ-periodic, α^ε and β^ε are bounded on $[0, n] \times \bar{\mathcal{O}} \times \mathbb{R}^n$, uniformly with respect to ε.

Taking square and letting ε tend to 0, we obtain easily the result. ∎

Now, we can state our results. First, define the homogenized coefficients. Set, for (x, y, z) in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{k \times n}$,

$$\bar{a} = \int_\Xi (\mathrm{I_n} - D\chi^1(\eta))\, a(\eta)(\mathrm{I_n} - D\chi^1(\eta))^t\, m(\eta)\, d\eta,$$

$$\bar{f}(x, y, z) = \int_\Xi f(x, \eta, y, [z(\mathrm{I_n} - D\chi^1(\eta)) - D\chi^2(\eta)]\, \sigma(\eta))\, m(\eta)\, d\eta,$$

and consider the system of semilinear elliptic PDEs,

$$\begin{cases} \frac{1}{2}\bar{a}_{i,j}\partial^2_{i,j} u_m + \bar{f}_m(x, u, Du) = 0, & \text{in } \mathcal{O}, \qquad m = 1, ..., k, \\ u_{|\partial\mathcal{O}} = h_{|\partial\mathcal{O}}. \end{cases} \tag{20}$$

THEOREM 4.5. *Under the assumptions* (A9)–(A12), *the system* (20) *has a unique solution* $u \in \mathcal{C}^4(\bar{\mathcal{O}})$ *and moreover*

$$\forall x \in \bar{\mathcal{O}}, \qquad u^\varepsilon(x) \to u(x), \qquad \textit{as} \quad \varepsilon \to 0.$$

Proof. The existence and uniqueness of u, the solution of (20), and its regularity come from the smoothness of the data and the fact that $\bar{a}$ is elliptic. Indeed,

$$\begin{aligned} \bar{a} &\geqslant \beta \underline{m} \int_\Xi (\mathrm{I_n} - D\chi^1(\eta))(\mathrm{I_n} - D\chi^1(\eta))^t\, d\eta \\ &= \beta \underline{m} \left(\mathrm{I_n} + \int_\Xi D\chi^1(\eta)\, D\chi^1(\eta)^t\, d\eta \right) \\ &\geqslant \beta \underline{m}\, \mathrm{I_n}. \end{aligned}$$

Fix $m = 1, ..., k$. We consider the asymptotic expansion of u^ε_m

$$u^\varepsilon_m(x) = u_m(x) + \varepsilon u^1_m(x, \eta) + \varepsilon^2 \cdots,$$

and substituting it into (17) and comparing the terms of order ε^{-1}, we get, taking into consideration the definition of χ^1 and χ^2,

$$u^1_m(x, \eta) = -\chi^1_l(\eta)\, \partial_l u_m(x) - \chi^2_m(\eta).$$

This is why we think to calculate

$$d\chi_m^2\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) \qquad \text{and} \qquad d\left[\chi_l^1\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\partial_l u_m(X_x^\varepsilon(t))\right].$$

Using Itô's formula, we obtain

$$\begin{aligned}
du^\varepsilon(X_x^\varepsilon(t)) &= dY_x^\varepsilon(t) \\
&= -\left[\frac{1}{\varepsilon}\, g\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) + f\left(X_x^\varepsilon(t), \frac{X_x^\varepsilon(t)}{\varepsilon}, Y_x^\varepsilon(t), Z_x^\varepsilon(t)\right)\right] dt + Z_x^\varepsilon(t)\, dW_t,
\end{aligned}$$

and

$$\begin{aligned}
du_m(X_x^\varepsilon(t)) = {}& \left[\frac{1}{2}\, a_{i,j}\partial_{i,j}^2 u_m + \frac{1}{\varepsilon}\, b_i \partial_i u_m\right]\left(X_x^\varepsilon(t), \frac{X_x^\varepsilon(t)}{\varepsilon}\right) dt \\
& + Du_m(X_x^\varepsilon(t))\, \sigma\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) dW_t,
\end{aligned}$$

and taking into account (20), for $t \leqslant \tau_x^\varepsilon$,

$$\begin{aligned}
du_m(X_x^\varepsilon(t)) = {}& \left[\frac{1}{2}\,(a_{i,j} - \bar{a}_{i,j})\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\partial_{i,j}^2 u_m(X_x^\varepsilon(t))\right. \\
& \left. + \frac{1}{\varepsilon}\, b_i\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\partial_i u_m(X_x^\varepsilon(t))\right] dt \\
& - \bar{f}_m(X_x^\varepsilon(t), u(X_x^\varepsilon(t)), Du(X_x^\varepsilon(t)))\, dt \\
& + Du_m(X_x^\varepsilon(t))\, \sigma\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) dW_t.
\end{aligned}$$

In addition,

$$d\chi_m^2\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) = \frac{1}{\varepsilon^2}\, L\chi_m^2\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) dt + \frac{1}{\varepsilon}\, D\chi_m^2\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\sigma\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) dW_t,$$

and

$$\begin{aligned}
d\left[\chi_l^1\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\partial_l u_m(X_x^\varepsilon(t))\right] = {}& \frac{1}{\varepsilon^2}\, L\chi_l^1\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\partial_l u_m(X_x^\varepsilon(t))\, dt \\
& + \frac{1}{\varepsilon}\, dU_m^\varepsilon(t) + dA_m^\varepsilon(t),
\end{aligned}$$

where we have set, for $m=1, ..., k$,

$$A_m^\varepsilon(t)=\frac{1}{2}\int_0^t \chi_l^1\left(\frac{X_x^\varepsilon(s)}{\varepsilon}\right) a_{i,j}\left(\frac{X_x^\varepsilon(s)}{\varepsilon}\right) \partial^3_{i,j,l} u_m(X_x^\varepsilon(s))\,ds$$

$$+\int_0^t \chi_l^1\left(\frac{X_x^\varepsilon(s)}{\varepsilon}\right) D(\partial_l u_m)(X_x^\varepsilon(s))\,\sigma\left(\frac{X_x^\varepsilon(s)}{\varepsilon}\right) dW_s,$$

and

$$U_m^\varepsilon(t)=\int_0^t [b_j\chi_l^1+a_{i,j}\partial_{\eta_i}\chi_l^1]\left(\frac{X_x^\varepsilon(s)}{\varepsilon}\right) \partial^2_{l,j} u_m(X_x^\varepsilon(s))\,ds$$

$$+\int_0^t \partial_l u_m(X_x^\varepsilon(s))\, D\chi_l^1\left(\frac{X_x^\varepsilon(s)}{\varepsilon}\right) \sigma\left(\frac{X_x^\varepsilon(s)}{\varepsilon}\right) dW_s.$$

Taking into account Lemma 4.3, we get, for $m=1, ..., k$,

$$d\chi_m^2\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)=\frac{1}{\varepsilon^2} g_m\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) dt+\frac{1}{\varepsilon} D\chi_m^2\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) \sigma\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) dW_t,$$

and

$$d\left[\chi_l^1\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) \partial_l u_m(X_x^\varepsilon(t))\right]$$

$$=\frac{1}{\varepsilon^2} b_l\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right) \partial_l u_m(X_x^\varepsilon(t))\,dt+\frac{1}{\varepsilon}\,dU_m^\varepsilon(t)+dA_m^\varepsilon(t).$$

Finally, we set $C_m^\varepsilon(t)=\chi_m^2(X_x^\varepsilon(t)/\varepsilon)+\chi_l^1(X_x^\varepsilon(t)/\varepsilon)\,\partial_l u_m(X_x^\varepsilon(t))$, then

$$d[u_m^\varepsilon(X_x^\varepsilon(t))-u_m(X_x^\varepsilon(t))]$$

$$=-\frac{1}{2}\left[a_{i,j}\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)-\bar{a}_{i,j}\right] \partial^2_{i,j} u_m(X_x^\varepsilon(t))\,dt$$

$$-\left[f_m\left(X_x^\varepsilon(t), \frac{X_x^\varepsilon(t)}{\varepsilon}, Y_x^\varepsilon(t), Z_x^\varepsilon(t)\right)\right.$$

$$\left.-\bar{f}_m(X_x^\varepsilon(t), u(X_x^\varepsilon(t)), Du(X_x^\varepsilon(t)))\right] dt$$

$$+\left[Z_x^\varepsilon(t)-\left[Du_m(X_x^\varepsilon(t))-D\chi_m^2\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\right]\sigma\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\right] dW_t$$

$$+dU_m^\varepsilon(t)+\varepsilon d[A_m^\varepsilon(t)-C_m^\varepsilon(t)].$$

As a consequence, putting, in view of the definition of U^ε_m,

$$\widetilde{Y^\varepsilon_n}(t) = Y^\varepsilon_x(t) - u(X^\varepsilon_x(t \wedge \tau^\varepsilon_x)), \tag{21}$$

and

$$\widetilde{Z^\varepsilon_x}(t) = Z^\varepsilon_x(t) - \mathbb{1}_{t \leqslant \tau^\varepsilon_x}\left[Du(X^\varepsilon_x(t)) \left[\mathrm{I_n} - D\chi^1\left(\frac{X^\varepsilon_x(t)}{\varepsilon}\right)\right] - D\chi^2\left(\frac{X^\varepsilon_x(t)}{\varepsilon}\right)\right] \sigma\left(\frac{X^\varepsilon_x(t)}{\varepsilon}\right),$$

$(\widetilde{Y^\varepsilon_x}, \widetilde{Z^\varepsilon_x})$ solves the BSDE

$$-d\widetilde{Y^\varepsilon_n}(t) = \mathbb{1}_{t \leqslant \tau^\varepsilon_x}\left[F\left(X^\varepsilon_x(t), \frac{X^\varepsilon_x(t)}{\varepsilon}, \widetilde{Y^\varepsilon_x}(t), \widetilde{Z^\varepsilon_x}(t)\right) dt - \widetilde{Z^\varepsilon_x}(t)\, dW_t + \varepsilon dB^\varepsilon_x(t)\right], \tag{22}$$

with the terminal condition $\widetilde{Y^\varepsilon_x}(\tau^\varepsilon_x) = 0$, where we put $B^\varepsilon_x(t) = -A^\varepsilon(t) + C^\varepsilon(t)$, and for $m = 1, ..., k$, we set

$$\begin{aligned} F_m(x, \eta, y, z) = {} & f_m(x, \eta, y + u(x), z + [Du(x)[\mathrm{I_n} - D\chi^1(\eta)] - D\chi^2(\eta)]\, \sigma(\eta)) \\ & - \bar{f}_m(x, u(x), Du(x)) + \tfrac{1}{2}[a_{i,j}(\eta) - \bar{a}_{i,j} - 2b_j(\eta)\, \chi^1_i(\eta) \\ & - 2a_{k,j}(\eta)\, \partial_{\eta_k} \chi^1_i(\eta)]\, \partial^2_{i,j} u_m(x). \end{aligned}$$

Remark that we have, see [2, p. 416–417],

$$\int_\Xi [a_{i,j} - (b_j \chi^1_i + b_i \chi^1_j) - a_{k,j} \partial_{\eta_k} \chi^1_i - a_{k,i} \partial_{\eta_k} \chi^1_j](\eta)\, m(\eta)\, d\eta = \bar{a}_{i,j}.$$

Thus, from the definition of $\bar{f}$, for each x in $\mathbb{R}^n$,

$$\int_\Xi F(x, \eta, 0, 0)\, m(\eta)\, d\eta = 0. \tag{23}$$

Moreover, in view of assumption (A9) and the smoothness of u, χ^1 and χ^2, F belongs to $\mathscr{C}^4_{\mathrm{b}}$.

We want to apply Proposition 2.5 to $(\widetilde{Y^\varepsilon_n}, \widetilde{Z^\varepsilon_x})$ which solves the BSDE (22).

First, let us note that $(y, z) \to F(X^\varepsilon_x(t), X^\varepsilon_x(t)/\varepsilon, y, z)$ is uniformly Lipschitz with constants $C = \sup |D_y f(x, \eta, y, z)|$ and $\gamma = \sup |D_z(x, \eta, y, z)|$, and also μ-monotone, where μ is the constant appearing in assumption (A10). Hence, assumption (A11) says that $2\mu > \gamma^2$.

Consider ρ such that $\gamma^2-2\mu<\rho<0$. Since F is bounded, say by M, we easily show that

$$\mathbb{E}\left[\int_0^{\tau_x^\varepsilon} e^{\rho s}\left|F\left(X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}, 0, 0\right)\right|^2 ds\right] \leqslant -\frac{M^2}{\rho}.$$

Recall that $B_x^\varepsilon(t)=C_x^\varepsilon(t)-\int_0^t \varphi_x^\varepsilon(s)\,ds-\int_0^t \psi_x^\varepsilon(s)\,dW_s$, where

$$(C_x^\varepsilon)_m(t)=\chi_m^2\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)+\chi_l^1\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\partial_l u_m(X_x^\varepsilon(t)),$$

$$(\varphi_x^\varepsilon)_m(t)=\frac{1}{2}\chi_l^1\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)\partial^3_{i,j,l}u_m(X_x^\varepsilon(t))\,a_{i,j}\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right),$$

$$(\psi_x^\varepsilon)_m(t)=\chi_l^1\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right)D(\partial_l u_m)(X_x^\varepsilon(t))\,\sigma\left(\frac{X_x^\varepsilon(t)}{\varepsilon}\right).$$

It is worth noting that in view of the smoothness of the coefficients, $(\varphi_x^\varepsilon(t))$, $(\psi_x^\varepsilon(t))$ and $(C_x^\varepsilon(t))$ are bounded processes uniformly with respect to ε. We derive easily from this remark that, setting $V_x^\varepsilon(t)=\varepsilon B_x^\varepsilon(t)$,

$$\mathbb{E}\left[\int_0^{\tau_x^\varepsilon} e^{\rho s}\,|V_x^\varepsilon(s)|^2\,ds\right]\leqslant \varepsilon^2 C\left(1+\mathbb{E}\left[\int_0^{\tau_x^\varepsilon} e^{\rho s}\left|\int_0^s \psi_x^\varepsilon(u)\,dW_u\right|^2 ds\right]\right).$$

But, on the other hand,

$$\begin{aligned}\mathbb{E}\left[\int_0^{\tau_x^\varepsilon} e^{\rho s}\left|\int_0^s \psi_x^\varepsilon(u)\,dW_u\right|^2 ds\right] &\leqslant \int_0^\infty e^{\rho s}\,\mathbb{E}\left[\left|\int_0^{s\wedge\tau_x^\varepsilon} \psi_x^\varepsilon(u)\,dW_u\right|^2\right] ds\\ &\leqslant \int_0^\infty e^{\rho s}\,\mathbb{E}\left[\int_0^{s\wedge\tau_x^\varepsilon} \|\psi_x^\varepsilon(u)\|^2\,du\right] ds.\end{aligned}$$

$(\psi_x^\varepsilon(\cdot))$ being uniformly bounded, the previous inequality implies that

$$\mathbb{E}\left[\int_0^{\tau_x^\varepsilon} e^{\rho s}\,|V_x^\varepsilon(s)|^2\,ds\right]\leqslant C\varepsilon^2.$$

Moreover,

$$\begin{aligned}\mathbb{E}[e^{\rho\tau_x^\varepsilon}\,|V_x^\varepsilon(\tau_x^\varepsilon)|^2]\leqslant 3\varepsilon^2\,\mathbb{E}\Bigg[e^{\rho\tau_x^\varepsilon}\Bigg[&|C_x^\varepsilon|^2+\left(\int_0^{\tau_x^\varepsilon}|\varphi_x^\varepsilon(s)|\,ds\right)^2\\ &+\left|\int_0^{\tau_x^\varepsilon}\psi_x^\varepsilon(s)\,dW_s\right|^2\Bigg]\Bigg],\end{aligned}$$

and since $t \to t^2 e^{\rho t}$ is a bounded function on $\mathbb{R}_+$ (since $\rho < 0$), we get

$$\mathbb{E}[e^{\rho\tau_x^\varepsilon} |V_x^\varepsilon(\tau_x^\varepsilon)|^2] \leqslant C\varepsilon^2 \left(1 + \mathbb{E}\left[e^{\rho\tau_x^\varepsilon} \left| \int_0^{\tau_x^\varepsilon} \psi_x^\varepsilon(s)\, dW_s \right|^2 \right]\right).$$

Using Itô's formula and setting $\alpha_x^\varepsilon(t) = \int_0^t \psi_x^\varepsilon(s)\, dW_s$, we obtain, for t in $\mathbb{R}_+$,

$$e^{\rho(t \wedge \tau_x^\varepsilon)} |\alpha_x^\varepsilon(t \wedge \tau_x^\varepsilon)|^2 = \int_0^{t \wedge \tau_x^\varepsilon} e^{\rho s}(\rho\, |\alpha_x^\varepsilon(s)|^2 + \|\psi_x^\varepsilon(s)\|^2)\, ds$$
$$+ \int_0^{t \wedge \tau_x^\varepsilon} e^{\rho s} \alpha_x^\varepsilon(s) \cdot \psi_x^\varepsilon(s)\, dW_s.$$

Taking into account that $\rho < 0$, Burkholder, Davis, and Gundy's inequality yields

$$\begin{aligned}
&\mathbb{E}[\sup e^{\rho(t \wedge \tau_x^\varepsilon)} |\alpha_x^\varepsilon(t \wedge \tau_x^\varepsilon)|^2] \\
&\quad \leqslant -\frac{M^2}{\rho} + C\mathbb{E}\left[\left(\int_0^{\tau_x^\varepsilon} e^{2\rho s}\, |\alpha_x^\varepsilon(s)|^2\, \|\psi_x^\varepsilon(s)\|^2\, ds\right)^{1/2}\right] \\
&\quad \leqslant -\frac{M^2}{\rho} + C\left(\mathbb{E}\left[\int_0^{\tau_x^\varepsilon} e^{2\rho s}\, |\alpha_x^\varepsilon(s)|^2\, \|\psi_x^\varepsilon(s)\|^2\, ds\right]\right)^{1/2}.
\end{aligned}$$

From the boundedness of $(\psi_x^\varepsilon(\cdot))$, we get $\mathbb{E}[|\alpha_x^\varepsilon(t)|^2] \leqslant M^2 t$ and thus

$$\mathbb{E}[\sup e^{\rho(t \wedge \tau_x^\varepsilon)} |\alpha_x^\varepsilon(t \wedge \tau_x^\varepsilon)|^2] \leqslant C,$$

which implies that

$$\mathbb{E}[e^{\rho\tau_x^\varepsilon} |V_x^\varepsilon(\tau_x^\varepsilon)|^2] \leqslant C\varepsilon^2.$$

In particular, $e^{(\rho/2)\tau_x^\varepsilon} V_x^\varepsilon(\tau_x^\varepsilon)$ converges to 0 in L^2 and the same is true for $e^{\lambda\tau_x^\varepsilon} V_x^\varepsilon(\tau_x^\varepsilon)$.

In the same way, we get

$$\mathbb{E}[|V_x^\varepsilon(t \wedge \tau_x^\varepsilon)|^2] \leqslant C(1+t^2)\varepsilon^2.$$

Finally, in view of (23), since F is smooth, Lemma 4.4 implies that

$$\mathbb{E}\left[\left|\int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} F\left(X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}, 0, 0\right) ds\right|^2\right] \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Thus, all assumptions of Proposition 2.5 are satisfied, and therefore

$$\mathbb{E}[|\widetilde{Y_n^\varepsilon}(t \wedge \tau_x^\varepsilon)|^2] \to 0, \qquad \text{as} \quad \varepsilon \to 0.$$

This is true, in particular for $t=0$. Coming back to (21), we obtain

$$u^{\varepsilon}(x) \to u(x), \qquad \text{as} \quad \varepsilon \to 0.$$

The proof is complete. ∎

4.3. *The Case of Dimension One*

We finish with a stronger result for the case of a single equation.

THEOREM 4.6. *Suppose* $k=1$ *and let* (A9), (A10), *and* (A12) *hold. Then*

$$\forall x \in \overline{\mathcal{O}}, \qquad u^{\varepsilon}(x) \to u(x), \qquad \textit{as} \quad \varepsilon \to 0.$$

Proof. We keep the same notations as in the proof of the previous result.

Recall that $(\widetilde{Y^{\varepsilon}_x}, \widetilde{Z^{\varepsilon}_x})$ solves the BSDE (22) and we need to show that $\widetilde{Y^{\varepsilon}_x}(0) \to 0$.

Here, we cannot apply directly Proposition 3.5 because $B^{\varepsilon}_x(\tau^{\varepsilon}_x)$ is not bounded. Once again we overcome this difficulty by a change of variables.

Fix $0<\delta<\mu$ and set

$$\overline{Y^{\varepsilon}_x}(t) = e^{-\delta t}\left(\widetilde{Y^{\varepsilon}_x}(t) + \varepsilon C^{\varepsilon}_x(t) - \varepsilon \int_0^t \varphi^{\varepsilon}_x(s)\, ds\right)$$

$$\overline{Z^{\varepsilon}_x}(t) = e^{-\delta t}(\widetilde{Z^{\varepsilon}_x}(t) + \varepsilon \psi^{\varepsilon}_x(t)).$$

Itô's formula shows that $(\overline{Y^{\varepsilon}_x}, \overline{Z^{\varepsilon}_x})$ solves the BSDE

$$-d\overline{Y^{\varepsilon}_x}(t) = \mathbb{1}_{t \leqslant \tau^{\varepsilon}_x}[G^{\varepsilon}(t, \overline{Y^{\varepsilon}_x}(t), \overline{Z^{\varepsilon}_x}(t))\, dt - \overline{Z^{\varepsilon}_x}(t)\, dW_t],$$

with the final condition $\overline{Y^{\varepsilon}_x}(\tau^{\varepsilon}_x) = \varepsilon e^{-\delta \tau^{\varepsilon}_x}(C^{\varepsilon}_x(\tau^{\varepsilon}_x) - \int_0^{\tau^{\varepsilon}_x} \varphi^{\varepsilon}_x(s)\, ds)$, where we have set

$$G^{\varepsilon}(t, y, z)$$
$$= \delta y + e^{-\delta t} F\left(X^{\varepsilon}_x(t), \frac{X^{\varepsilon}_x(t)}{\varepsilon}, e^{\delta t} y - \varepsilon C^{\varepsilon}_x(t) + \varepsilon \int_0^t \varphi^{\varepsilon}_x(s)\, ds, e^{\delta t} z - \varepsilon \psi^{\varepsilon}_x(t)\right).$$

Since F is bounded, $G^{\varepsilon}(t, 0, z)$ is uniformly bounded. Moreover, G^{ε} is Lipschitz with constant $K+\delta$ and monotone; the constant is $\mu-\delta>0$.

Remember that $(C^{\varepsilon}_x(\cdot))$ and $(\varphi^{\varepsilon}_x(\cdot))$ are bounded processes uniformly with respect to ε. Hence,

$$\left\| \varepsilon e^{-\delta \tau^{\varepsilon}_x}\left(C^{\varepsilon}_x(\tau^{\varepsilon}_x) - \int_0^{\tau^{\varepsilon}_x} \varphi^{\varepsilon}_x(s)\, ds \right) \right\|_{\infty} \leqslant K\varepsilon,$$

and thus

$$\left\| \varepsilon e^{-\delta\tau_x^\varepsilon} \left(C_x^\varepsilon(\tau_x^\varepsilon) - \int_0^{\tau_x^\varepsilon} \varphi_x^\varepsilon(s)\, ds \right) \right\|_2 \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

The last point to establish is that for each $0 \leqslant t \leqslant n$,

$$\mathbb{E}\left[\left| \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} G^\varepsilon(s, 0, 0)\, ds \right|^2 \right] \to 0, \qquad \text{as} \quad \varepsilon \to 0.$$

We have

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} G^\varepsilon(s, 0, 0)\, ds \right|^2 \right] \\
&\quad \leqslant 2\mathbb{E}\left[\left| \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} e^{-\delta s} F\left(X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}, 0, 0 \right) ds \right|^2 \right] \\
&\qquad + 2\mathbb{E}\left[\left| \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} \left[G^\varepsilon(s, 0, 0) - e^{-\delta s} F\left(X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}, 0, 0 \right) \right] ds \right|^2 \right].
\end{aligned}$$

The first term tends to 0 as $\varepsilon \to 0$ by Lemma 4.4. Moreover, since

$$G^\varepsilon(t, 0, 0) = e^{-\delta t} F\left(X_x^\varepsilon(t), \frac{X_x^\varepsilon(t)}{\varepsilon}, -\varepsilon C_x^\varepsilon(t) + \varepsilon \int_0^t \varphi_x^\varepsilon(s)\, ds, -\varepsilon \psi_x^\varepsilon(t) \right)$$

we get, using Hölder's inequality and the fact that F is Lipschitz,

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} \left[G^\varepsilon(s, 0, 0) - e^{-\delta s} F\left(X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}, 0, 0 \right) \right] ds \right|^2 \right] \\
&\quad \leqslant K\varepsilon^2 n\, \mathbb{E}\left[\int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} e^{-\delta s} \left[|C_x^\varepsilon(s)|^2 + \left| \int_0^s \varphi_x^\varepsilon(u)\, du \right|^2 + |\psi_x^\varepsilon(s)|^2 \right] ds \right].
\end{aligned}$$

As the processes $(C_x^\varepsilon(\cdot))$, $(\varphi_x^\varepsilon(\cdot))$ and $(\psi_x^\varepsilon(\cdot))$ are uniformly bounded, we finally get, for another constant K,

$$\mathbb{E}\left[\left| \int_{t \wedge \tau_x^\varepsilon}^{n \wedge \tau_x^\varepsilon} \left[G^\varepsilon(s, 0, 0) - e^{-\delta s} F\left(X_x^\varepsilon(s), \frac{X_x^\varepsilon(s)}{\varepsilon}, 0, 0 \right) \right] ds \right|^2 \right] \leqslant K\varepsilon^2 n.$$

From Proposition 3.5, we get $\lim_{\varepsilon \to 0} \mathbb{E}[|\overline{Y_x^\varepsilon}(0)|^2] = 0$. Since

$$\overline{Y_x^\varepsilon}(0) = u^\varepsilon(x) - u(x) + \varepsilon \left[\chi^2\left(\frac{x}{\varepsilon}\right) + Du(x)\, \chi^1\left(\frac{x}{\varepsilon}\right) \right],$$

we have, from the boundedness of χ^1 and χ^2, that,

$$u^{\varepsilon}(x) \to u(x), \qquad \text{as} \quad \varepsilon \to 0,$$

which is the desired result. ∎

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