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On the splitting-up method for rough (partial) differential equations

Peter Friz [a,b], Harald Oberhauser [a,*]

[a] TU Berlin, Germany
[b] WIAS, Germany

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ABSTRACT

This article introduces the splitting method to systems driven by rough paths. The focus is on (nonlinear) partial differential equations with rough noise but we also cover rough differential equations. Applications to stochastic partial differential equations arising in control theory and nonlinear filtering are given.

1. Introduction

This article introduces the splitting-up method for evolution equations with perturbations/noise of *rough path* type. Our focus is on possibly nonlinear *rough partial differential equations* (RPDEs), formally written as

$$du = F\left(t, x, u, Du, D^2u\right) dt + \Lambda(t, x, u, Du)\, d\mathbf{z} \quad \text{on } (0, T] \times \mathbb{R}^e, \; u(0, \cdot) = u_0(\cdot); \tag{1}$$

but we also cover *rough differential equations* (RDEs) of the form

$$dy = V(y)\, dt + W(y)\, d\mathbf{z}, \quad y(0) \in \mathbb{R}^e.$$

In both examples, $\mathbf{z} = \mathbf{z}(t)$ is a rough path (of finite p-variation, some $p \geqslant 1$) in the sense of Lyons [21,19,13]; rough partial differential equations were introduced independently in [16] and [5]; see also [30]. Our interpretation of (1) is taken from the recent [6] (we give some recalls in Section 3), inspired by the pathwise SPDE theory of Lions and Souganidis [22–25].

Readers not familiar with rough paths may think of $d\mathbf{z}$ in a typical application as $\circ dB$, the Stratonovich differential of a multi-dimensional Brownian motion (with $p = 2 + \varepsilon$); the point is that

* Corresponding author.
E-mail address: h.oberhauser@gmail.com (H. Oberhauser).

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z contains *more* information (e.g. the iterated integral of Brownian motion, resp. the Lévy area) which then allows for a truly pathwise and robust treatment of the stochastic equations under consideration. Specific recalls on rough path theory will be given later as necessary in order to keep the article reasonably self-contained. Similarly, we will give recalls on PDE theory as necessary; it suffices to say that we have adopted here, as in [6], the viscosity point of view (Crandall–Ishii–Lions ...) which has the benefit of numerous stability properties.

Although, ultimately, there is the interaction of rough paths, viscosity and splitting ideas that forms the heart of this paper, it may be helpful to the reader to see the splitting argument separated from the rest. To this end, let us first consider the **case of ordinary differential equations** of the form

$$dy = V(y)\,dt + W(y)\,dt,$$

started at $y(0) \in \mathbb{R}^e$. We take the point of view that this is a special case of the controlled ordinary differential equation

$$dy = V(y)\,da + W(y)\,db;$$

it suffices to take the input signal $a(t) \equiv t$, $b(t) \equiv t$. The main observation here is that the "diagonal" input signal $t \mapsto (t, t)$ may be approximated by a "step" signal with the effect of following the vector field V for some (small) time Δ, then W for some time Δ, then V and so on. Thus, solving this ODE driven by a "step" input signal, amounts precisely to implement a splitting scheme for this equation. Since the step signals approximate the diagonal signal, as $\Delta \to 0$, one expects (correctly) that a *good understanding of how differential equations react to input signals yields convergence of the splitting scheme*. This observation was made by Terry Lyons (during the IRTG SMCP Summer School 2009 in Chorin, and quite possibly earlier and elsewhere). It is worth pointing out, that an ODE continuity result[1] which asserts convergence $y^n \to y$ under the assumption that[2]

$$\left|\left(a^n, b^n\right) - (a, b)\right|_{1\text{-var};[0,T]} \to 0$$

is *not* good enough in the present setting: the length of the step-signal approximations differs by a factor $\sqrt{2}$ from the length of the limiting diagonal signal. As a consequence, convergence of the driving signals does not take place in 1-variation but (uniform convergence with uniform 1-variation bounds, combined with an interpolation argument) only in $(1+\varepsilon)$-variation (which is good enough). We shall return to this example in full detail in Section 2.2 below.

Since the "*understanding of how differential equations react to input signals* (*in p-variation metrics*)" is the very goal and purpose of rough path theory, one should not be surprised that this line of reasoning can be pushed much further:

In the **case of rough differential equations**, of the form

$$dy = V(y)\,d\xi + W(y)\,d\mathbf{z}, \quad y(0) \in \mathbb{R}^e$$

with $\xi(t) = t$, the ideas used in the ODE example extend without too much difficulty. One considers "generalized step" approximations,[3] as formalized in Section 2.1 below, which amount to: follow the vector field V for some (small) time Δ, then solve the RDE driven by the vector field W along the p-variation rough path $\mathbf{z}$ restricted to $[0, \Delta]$ but run a double speed, then follow again V and so on. Since we can show that rough differential equations driven by $t \mapsto (\xi_t, \mathbf{z}_t)$ depend continuously on ξ resp. $\mathbf{z}$ in the appropriate sense, the argument is then completed essentially as in the ODE case. The details of this discussion are found in Section 2.3.

[1] For instance, [13, Theorem 3.18].

[2] We write $|z|_{1\text{-var};[0,T]} := \sup_{0 \leqslant t_1 < \cdots < t_n \leqslant T} \sum_{t_i} |z_{t_{i+1}} - z_{t_i}|$.

[3] ...already applied in the SPDE context by Krylov and Gyöngy [18] ... .

In the **case of rough partial differential equations**, of the form

$$du = F\big(t, x, u, Du, D^2u\big)\,d\xi + \Lambda(t, x, u, Du)\,d\mathbf{z} \quad \text{on } (0, T] \times \mathbb{R}^e,\ u(0,\cdot) = u_0(\cdot), \tag{2}$$

leaving aside for the moment any details about in which function spaces we seek to solve this equation, a new difficulty arises. Namely, we cannot hope to establish continuity of such solutions as function of ξ in $(1+\varepsilon)$-variation. Indeed, as is illustrated in the simple example of the heat-equation

$$\frac{\partial}{\partial t}u(t,x) = \left[\frac{\partial^2}{\partial x^2}u(t,x)\right]\dot{\xi}(t)$$

something like $\dot{\xi} \geqslant 0$ is required to make this PDE a well-posed initial value problem. Since $(1+\varepsilon)$-variation does not control the sign of $\dot{\xi}$, leaving alone existence of this derivative, we are forced to take a step back and only assume ξ to be continuous, non-decreasing (a side-effect of which is $|\xi|_{1\text{-var};[0,T]} = \xi_T - \xi_0 \equiv \xi_{0,T} \geqslant 0$). Thus, the best we can hope for is *stability of RPDEs* in the sense that solutions to the above RPDEs driven by $(\xi^n, \mathbf{z}^n)$ converge to the RPDE solution driven by $(\xi, \mathbf{z})$ under the assumption of uniform convergence with "uniform variation bounds" where the latter means $\sup_n \xi^n_{0,T} < \infty$ *and* uniform p-variation bounds on the rough paths $(\mathbf{z}^n)$. Stability of RPDEs in this sense will then be enough to conclude that "generalized step-approximations" cause convergence to the correct limit as $\Delta \to 0$. But this means exactly convergence of the RPDE splitting scheme. What remains to be done here, of course, is to *exhibit a setting and structural assumptions* (on the nonlinearity F resp. the noise structure Λ) *such that the necessary stability of solutions in* $(\xi, \mathbf{z})$ *is guaranteed.* Our choice here has been the setting of [6] where RPDEs are solved by a mixture of rough path and viscosity ideas. All this will be done in Section 4; some auxiliary material concerning parabolic viscosity solutions where $F = F(t, x, u, Du, D^2u)$ depends measurably, but not everywhere continuously in t, can be found in Appendix B.

At last, in Section 5 we shall apply this (purely deterministic!) splitting result to concrete stochastic partial differential equations where F is a possibly nonlinear (degenerate, HJB type) elliptic operator, Λ is a collection of first order operators, i.e.

$$\Lambda_k(t, x, u, Du) = \big(Du \cdot \sigma_k(t,x)\big) + u\nu_k(t,x) + g_k(t,x)$$

and σ, ν, g resp. $V, W = (W_i)$ are (collections) of vector fields on $[0,T] \times \mathbb{R}^e$ resp. $\mathbb{R}^e$. What follows are splitting results for stochastic HJB partial differential equations, with the additional freedom that the stochastic source can be non-Brownian (and even a non-semimartingale; e.g. when taking $\mathbf{z}$ as rough path lift of a fractional Brownian motion with Hurst parameter $\in (1/4, 1/2)$). Linear SPDEs are also covered of course, in particular we can handle the Zakai equation from nonlinear filtering written in the form

$$du = L\big(t, x, u, Du, D^2u\big)\,dt + \sum_{k=1}^{d} \Lambda_k(t, x, u, Du) \circ dB^k, \quad u(0,\cdot) = u_0(\cdot); \tag{3}$$

here L is here a linear (degenerate elliptic) operator,

$$L(t, x, r, p, X) = \operatorname{Trace}\big[A(t,x) \cdot X\big] + b(t,x) \cdot p + c(t, x, r),$$

and B a standard d-dimensional Brownian motion (for a detailed discussion and applications to the robustness problem in nonlinear filtering cf. [11]). At the risk of spelling out the obvious, let us recall explicitly what is meant by splitting in this context. For $n \in \mathbb{N}$ consider the partition $D^n = \{t^n_i = in^{-1}T,\ i = 0, \dots, n\}$ of the interval $[0,T]$ and define the approximation u_n recursively

$$u_n\big(t^n_{i+1}, \cdot\big) := [\mathbf{Q}_{t^n_i t^n_{i+1}} \circ \mathbf{P}_{t^n_i t^n_{i+1}}]\big(u_n\big(t^n_i, \cdot\big)\big)$$

with $\{\mathbf{P}_{st},\ 0 \leqslant s \leqslant t \leqslant T\}$ and $\{\mathbf{Q}_{st},\ 0 \leqslant s \leqslant t \leqslant T\}$ the solution operators of

$$dv = L\big(t, x, v, Dv, D^2 v\big)\,dt, \quad v(s,x) = v(x) \quad \text{and} \tag{4}$$

$$dw = \sum_{k=1}^{d} \Lambda_k(t, x, w, Dw) \circ dB_t^k, \quad w(s,x) = v(x). \tag{5}$$

That is, on each interval $[t_i^n, t_{i+1}^n]$ one solves first the PDE (4) on $[t_i^n, t_{i+1}^n]$ with initial data $u_n(t_i^n, \cdot)$ and then one uses its solution as initial value for Eq. (5) (so-called "predictor" and "corrector" steps in [10]). Under appropriate conditions, one can show that u_n converges to u and also derive rates of convergence [18].

At last, a few words about the splitting methods in general and its past applications to SPDEs in particular. The splitting-up method (which runs under many names: dimensional splitting, operator splitting, Lie–Trotter–Kato formula, Baker–Campbell–Hausdorff formula, Chernoff formula, leapfrog method, predictor–corrector method, etc.) is one of the most standard methods for calculating solutions of (stochastic, ordinary, partial) differential equations numerically; for a survey we recommend [27]. For S(P)DEs a splitting-up method was introduced in [2] for the Zakai equation in filtering and has received much attention since. We explicitly mention [18] which extends the previous results to general linear SPDEs of the form (3). All the above mentioned authors use (to the best of our knowledge) either semigroup theory or stochastic calculus to prove splitting results but neither is available for (1) due to the nonlinear form of F and the pathwise noise $\mathbf{z}$ which does not allow for semimartingale techniques.

To summarize the contribution of this article: we take the novel point of view that *splitting-up results of RPDEs* (*and then*: *SPDEs*) *follow from stability in rough path sense*. Old techniques (such as the generalized step-approximations of Krylov and Gyöngy [18]) remain important. The key difficulty is to exhibit a setting of RPDEs in which the required rough path stability holds; at present (and this is the only place where viscosity techniques matter!) we are only able to do this in the setting of rough viscosity solutions as introduced in [6]. A splitting result for the RPDEs considered by Gubinelli and Tindel [16], for instance, would follow from the same arguments provided the required stability can be established in their setting (we suspect this is possible, but the technicalities will be completely different; in particular, viscosity theory would not play any role). At last, due to the generality of Eq. (1) we do not give rates of convergence but hope to return to this question, under additional structural assumptions on the non-linearity, in the future.

2. Splitting: The case of ODEs and RDEs

2.1. Generalized step-approximations

For fixed $\Delta > 0$ and $t \in [0,T]$ set[4] $t_\Delta = \lfloor t/\Delta \rfloor \Delta$ and $t^\Delta = \lfloor t/\Delta \rfloor \Delta + \Delta$ (i.e. $[t_\Delta, t^\Delta)$ is the interval in the equidistant partition of $[0,T]$ with mesh size Δ that contains t). Define two time changes

$$a(\Delta, t) = \begin{cases} t_\Delta + 2(t - t_\Delta), & t_\Delta \leqslant t \leqslant t_\Delta + \Delta/2, \\ t^\Delta, & t_\Delta + \Delta/2 < t \leqslant t^\Delta, \end{cases} \qquad b(\Delta, t) = a\left(\Delta, t + \frac{\Delta}{2}\right).$$

That is, $a(\Delta, \cdot)$ runs on the first half of each interval $[t_\Delta, t^\Delta)$ with double speed from t_Δ to t^Δ and stays still in the second half, whereas $b(\Delta, \cdot)$ does this in opposite order. Clearly,

$$\big(a(\Delta, \cdot), b(\Delta, \cdot)\big) \to id_2(\cdot)$$

[4] $\lfloor . \rfloor$ denotes the lower floor function.

uniformly on $[0,T]$ where $id_2 : t \mapsto (t,t) \in \mathbb{R}^2$. It is also clear that, if $\mathbb{R}^2$ is equipped with Euclidean structure,

$$\forall \Delta > 0: \quad \left|\left(a(\Delta,\cdot), b(\Delta,\cdot)\right)\right|_{1\text{-var};[0,T]} = \sqrt{2}T;$$

from basic interpolation [13, Lemma 5.27] we see that

$$\left(a(\Delta,\cdot), b(\Delta,\cdot)\right) \to id_2(\cdot) \quad \text{in } (1+\varepsilon)\text{-variation, any } \varepsilon > 0. \tag{6}$$

2.2. Splitting ODEs

For given $n \geqslant 1$ denote by D^n the partition $\{\frac{k}{n}T,\ k = 0,\dots,n\}$ of $[0,T]$. Let $V, W \in \mathrm{Lip}^1(\mathbb{R}^e, \mathbb{R}^e)$. We are interested in splitting of the ODE

$$dy_t = V(y_t)\,dt + W(y_t)\,dt, \quad y(0) = y_0 \in \mathbb{R}^e. \tag{7}$$

Denote the solution of (7) by $\pi_{V,W}(0, y_0; id_2)$. Classic Lie-splitting corresponds to the approximation of the path id_2 by the sequence of paths $t \mapsto (a(n^{-1}, t), b(n^{-1}, t))$. Therefore let y^n be the ODE solution of

$$dy_t^n = V(y_t)\,da\left(n^{-1}, t\right) + W(y_t)\,db\left(n^{-1}, t\right), \quad y(0) = y_0 \in \mathbb{R}^e,$$

i.e. $y^n = \pi_{V,W}(0, y_0; (a(n^{-1},\cdot), b(n^{-1},\cdot)))$. Define the solution operators $\{\mathbf{P}_{s,t}^{n;V},\ 0 \leqslant s \leqslant t \leqslant T\}$ and $\{\mathbf{P}_u^V,\ 0 \leqslant u \leqslant T\}$, as maps from $\mathbb{R}^e$ to $\mathbb{R}^e$, as

$$\mathbf{P}_{s,t}^{n,V}(x) := \pi_V\left(s, x; a\left(n^{-1},\cdot\right)\right)_t \quad \text{and} \quad \mathbf{P}_{t-s}^V(x) := \pi_V(0, x; id_1)_{t-s} = \pi_V(s, x; id_1)_t$$

(here $id_1 : t \mapsto t$ and $\mathbf{P}^V$ is a one parameter group); similarly define $\mathbf{Q}^{n,W}$ and $\mathbf{Q}^W$. By the definition of a and b we have

$$y^n\left(t + \frac{1}{n}\right) = \left[\mathbf{Q}_{t,t+1/n}^{n;W} \circ \mathbf{P}_{t,t+1/n}^{n;V}\right]\left(y^n(t)\right) \tag{8}$$

whenever t is a point in the dissection D^n. Also note that $\mathbf{P}_{s,t}^{n,V} \equiv \mathbf{P}_{t-s}^V$ resp. $\mathbf{Q}_{s,t}^{n,W} \equiv \mathbf{Q}_{t-s}^W$ for $s,t \in D^n$. Since

$$\left(a\left(n^{-1},\cdot\right), b\left(n^{-1},\cdot\right)\right) \to id_2$$

in $(1+\varepsilon)$-variation, it follows from "continuity of the Itô-map: $z \mapsto \pi_V(0, y_0; z)$ in q-variation, $1 \leqslant q < 2$" (see [26,21] or [19, Chapter 1.4]), that

$$\pi_{V,W}\left(0, y_0; \left(a\left(n^{-1},\cdot\right), b\left(n^{-1},\cdot\right)\right)\right) = y^n \to y = \pi_{V,W}(0, y_0; id_2) \quad \text{as } n \to \infty$$

in $|.|_{\infty;[0,T]}$ norm[5] (and even in $(1+\varepsilon)$-variation semi-norm). Using the identity (8) one recovers the "classic Lie-splitting"

$$\left[\mathbf{Q}_{1/n}^W \circ \mathbf{P}_{1/n}^V\right]^{\lfloor t/n \rfloor}(y_0) \to y_t \quad \text{as } n \to \infty \text{ for every } t \in [0,T]$$

where y is the ODE solution of (7). Moreover, the convergence holds in $|.|_{\infty;[0,T]}$ norm and by interpolation even in stronger $(1+\varepsilon)$-variation norm for every $\varepsilon > 0$.

[5] Strictly speaking this argument requires $V, W \in \mathrm{Lip}^{1+\varepsilon}$ for some $\varepsilon > 0$; but see Remark 1 below.

Remark 1. Although the above example, due to Terry Lyons, is a pretty illustration how q-variation can matter in simple situations, it will be crucial for our applications to RPDEs to understand how one can get away with $q = 1$ in the above argument. The problem is, of course, that

$$\big(a(\Delta,\cdot), b(\Delta,\cdot)\big) \nrightarrow id_2 \quad \text{in 1-variation.}$$

Nonetheless, recent variations on the theme [13, Chapter 10] imply that the Itô map is also continuous “under uniform convergence with uniform p-variation bounds $1 \leqslant p < 2$” (and for $p \geqslant 2$ in the appropriate rough path sense). The case $p = 1$ is much simpler [13, Theorem 3.15] and valid for $V, W \in \mathrm{Lip}^1(\mathbb{R}^e, \mathbb{R}^e)$. Since

$$\big(a\big(n^{-1},\cdot\big), b\big(n^{-1},\cdot\big)\big) \to id_2$$

uniformly with uniform 1-variation bounds (cf. above derivation of (6) above) we can already assert that $y^n \to y$ uniformly on $[0,T]$. (If required, convergence in $(1+\varepsilon)$-variation is recovered by interpolation, using that one has uniform 1-variation bounds on the y^n.)

2.3. Splitting RDEs

We begin with some recalls on rough differential equations (with drift) of the form

$$dy_t = V(y_t)\,d\xi_t + W(y_t)\,d\mathbf{z}_t, \quad y(0) = y_0 \in \mathbb{R}^e, \tag{9}$$

where $p \in [1,\infty)$, $\mathbf{z} \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$ is a p-rough path in the sense of [19] or [13, Chapter 9] and $\xi \in C^{1\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$. Let us remark straight away that for sufficiently smooth (e.g. Lip^p) vector fields V, W this equation is well posed. Indeed, there is a canonical way of viewing the pair $(\xi, \mathbf{z})$ as a rough path, say $S(\xi,\mathbf{z})$; the reason is that all additionally required iterated integrals are canonically defined as Young (in fact: Riemann–Stieltjes) integrals. The pairing map $(\xi,\mathbf{z}) \mapsto S(\xi,\mathbf{z}) \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^{1+d}))$ can be seen to be continuous [13, Remark 9.32]; in particular, one can use the standard theory (e.g. [13, Chapter 10]) of rough differential equations, driven by $S(\xi,\mathbf{z})$. It should come as no surprise that weaker regularity assumptions on V are possible if one exploits the fact that ξ is much more regular than $\mathbf{z}$. Indeed, following [13, Chapter 12], for

$$V \in \mathrm{Lip}^1\big(\mathbb{R}^e, \mathbb{R}^e\big), \qquad W = (W_i) \subset \mathrm{Lip}^p\big(\mathbb{R}^e, \mathbb{R}^e\big)$$

there exists a unique global solution to (9), denoted by $\pi_{V,(W)}(0, y_0; (\xi,\mathbf{z}))$. As in the previous section on splitting for ODEs, we will deal with approximations which converge uniformly with uniform 1- (resp. p-) variation bounds, but in general do not converge in 1- (resp. p-) variation. We shall thus appeal straight away to the recent “variations on the theme” [13, Chapter 10] which imply that the Itô map is also continuous “under uniform convergence with uniform p-variation bounds”. In particular [13, Theorem 12.11] if $(\xi^n, z^n)_n \subset C^{1\text{-var}}([0,T],\mathbb{R}) \times C^{1\text{-var}}([0,T],\mathbb{R}^d)$ converges to $(\xi,\mathbf{z})$ in the sense[6]

$$\begin{gathered}\sup_n \big\| S_{[p]}\big(z^n\big)\big\|_{p\text{-var};[0,T]} + \sup_n \big\|\xi^n\big\|_{1\text{-var};[0,T]} < \infty,\\ d_{\infty;[0,T]}\big(S_{[p]}\big(z^n\big), \mathbf{z}\big) + \big|\xi^n - \xi\big|_{\infty;[0,T]} \to 0 \quad \text{as } n \to \infty\end{gathered} \tag{11}$$

[6] $S_{[p]}$ denotes the canonical lift to $G^{[p]}(\mathbb{R}^d)$-valued path given by iterated Riemann–Stieltjes integration;

$$S_{[p]}\big(z^n\big)_{s,t} = 1 + \int_s^t dz^n_{u_1} + \cdots + \int_{s\leqslant u_1 \leqslant \cdots \leqslant u_{[p]} \leqslant t} dz^n_{u_1} \otimes \cdots \otimes dz^n_{u_{[p]}}. \tag{10}$$

$d_{0;[0,T]}(S_{[p]}(z^n), \mathbf{z}) := \sup_{s,t\in[0,T]} d(S_{[p]}(z^n)_{s,t}, \mathbf{z}_{s,t})$ where d is the Carnot–Caratheodory metric on $G^{[p]}(\mathbb{R}^d)$.

and $(y^n)_n$ is obtained by solving, for each fixed $n \in \mathbb{N}$,

$$dy_t^n = V\big(y_t^n\big)\,d\xi_t^n + \sum_{i=1}^d W_i\big(y_t^n\big)\,dz_t^{n,i}, \quad y^n(0) = y_0 \in \mathbb{R}^e$$

then y^n converges (uniformly on $[0,T]$, with uniform p-variation bounds) to $\pi_{V,(W)}(0, y_0; (\xi, \mathbf{z}))$, the unique solution of (9).

Let us now turn to the problem of establishing convergence of the splitting method for a rough differential equation of the form

$$dy_t = V(y_t)\,dt + W(y_t)\,d\mathbf{z}_t, \quad y(0) = y_0 \in \mathbb{R}^e;$$

we want to solve, alternating, an ODE along V, a (drift free) rough differential equation along W driven by $\mathbf{z}$, again an ODE along V and so on. As a first step (recall that definition of a, b as given in Section 2.1) we establish

Lemma 2. *Let $\mathbf{z} \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$, $\xi \in C^{1\text{-var}}([0,T], \mathbb{R})$. If we define $\xi^\Delta(t) = \xi(a(\Delta, t))$, $\mathbf{z}^\Delta(t) = \mathbf{z}(b(\Delta, t))$ then $\xi^\Delta \in C^{1\text{-var}}([0,T], \mathbb{R})$, $\mathbf{z}^\Delta \in C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$ and*

$$\sup_{\Delta>0}\big\|\mathbf{z}^\Delta\big\|_{p\text{-var};[0,T]} + \sup_{\Delta>0}\big|\xi^\Delta\big|_{1\text{-var};[0,T]} < \infty,$$
$$d_0\big(\mathbf{z}^\Delta, \mathbf{z}\big) + \big|\xi^\Delta - \xi\big|_{\infty;[0,T]} \to 0 \quad \textit{as } \Delta \to 0.$$

Proof. First note that the variation norm is invariant under parametrization which implies the first two statements. The second statement concerning uniform convergence of $\mathbf{z}^\Delta$ resp. ξ^Δ follows easily from (uniform) continuity of $\mathbf{z}$ resp. ξ on $[0,T]$. □

Ultimately we are interested in a sequence of paths giving a Lie-splitting scheme and therefore we define

$$\xi^n(t) := \xi\big(a\big(n^{-1}, t\big)\big) \quad \text{and} \quad \mathbf{z}_t^n := \mathbf{z}\big(b\big(n^{-1}, t\big)\big)$$

(this is a slight abuse of the notation of Lemma 2 where $\xi^n, \mathbf{z}^n$ would be denoted as $\xi^{n^{-1}}, \mathbf{z}^{n^{-1}}$). Similar to the ODE example, define the solution operator $\{\mathbf{P}_{s,t}^{n;V},\ 0 \leqslant s \leqslant t \leqslant T\}$ mapping points in $\mathbb{R}^e$ to $\mathbb{R}^e$ as $\mathbf{P}_{s,t}^{n,V}(x) := \pi_V(s, x; \xi^n)_t$ and in an analogous way the operators $\mathbf{P}^V$, $\mathbf{Q}^{n,W}$ and $\mathbf{Q}^W$. It remains to show that

$$\mathbf{P}_{s,t}^{n;V} \equiv \mathbf{P}_{s,t}^V \quad \text{and} \quad \mathbf{Q}_{s,t}^{n;W} \equiv \mathbf{Q}_{s,t}^W \quad \text{for } s, t \in D^n$$

(in contrast with the ODE example, $\mathbf{P}^V$ and $\mathbf{Q}^W$ are now "two parameter" groups due to the explicit time-dependence of ξ and $\mathbf{z}$): since $G^{[p]}(\mathbb{R}^d)$ is a geodesic space, there exists a sequence of paths (concatenations of geodesics on the sequence of dissections D^m), $(z^m)_m \subset C^{1\text{-var}}([0,T], \mathbb{R}^d)$ with $S_{[p]}(z_{s,t}^m) = \mathbf{z}_{s,t}$ for $s, t \in D^m$, such that

$$\sup_m \big\|S_{[p]}\big(z^m\big)\big\|_{p\text{-var}} < \infty,$$
$$d_0\big(S_{[p]}\big(z^m\big), \mathbf{z}\big) \to 0 \quad \text{as } m \to \infty,$$

with $S_{[p]}$ as in (10). Now define $\mathbf{z}^{n,m}(t) := \mathbf{z}^m(a(n^{-1}, t))$ and note that $\mathbf{z}^m$ and $\mathbf{z}^{n,m}$ have bounded 1-variation ($\mathbf{z}^m$ by construction, $\mathbf{z}^{n,m}$ because the variation norm is invariant under parametrization). Hence, for m, n fixed we deal with an ODE as in the example above and therefore

$$\pi_W\left(s, x; \mathbf{z}^{n,m}\right)_{s,t} = \pi_W\left(s, x; \mathbf{z}^m\right)_{s,t}$$

for $s, t \in D^n$. Keeping n fixed and letting $m \to \infty$, the LHS converges to $\pi_W(s, x; \mathbf{z}^n)_{s,t}$ by Lemma 2 and Lyons' limit theorem and the RHS to $\pi_W(s, x; \mathbf{z})_{s,t}$; we can conclude $\mathbf{Q}^{n;W}_{s,t} \equiv \mathbf{Q}^W_{s,t}$ for $s, t \in D^n$. A similar argument shows $\mathbf{P}^{n;V}_{s,t} \equiv \mathbf{P}^V_{s,t}$ for $s, t \in D^n$. We can now finish the argument in the same way as in the previous example: solutions of $\pi_{V,(W)}(s, x; (\xi^n, \mathbf{z}^n))$ converge uniformly to $\pi_{V,(W)}(s, x; (\xi, \mathbf{z}))$. On neighboring points $s, t \in D^n$, $\pi_{V,(W)}(s, x; (\xi^n, \mathbf{z}^n))_{s,t}$ can be identified as

$$\left[\mathbf{Q}^{n,W}_{s,t} \circ \mathbf{P}^{n,V}_{s,t}\right](x) = \left[\mathbf{Q}^W_{s,t} \circ \mathbf{P}^V_{s,t}\right](x).$$

Hence, for every $t \in [0, T]$

$$y^{n;\mathrm{Split}}_t := \prod_{k=0}^{\lfloor t/n \rfloor - 1} \left[\mathbf{Q}^W_{k/n,(k+1)/n} \circ \mathbf{P}^V_{k/n,(k+1)/n}\right](y_0) \to \pi_{V,(W)}\left(0, y_0; (\xi, \mathbf{z})\right)_t \quad \text{as } n \to \infty$$

and it is easy to see that this convergence is uniform on $[0, T]$. Moreover, $\sup_n |y^{n;\mathrm{Split}}|_{p\text{-var};[0,T]} < \infty$ which implies by interpolation convergence in $(p + \varepsilon)$-variation norm of $y^{n;\mathrm{Split}}$ for every $\varepsilon > 0$.

Remark 3. Similarly, one shows convergence of a splitting scheme based on the different order $\mathbf{P}^V \circ \mathbf{Q}^W$ (instead of $\mathbf{Q}^W \circ \mathbf{P}^V$, as above). Note also, that we restrict ourselves in this article to Lie-splitting schemes but the methods can be easily modified to include Strang-splitting (see [27] for the difference between Lie- and Strang-splitting schemes) by using an appropriate modification of the time change. Further, we just deal with equidistant partitions. Numerous variations of all this are possible (as long as one can show convergence in a rough path topology of the approximating sequence) and such modifications are of great importance for rates of convergence; we shall return to this in future work.

3. Rough partial differential equations

We now turn to (scalar) rough partial differential equations (RPDEs) of the form

$$du = F\left(t, x, u, Du, D^2u\right) dt + \Lambda(t, x, u, Du)\, d\mathbf{z} \quad \text{on } (0, T] \times \mathbb{R}^e,\ u(0, \cdot) = u_0(\cdot). \tag{12}$$

Even in the smooth case, i.e. when the "rough differential" $d\mathbf{z}$ is replaced by the classical differential $dz = \dot{z}(t)\, dt$, or the term $\Lambda(t, x, u, Du)\, d\mathbf{z}$ is omitted altogether, it is not a trivial matter to solve this equation. Under structural assumptions on F satisfied in particular by examples from optimal (stochastic) control theory, viscosity theory [7,12] provides a convenient framework to discuss existence, uniqueness and stability of solutions. Since our approach to (12) is based on some transformation under which, loosely speaking some pointwise transform $v(t, x)$ of $u(t, x)$ satisfies a PDE of the above form without noise term $\Lambda(t, x, u, Du)\, d\mathbf{z}$, it may be helpful to recall some basic ideas of (second order) viscosity solutions; this will be done in Section 3.1 below.

Let us informally discuss the idea of an RPDE before we give the precise definition in Section 4. As in previous section we need to replace dt by $d\xi$ for a sufficiently big class of ξ's (in particular, we wish to include $a(n^{-1}, \cdot) \notin C^1$). A real-valued, bounded and continuous function u on $[0, T] \times \mathbb{R}^e$ is called a solution to

$$du = F\left(t, x, u, Du, D^2u\right) d\xi + \Lambda(t, x, u, Du)\, d\mathbf{z} \tag{13}$$

if u is the uniform limit (locally on compacts) of any sequence (u^n) of (standard) *viscosity solutions*

$$du^n = F\big(t,x,u^n,Du^n,D^2u^n\big)\,d\xi_n + \sum_{i=1}^{d} \Lambda_k\big(t,x,u^n,Du^n\big)\,dz_n^i \quad \text{on } (0,T]\times\mathbb{R}^e,$$

$$u^n(0,x) = u_0(x) \quad \text{on } \mathbb{R}^e,$$

where $(z_n) \subset C^\infty([0,T],\mathbb{R}^d)$ and $(\xi_n) \subset C^\infty([0,T],\mathbb{R})$ are sequences of smooth driving signals, converging to $(\xi,\mathbf{z})$ "uniformly, with uniform variation bounds" in the sense of (11). However, to apply the methods outlined in the sections above to derive a splitting method, care has to be taken: firstly, in the RDE case sequences (ξ_n, z_n) of smooth paths converging to $(\xi,\mathbf{z})$ gave rise to a solution, but in the RPDE case, if $d\xi_n/dt < 0$, one cannot expect to treat even the simple heat-equation (in the language of viscosity theory: $F\dot{\xi}_n$ ceases to be degenerate elliptic). Secondly, as already noted above, the typical choice $\xi^n(t) = a(n^{-1},t)$ for Lie-splitting leads to $F\dot{\xi}^n$ which is not continuous in t ($t \mapsto a(n^{-1},t) = \xi^n(t)$ has kinks, hence $d\xi^n$ does not even exist on points of the partition).[7] Such time-discontinuities are in general difficult to handle in a viscosity setting. Thirdly, one has to show the continuous dependence of the solution[8] of (13) on not only $\mathbf{z}$ but continuous dependence on $(\xi,\mathbf{z})$ in a rough path sense.

The first point is dealt with by characterizing the class of admissible approximations to ξ, leading to the path space $C_0^{1\text{-var};+}([0,T],\mathbb{R})$, described in Section 3.2 below. Section 3.3 deals with approximations of nonlinear PDEs. At last, Section 4 gives the precise definitions of rough viscosity solutions and stability, contains the main theorem and examples of RPDEs which exhibit such a stability.

3.1. Viscosity solutions

Consider a real-valued function $u = u(t,x)$ with $t \in [0,T]$, $x \in \mathbb{R}^e$ and assume $u \in C^2$ is a classical subsolution,

$$\partial_t u - F\big(t,x,u,Du,D^2u\big) \leqslant 0,$$

where F is a continuous function, *proper* in the sense of *degenerate ellipticity* ($F(t,x,r,p,A) \leqslant F(t,x,r,p,A+B)$ whenever $B \geqslant 0$ in the sense of symmetric matrices) and F is non-increasing in r (this rules out conservation laws). The idea is to consider a (smooth) test function φ and look at a local maxima $(\hat{t},\hat{x})$ of $u-\varphi$. Basic calculus implies that $Du(\hat{t},\hat{x}) = D\varphi(\hat{t},\hat{x})$, $D^2u(\hat{t},\hat{x}) \leqslant D\varphi(\hat{t},\hat{x})$ and, from degenerate ellipticity,

$$\partial_t\varphi - F\big(\hat{t},\hat{x},u,D\varphi,D^2\varphi\big) \leqslant 0. \tag{14}$$

This suggests to define a *viscosity subsolution* (at the point $(\hat{x},\hat{t})$) to $\partial_t - F = 0$ as a (possibly upper-semi-) continuous function u with the property that (14) holds for any test function. Similarly, *viscosity supersolutions* are defined by reversing inequality in (14); *viscosity solutions* are both super- and subsolutions. A different point of view is to note that $u(t,x) \leqslant u(\hat{t},\hat{x}) - \varphi(\hat{t},\hat{x}) + \varphi(t,x)$ for (t,x) near $(\hat{t},\hat{x})$. A simple Taylor expansion then implies

$$u(t,x) \leqslant u(\hat{t},\hat{x}) + a(t-\hat{t}) + p\cdot(x-\hat{x}) + \frac{1}{2}(x-\hat{x})^T\cdot X\cdot(x-\hat{x}) + o\big(|\hat{x}-x|^2 + |\hat{t}-t|\big) \tag{15}$$

[7] One could avoid discontinuous time-dependence by restricting the class of splitting schemes (s.t. $(\dot{\xi}^n) \subset C^1$). However, many popular schemes (Strang, Lie, etc.) would then not be covered.

[8] The results in [6] and [11] do not cover this due the time-discontinuity of the approximating sequence $(d\xi^n)$.

as $|\hat{x}-x|^2+|\hat{t}-t|\to 0$ with $a=\partial_t\varphi(\hat{t},\hat{x})$, $p=D\varphi(\hat{t},\hat{x})$, $X=D^2\varphi(\hat{t},\hat{x})$. Moreover, if (15) holds for some (a,p,X) and u is differentiable, then $a=\partial_t u(\hat{t},\hat{x})$, $p=Du(\hat{t},\hat{x})$, $X\leqslant D^2u(\hat{t},\hat{x})$, hence by degenerate ellipticity

$$\partial_t\varphi+F(\hat{t},\hat{x},u,p,X)\leqslant 0.$$

Pushing this idea further leads to a definition of viscosity solutions based on a generalized notion of "$(\partial_t u, Du, D^2u)$" for non-differentiable u, the so-called parabolic semijets, and it is a well-known fact that both definitions are equivalent. As a typical result,[9] the initial value problem $(\partial_t-F)u=0$, $u(0,\cdot)=u_0\in \mathrm{BUC}(\mathbb{R}^e)$ has a unique solution in $\mathrm{BUC}([0,T]\times\mathbb{R}^e)$ provided $F=F(t,x,u,Du,D^2u)$ is continuous, proper and satisfies a (well-known) technical condition.[10] In fact, uniqueness follows from a stronger property known as *comparison*: assume u (resp. v) is a supersolution (resp. subsolution) and $u_0\geqslant v_0$; then $u\geqslant v$ on $[0,T]\times\mathbb{R}^e$. A key feature of viscosity theory is what workers in the field simply call *stability properties*. For instance, it is relatively straightforward to study $(\partial_t-F)u=0$ via a sequence of approximate problems, say $(\partial_t-F^n)u^n=0$, provided $F^n\to F$ locally uniformly and some a priori information on the u^n (e.g. locally uniform convergence, or locally uniform boundedness).[11] Note the stark contrast to the classical theory where one has to control the actual derivatives of u^n.

3.2. The space $C_0^{1\text{-var},+}([0,T],\mathbb{R})$

As pointed out above, we have to avoid to fall outside the scope of (degenerate) elliptic PDE theory by selecting a reasonable class of approximations to ξ. Using the notation $C^{0,1\text{-var}}([0,T],\mathbb{R})$ for the closure of the space of smooth paths in variation norm $(\overline{C^\infty}^{|\cdot|_{1\text{-var}}}([0,T],\mathbb{R}))$ we recall that

$$\begin{aligned} W_0^{1,1}\big([0,T],\mathbb{R}\big) &\equiv \left\{x\colon [0,T]\to\mathbb{R},\ \exists y\in L^1\big([0,T],\mathbb{R}\big)\ \text{s.t.}\ x(t)=\int_0^t y(u)\,du\right\}\\ &=\big\{x\colon [0,T]\to\mathbb{R},\ x\ \text{absolutely continuous},\ x(0)=0\big\}\\ &=\overline{\big\{x\colon [0,T]\to\mathbb{R},\ x\in C^\infty,\ x(0)=0\big\}}^{|\cdot|_{1\text{-var}}}\\ &\equiv C_0^{0,1\text{-var}}\big([0,T],\mathbb{R}\big)\subsetneq C_0^{1\text{-var}}\big([0,T],\mathbb{R}\big). \end{aligned}$$

The above discussion motivates the following definition.

Definition 4. $C_0^{1,+}([0,T],\mathbb{R})=\{\xi\in C_0^1([0,T],\mathbb{R})\colon\ \xi_T=T,\dot{\xi}>0\}$.

Note that for a fixed $\Delta>0$ the paths $a(\Delta,\cdot)$ and $b(\Delta,\cdot)$ are not elements of $C_0^{1,+}([0,T],\mathbb{R})$, but elements of the closure in supremum norm, $\overline{C_0^{1,+}}^{|\cdot|_\infty}([0,T],\mathbb{R})$. Working with $C_0^{1,+}$ enables us in Section 3.3 below to give a short proof of existence, uniqueness and stability for PDEs of the form $\partial_t u=F(x,u,Du,D^2u)\dot{\xi}_t$ with paths $\xi\in\overline{C_0^{1,+}}^{|\cdot|_\infty}([0,T],\mathbb{R})$.

Proposition 5. *Denote* $C_0^{1\text{-var},+}([0,T],\mathbb{R})=\overline{C_0^{1,+}}^{|\cdot|_\infty}([0,T],\mathbb{R})$. *Then*

[9] BUC(...) denotes the space of bounded, uniformly continuous functions.
[10] (3.14) of the User's Guide [7] in the case of a bounded domain instead of $\mathbb{R}^e$; see also [6] when the domain is $\mathbb{R}^e$. In this case one needs the additional assumption that $F=F(t,x,u,p,X)$ is uniformly continuous whenever u,p,X are bounded. It is folklore of the subject that solutions can be seen to be BUC in time–space for BUC initial data and that one can work on the closed time interval $[0,T]$; the situation is summarized in [9]. Alternatively, one may replace $\mathrm{BUC}([0,T]\times\mathbb{R}^e,\mathbb{R})$ throughout by $\mathrm{BC}([0,T)\times\mathbb{R}^e,\mathbb{R})$.
[11] What we have in mind here is the *Barles–Perthame method of semi-relaxed limits* [12].

$$C_0^{1\text{-var},+}\big([0,T],\mathbb{R}\big) = \Bigg\{\xi_t \in C_0\big([0,T],\mathbb{R}\big)\colon\ \xi_T = T \text{ and } \exists \dot{\xi} \in L^1\big([0,T],\mathbb{R}_{\geqslant 0}\big),$$

$$\exists a \in C_0^{1\text{-var}}\big([0,T],\mathbb{R}_{\geqslant 0}\big),\ a \text{ increasing, } \dot{a} = 0 \text{ a.s. and } \xi_t = a_t + \int_0^t \dot{\xi}_u\,du\Bigg\}$$

and $C_0^{1\text{-var},+}([0,T],\mathbb{R}) \subsetneq C_0^{1\text{-var}}([0,T],\mathbb{R})$.

Proof. $\subset$: Let (ξ^ε) be a Cauchy sequence w.r.t. $|.|_\infty$. Since $(C_0([0,T],\mathbb{R}), |.|_\infty)$ is complete, ξ^ε converges uniformly to some $\xi \in C_0([0,T],\mathbb{R})$. This ξ is monotone (not necessarily strict) increasing and hence $|\xi|_{1\text{-var};[0,T]} < \infty$ (recall that $\xi_T = T$). Every function of finite 1-variation is Lebesgue-a.e. differentiable and has a representation of the form

$$\xi_t = a_t + \int_{[0,t]} \dot{\xi}_u\,du$$

where a is a function of 1-variation with $\dot{a} = 0$ Lebesgue-a.e. Now $\xi_{s,s+h} \geqslant 0$, for every $h > 0$; $s \in [0,1)$. Hence we have $a_{s,s+h} \geqslant -\int_s^{s+h} \dot{\xi}_u\,du$ and sending $h \to 0$ shows together with $\dot{a} = 0$ a.s. that a is monotone increasing and this implies $\dot{\xi}_u \geqslant 0$ Lebesgue-a.e.

$\supset$: $F(t) := \xi_t$ defines a continuous distribution function on $[0,T]$ and let X be a random variable with distribution F. For $\varepsilon > 0$ denote by F^ε the distribution function of the random variable $X + \varepsilon N$ where N is a standard normal, independent of X. Clearly, $X + \varepsilon N \to X$ a.s. as $\varepsilon \to 0$ and so the F^ε converges pointwise. By the lemma below this implies uniform convergence of F^ε to F. It remains to show that $\xi_t^\varepsilon := F^\varepsilon(t)$ is C^1 but this follows from

$$F^\varepsilon(t) = \int_0^t F(t-u)\,dF_{\varepsilon N}(u)$$

where $F_{\varepsilon N}$ is the distribution function of εN. □

We need a "Dini-type" lemma for the proof of Proposition 5.

Lemma 6. *Let* $(f^\eta)_{\eta>0} \subset C_0([0,T],\mathbb{R})$, $f^\eta(1) = 1$, *each* f^η *increasing* (*not necessarily strictly*) *and assume* $f^\eta \to f \in C_0([0,T],\mathbb{R})$ *pointwise as* $\eta \to 0$. *Then,* $|f^\eta - f|_{\infty;[0,T]} \to 0$ *as* $\eta \to 0$.

Proof. Given $\varepsilon > 0$ we can choose an $n \in \mathbb{N}$ big enough s.t.

$$\left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right| < \frac{\varepsilon}{2}$$

for every $i \in \{0,1,\dots,n\}$. Now choose η small enough such that

$$\left| f^\eta\left(\frac{i}{n}\right) - f\left(\frac{i}{n}\right)\right| < \frac{\varepsilon}{2}$$

for all $i \in \{0, 1, \dots, n\}$. This implies $|f^{\eta}(x) - f(x)| < \varepsilon$ since every x is an element of (at least one) interval $[\frac{i-1}{n}, \frac{i}{n}]$ and by monotonicity and using above estimates

$$f^{\eta}(x) \leqslant f^{\eta}\left(\frac{i}{n}\right) \leqslant f\left(\frac{i-1}{n}\right) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leqslant f(x) + \varepsilon.$$

Similarly

$$f^{\eta}(x) > f(x) + \varepsilon$$

and so $|f^{\eta}(x) - f(x)| < \varepsilon$ for all $x \in [0, T]$. □

Remark 7. Concerning the choice of notation $C_0^{1\text{-var},+}$, note that the space of paths of finite 1-variation $C_0^{1\text{-var}}$ is given as the closure of C^1-paths with uniformly bounded 1-variation. Since paths in $C_0^{1,+}$ have 1-variation bounded by T, the notation $C_0^{1\text{-var},+}([0, T], \mathbb{R})$ seems natural.

Remark 8. The paths $a(\Delta, \cdot)$ and $b(\Delta, \cdot)$ converge in $(1+\varepsilon)$-variation to the path $id_1 : t \mapsto t$ for every $\varepsilon > 0$ and therefore also uniformly (but not in 1-variation!).

Remark 9. $C_0^{1\text{-var},+}([0, T], \mathbb{R})$ is not a linear space but a convex subset of $C_0^{1\text{-var}}([0, T], \mathbb{R})$.

Remark 10. Despite the restriction of $C_0^{1\text{-var},+}([0, T], \mathbb{R})$ to paths with $\xi(T) = T$ which is convenient in the proofs, one can handle PDEs with general increasing processes by rescaling; e.g. replace ξ by $\tilde{\xi}(t) := \xi(t)\frac{T}{\xi(T)} \in C_0^{1\text{-var},+}([0, T], \mathbb{R})$ and write $du = F(t, x, u, Du, D^2u)\, d\xi = \tilde{F}(t, x, u, Du, D^2u)\, d\tilde{\xi}$ with $\tilde{F} := F\frac{\xi(T)}{T}$.

3.3. PDEs with discontinuous time-dependence

This section extends the notion of viscosity solutions to equations of the form

$$du = F\big(t, x, u, Du, D^2u\big)\, d\xi(t), \quad u(0, x) = u_0(x),$$

with F a continuous function and $\xi \in C_0^{1\text{-var},+}([0, T], \mathbb{R})$. In Appendix B (Proposition 25) we show that this solution concept coincides with the notion of generalized viscosity solutions (going back to [17]) whenever the latter exists. In view of applications in Section 5 and to keep technicalities down, we focus here on time-independent $F = F(x, u, Du, D^2u)$. However, a proof for general time-dependent F is given in Appendix A (Proposition 21).

Proposition 11. *Let $(\xi^{\varepsilon})_{\varepsilon} \subset C_0^{1,+}([0, T], \mathbb{R})$ converge uniformly to some $\xi \in C_0^{1\text{-var};+}([0, T], \mathbb{R})$ as $\varepsilon \to 0$. Assume $(v^{\varepsilon})_{\varepsilon} \subset \mathrm{BUC}([0, T] \times \mathbb{R}^e, \mathbb{R})$ are locally uniformly bounded viscosity solutions of*

$$\partial_t v^{\varepsilon} = F^{\varepsilon}\big(x, v^{\varepsilon}, Dv^{\varepsilon}, D^2v^{\varepsilon}\big)\dot{\xi}_t^{\varepsilon}, \quad v^{\varepsilon}(0, x) = v_0(x) \tag{16}$$

with $F^{\varepsilon} : \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e \to \mathbb{R}$, $\mathbb{S}^e$ denoting the space of symmetric $(e \times e)$-matrices, a continuous and degenerate elliptic function. Further, assume that F^{ε} converges locally uniformly to a continuous, degenerate elliptic function F and that a comparison result holds for $\partial_t - F = 0$. Then there exists a v such that

$$v^{\varepsilon} \to v \quad \text{locally uniformly as } \varepsilon \to 0.$$

Further, v does not depend on the choice of the sequence approximating ξ and we also write $v \equiv v^{\xi}$ to emphasize the dependence on ξ and say that v solves

$$dv = F\big(x, v, Dv, D^2 v\big)\, d\xi_t, \quad v(0,x) = v_0(x).$$

We prepare the proof with

Lemma 12. *Let $\xi \in C_0^{1,+}([0,T],\mathbb{R})$,*

$$F : [0,T] \times \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^n \to \mathbb{R}$$

and let $\tilde{F}(t,x,r,p,X) = F(\xi^{-1}(t), r, x, p, X)$ for $(t,x,r,p,X) \in [0,T] \times \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^n$. Then

1. *if u is a sub- (resp. super)solution of $\partial_t - F\dot{\xi} = 0$, $u(0,\cdot) = u_0(\cdot)$ then $w(t,x) := u(\xi_t^{-1}, x)$ is a sub- (resp. super)solution of $\partial_t - \tilde{F} = 0$, $w(0,\cdot) = u_0(\cdot)$;*
2. *if w is a sub- (resp. super)solution of $\partial_t - \tilde{F} = 0$, $w(0,\cdot) = w_0(\cdot)$ then $u(t,x) := w(\xi_t, x)$ is a sub- (resp. super)solution of $\partial_t - F\dot{\xi}$, $u(0,\cdot) = w_0(\cdot)$.*

Proof. 1. Let φ be a $C^{1,2}$ function defined on an open neighborhood of $[0,T] \times \mathbb{R}^e$ and assume $w - \varphi$ attains a local maximum at $(\hat{t},\hat{x})$. Then

$$w(\hat{t},\hat{x}) - \varphi(\hat{t},\hat{x}) = u\big(\xi^{-1}(\hat{t}),\hat{x}\big) - \varphi(\hat{t},\hat{x}) = u\big(\xi^{-1}(\hat{t}),\hat{x}\big) - \tilde{\varphi}\big(\xi^{-1}(\hat{t}),\hat{x}\big)$$

where $\tilde{\varphi}(\hat{t},\hat{x}) := \varphi(\xi_{\hat{t}},\hat{x})$. Using that u is a subsolution gives

$$\begin{aligned}
\partial_t \hat{\varphi}|_{\xi_{\hat{t}}^{-1},\hat{x}} &\leqslant F\big(\hat{x}, \xi^{-1}(\hat{t}), u\big(\xi^{-1}(\hat{t}),\hat{x}\big), D\tilde{\varphi}|_{\xi_{\hat{t}}^{-1},\hat{x}}, D^2\tilde{\varphi}|_{\xi_{\hat{t}}^{-1},\hat{x}}\big)\dot{\xi}_{\xi_{\hat{t}}^{-1}} \\
&= F\big(\hat{x}, \xi^{-1}(\hat{t}), w(\hat{t},\hat{x}), D\varphi|_{\hat{t},\hat{x}}, D^2\varphi|_{\hat{t},\hat{x}}\big)\dot{\xi}_{\xi_{\hat{t}}^{-1}} \\
&= \tilde{F}\big(\hat{x}, \hat{t}, w(\hat{t},\hat{x}), D\varphi|_{\hat{t},\hat{x}}, D^2\varphi|_{\hat{t},\hat{x}}\big)\dot{\xi}_{\xi_{\hat{t}}^{-1}}
\end{aligned}$$

where we have used that $D\tilde{\varphi}|_{\xi_{\hat{t}}^{-1},\hat{x}} = D\varphi|_{\hat{t},\hat{x}}$ and $D^2\tilde{\varphi}|_{\xi_{\hat{t}}^{-1},\hat{x}} = D^2\varphi|_{\hat{t},\hat{x}}$. Since $\partial_t \hat{\varphi}|_{\xi_t^{-1},x} = \partial_t \varphi|_{t,x}\ \dot{\xi}|_{\xi_t^{-1}}$ and $\dot{\xi} > 0$ it follows that

$$\partial_t \varphi|_{\hat{t},x} \leqslant \tilde{F}\big(\hat{x}, \hat{t}, w(\hat{t},\hat{x}), D\varphi|_{\hat{t},\hat{x}}, D^2\varphi|_{\hat{t},\hat{x}}\big).$$

The same argument applies when u is a supersolution.

2. Assume $u - \varphi$ attains a local maximum at $(\hat{t},\hat{x}) \in [0,T] \times \mathbb{R}^e$. Then

$$u(\hat{t},\hat{x}) - \varphi(\hat{t},\hat{x}) = w\big(\xi(\hat{t}),\hat{x}\big) - \varphi(\hat{t},\hat{x}) = w\big(\xi(\hat{t}),\hat{x}\big) - \tilde{\varphi}\big(\xi(\hat{t}),\hat{x}\big)$$

where $\tilde{\varphi}(t,x) := \varphi(\xi^{-1}(t),x)$. Using that w is a subsolution gives

$$\begin{aligned}
\partial_t \hat{\varphi}|_{\xi_{\hat{t}},\hat{x}} &\leqslant \tilde{F}\big(\hat{x}, \xi(\hat{t}), w\big(\xi(\hat{t}),\hat{x}\big), D\tilde{\varphi}|_{\xi_{\hat{t}},\hat{x}}, D^2\tilde{\varphi}|_{\xi(\hat{t}),\hat{x}}\big) \\
&= F\big(\hat{x}, \hat{t}, u(\hat{t},\hat{x}), D\varphi|_{\hat{t},\hat{x}}, D^2\varphi|_{\hat{t},\hat{x}}\big)
\end{aligned}$$

where we have used that $D\tilde{\varphi}|_{\xi_{\hat{t}}^{-1},\hat{x}} = D\varphi|_{\hat{t},\hat{x}}$ and $D^2\tilde{\varphi}|_{\xi_{\hat{t}}^{-1},\hat{x}} = D^2\varphi|_{\hat{t},\hat{x}}$. Since $\dot{\xi} > 0$ and $\partial_t\hat{\varphi}|_{\xi_t,x} = \partial_t\varphi|_{t,x}$ $(\xi^{-1})'|_{\xi_t} = \partial_t\varphi|_{t,x}(\dot{\xi}(t))^{-1}$ it follows that

$$\partial_t\varphi|_{\hat{t},x} \leqslant F\big(\hat{x},\hat{t},u(\hat{t},\hat{x}),D\varphi|_{\hat{t},\hat{x}},D^2\varphi|_{\hat{t},\hat{x}}\big)\dot{\xi}(\hat{t}).$$

The same argument applies when w is a supersolution. □

Proof of Proposition 11. Set $w^\varepsilon(t,x) := v^\varepsilon((\xi^\varepsilon)^{-1}(t),x)$, by Lemma 12,

$$v^\varepsilon \text{ is a solution of } \partial_t - F^\varepsilon\dot{\xi}^\varepsilon = 0 \quad \text{iff} \quad w^\varepsilon \text{ is a solution of } \partial_t - F^\varepsilon = 0.$$

Let

$$\overline{w} := \limsup_{\varepsilon}{}^* w^\varepsilon \quad \text{and} \quad \underline{w} := \liminf_{\varepsilon}{}_* w^\varepsilon$$

and note that

$$F^\varepsilon(x,r,p,X) \to F(x,r,p,X) \quad \text{locally uniformly.}$$

Standard viscosity theory tells us that $\overline{w}$ and $\underline{w}$ are sub- resp. supersolutions of $\partial_t - F = 0$. Using the method of semi-relaxed limits (by the definition, $\overline{w} \geqslant \underline{w}$ and the reversed inequality follows from comparison which holds by the assumption) we conclude that $w(t,x) := \overline{w}(t,x) = \underline{w}(t,x)$. Further, use a Dini-type argument, for every compact set $K \subset \mathbb{R}^e$,

$$\big|w^\varepsilon - w\big|_{\infty;[0,T]\times K} = \sup_{t\in[0,T],\, x\in K}\big|w^\varepsilon(t,x) - w(t,x)\big| \to 0 \quad \text{as } \varepsilon \to 0.$$

Now define

$$v(t,x) := w\big(\xi(t),x\big)$$

and we get the claimed convergence $v^\varepsilon \to v$. □

4. Splitting RPDEs

4.1. Rough viscosity solutions and their stability

Solutions of RDEs can be defined via completion of the solution map in rough path metrics as limit points of ODEs. Similarly one can define solutions of partial differential equations with rough path noise.

Definition 13. Let $\mathbf{z} \in C_0^{0,p\text{-var}}([0,T],G^{[p]}(\mathbb{R}^d))$, $\xi \in C_0^{1\text{-var};+}([0,T],\mathbb{R})$. Further, let $(\xi^\varepsilon, z^\varepsilon) \subset C_0^{1,+}([0,T],\mathbb{R}) \times C_0^{1\text{-var}}([0,T],\mathbb{R}^d)$ converge to $(\xi,\mathbf{z})$ in the sense of (11) and assume the PDE

$$du^\varepsilon = F\big(x,u^\varepsilon,Du^\varepsilon,D^2u^\varepsilon\big)\,d\xi^\varepsilon + \sum_{k=1}^{d}\Lambda_k\big(t,x,u^\varepsilon,Du^\varepsilon\big)\,dz^{\varepsilon;k} \quad \text{on } (0,T]\times\mathbb{R}^n,$$

$$u(0,\cdot) = u_0(\cdot) \in \mathrm{BUC}\big(\mathbb{R}^e,\mathbb{R}\big)$$

has a unique solution $u^\varepsilon \in \mathrm{BUC}([0,T]\times\mathbb{R}^e;\mathbb{R})$ for every $\varepsilon > 0$. We call every limit point u of (u^ε) (in BUC topology) a solution of the RPDE

$$du = F\big(x,u,Du,D^2u\big)\,d\xi + \Lambda(t,x,u,Du)\,d\mathbf{z} \quad \text{on } (0,T]\times\mathbb{R}^e,\ u(0,x)=u_0(x). \tag{17}$$

If additionally, the limit is unique, does not depend on the choice of the approximating sequence $(\xi^\varepsilon, z^\varepsilon)$ and the map

$$(\xi,\mathbf{z}) \in C_0^{1\text{-var},+}\big([0,T],\mathbb{R}\big)\times C_0^{0,p\text{-var}}\big([0,T],G^{[p]}\big(\mathbb{R}^d\big)\big) \mapsto u \in \mathrm{BUC}\big([0,T]\times\mathbb{R}^e,\mathbb{R}\big)$$

is continuous then we say that the RPDE (17) is stable in a rough path sense and we also write $u = u^{\xi,\mathbf{z}}$ (or $u = u^{\mathbf{z}}$ when $\xi(t)=t$) to emphasize dependence on the rough path $(\xi,\mathbf{z})$.

Remark 14. Similar to the SDE situation where continuous dependence on the Brownian path holds if the commutator of the vector fields vanishes (or if the noise is one-dimensional), one can in special cases give pathwise meaning to (17) without rough path metrics (more precisely, if one chooses noise of the form $\sum_i H_i(Du)\,dz^i$ or one-dimensional noise, the methods of [22–25] apply); in [29] this is exploited to state a splitting scheme for the mean curvature evolution equation. However, note that rough path metrics are already needed to establish continuous dependence on the driving signal when $F=0$ and $\Lambda_k(t,x,r,p) = \langle \nu_k(x), p\rangle$, $k=1,\dots,d$, $d \geqslant 2$, ν_k bounded, smooth vector fields on $\mathbb{R}^e$; cf. [6].

4.2. The main theorem

We are now able to formulate our main theorem. The proof comes as an easy consequence of the results in the previous sections. In Section 5 we show that the regularity assumptions of the theorem are met by a large class of RPDEs/SPDEs.

Theorem 15. *Let $\mathbf{z} \in C_0^{0,p\text{-var}}([0,T],G^{[p]}(\mathbb{R}^d))$, $\xi \in C_0^{+,1\text{-var}}([0,T],\mathbb{R})$ and assume $u \in \mathrm{BUC}$ is the unique solution of the stable* (*in the sense of Definition* 13) *RPDE,*

$$du = F\big(x,u,Du,D^2u\big)\,d\xi + \Lambda(t,x,u,Du)\,d\mathbf{z} \quad \text{on } (0,T]\times\mathbb{R}^e,\ u(0,\cdot)=u_0(\cdot)\in\mathrm{BUC}\big(\mathbb{R}^e,\mathbb{R}\big). \tag{18}$$

Assume further that also the two (*R*)*PDEs given by setting either $F\equiv 0$ or $\Lambda\equiv 0$ in* (18) *are stable. Denote $\{\mathbf{P}_{s,t}, 0\leqslant u\leqslant T\}$ and $\{\mathbf{Q}_{s,t}, 0\leqslant s\leqslant t\leqslant T\}$ the solution operators*

$$\mathbf{P}_{s,t}: \mathrm{BUC}\big(\mathbb{R}^e,\mathbb{R}\big)\to \mathrm{BUC}\big([0,T]\times\mathbb{R}^e,\mathbb{R}\big), \quad \varphi \mapsto v,$$
$$\mathbf{Q}_{s,t}: \mathrm{BUC}\big(\mathbb{R}^e,\mathbb{R}\big)\to \mathrm{BUC}\big([0,T]\times\mathbb{R}^e,\mathbb{R}\big), \quad \phi \mapsto w,$$

with

$$dv = F\big(x,v,Dv,D^2v\big)\,d\xi, \quad v(0,x)=\varphi(x),$$
$$dw = \Lambda(t,x,w,Dw)\,d\mathbf{z}, \quad w(s,x)=\phi(x)$$

and set

$$u^{n;\mathrm{Split}}(t,x) := \prod_{i=0}^{\lfloor tn^{-1}\rfloor-1} [\mathbf{Q}_{in^{-1},(i+1)n^{-1}} \circ \mathbf{P}_{in^{-1},(i+1)n^{-1}}]\big(u_0(x)\big).$$

Then

$$u^{n;\mathrm{Split}} \to u \quad \textit{locally uniformly as } n \to \infty.$$

Proof. Define $\mathbf{z}^n = \mathbf{z}(b(n^{-1}, t))$, $\xi^n(t) = \xi(a(n^{-1}, t))$. By Lemma 2, $(\xi^n, \mathbf{z}^n) \to_n (\xi, \mathbf{z})$ in the sense of Lemma 2 and by stability, the solutions u^n of

$$du^n = F\big(x, u^n, Du^n, D^2 u^n\big)\, d\xi^n + \Lambda\big(t, x, u^n, Du^n\big)\, dz^n \quad \text{on } (0, T] \times \mathbb{R}^n,\ u(0, x) = u_0(x)$$

converge to u, the solution of (18). Now for each given n one can identify on points of the dissection $\{iT/n,\ i = 0, \ldots, n\}$ the solutions of $u^{n;\mathrm{Split}}$ with u^n and by the fact that the assumed stability u^n converges locally uniformly to u. □

4.3. Examples of stable RPDEs

This section shows stability in a rough path sense for a large class of RPDEs. Splitting results then follow readily by Theorem 15. Throughout this section $\mathbf{z}$ is a geometric p-rough path, i.e. $\mathbf{z} \in C_0^{0,p\text{-var}}([0, T], G^p(\mathbb{R}^d))$ for a $p \geqslant 1$.

Proposition 16. *Let*

$$F(x, r, p, X) = \inf_{\alpha \in \mathcal{A}} \big\{ \mathrm{Tr}\big[A(x; \alpha)^T X\big] + b(x; \alpha) \cdot p + f(x, r; \alpha) \big\}$$

where A, σ and b, f satisfy the assumption of Proposition 17 *uniformly with respect to $\alpha \in \mathcal{A}$ and $\nu = (\nu_k)_{k=1}^d \subset \mathrm{Lip}^\gamma(\mathbb{R}^e, \mathbb{R}^e)$ $\gamma > p + 2$. Then the RPDE*

$$du = F\big(x, u, Du, D^2 u\big)\, dt + Du \cdot \nu(x)\, d\mathbf{z}, \quad u(0, \cdot) \equiv u_0$$

has a unique solution $u^{\mathbf{z}} \in \mathrm{BUC}([0, T] \times \mathbb{R}^e)$ and is stable in a rough path sense.

Proof. The proof follows by combining the technique of "semi-relaxed rough path limits" as introduced in [6] in combination with Proposition 11. The steps are very similar to [6] so we do not spell out all the details but the key remark is that u^ε is a solution of

$$du^\varepsilon = F\big(x, u^\varepsilon, Du^\varepsilon, D^2 u^\varepsilon\big)\, d\xi^\varepsilon + Du^\varepsilon \cdot \nu(x)\, dz^\varepsilon, \quad u^\varepsilon(0, \cdot) \equiv u_0(\cdot)$$

iff $\tilde{v}^\varepsilon(t, x) := u(t, \psi^\varepsilon(t, x))$ is a "classical" viscosity solution of

$$d\tilde{v}^\varepsilon = \tilde{F}^\varepsilon\big(x, \tilde{v}^\varepsilon, D\tilde{v}^\varepsilon, D^2 \tilde{v}^\varepsilon\big)\, d\xi^\varepsilon, \quad \tilde{v}^\varepsilon(0, \cdot) \equiv u_0(\cdot)$$

where ψ^ε is the flow of an ODE driven by z^ε and F^ε is determined by F, ψ^ε and the Jacobian and Hessian of ψ^ε. One can verify that comparison holds for this PDE and the usual rough path stability results imply the convergence of $\psi^\varepsilon \to \psi^{\mathbf{z}}$, $\psi^{\mathbf{z}}$ being the flow of an RDE driven by $\mathbf{z}$; this together with the assumptions on F then implies $F^\varepsilon \to \check{F}$. Using Proposition 11 this is enough to conclude that $\tilde{v}^\varepsilon \to \tilde{v}$ as $\varepsilon \to 0$ locally uniformly, with $\tilde{v}$ being the solution of

$$d\tilde{v} = F\big(x, \tilde{v}, D\tilde{v}, D^2 \tilde{v}\big)\, d\xi, \quad \tilde{v}(0, \cdot) \equiv u_0.$$

Defining $u^{\mathbf{z}}(t, x) := \tilde{v}(t, (\psi^{\mathbf{z}})^{-1}(t, x))$ gives then the unique, stable RPDE solution. □

Proposition 17. *Let*

$$L(x,r,p,X) = \mathrm{Tr}\big[A(x)^T X\big] + b(x)\cdot p + f(x,r),$$
$$\Lambda_k(t,x,r,p) = \big(p\cdot\sigma_k(t,x)\big) + r\nu_k(t,x) + g_k(t,x)$$

with $A(x) = \bar\sigma(x)\bar\sigma^T(x) \in \mathbb{S}^e$ and $\bar\sigma : \mathbb{R}^e \to \mathbb{R}^{e\times e'}$, $b(x) : \mathbb{R}^e \to \mathbb{R}^e$ bounded, Lipschitz continuous in x. Also assume that $f : \mathbb{R}^e \times \mathbb{R} \to \mathbb{R}$ is continuous, bounded whenever r remains bounded, and with a lower Lipschitz bound, i.e. $\exists C < 0$ s.t.

$$f(x,r) - f(x,s) \geqslant C(r-s) \quad \text{for all } r \geqslant s,\ x\in\mathbb{R}^e$$

and that the coefficients of $\Lambda = (\Lambda_1,\ldots,\Lambda_d)$, that is σ, ν and g, have Lip^γ-regularity for $\gamma > p+2$. Then the RPDE

$$du = L\big(x,u,Du,D^2u\big)\,dt + \Lambda(t,x,u,Du)\,d\mathbf{z}, \quad u(0,\cdot)\equiv u_0(\cdot) \tag{19}$$

is stable in a rough path sense and has a unique solution $u^{\mathbf{z}} \in \mathrm{BUC}([0,T]\times\mathbb{R}^e,\mathbb{R})$.

Proof. Eq. (19) has been recently studied in [11]; the proof follows the same logic as the proof of the statement above; again the key remark being that u^ε is a solution of

$$du^\varepsilon = L\big(x,u^\varepsilon,Du^\varepsilon,D^2u^\varepsilon\big)\,d\xi^\varepsilon + \Lambda\big(t,x,u^\varepsilon,Du^\varepsilon\big)\,dz^\varepsilon, \quad u(0,\cdot)\equiv u_0(\cdot) \tag{20}$$

iff

$$\tilde v^\varepsilon(t,x) := \big(\phi^\varepsilon\big)^{-1}\big(t,u^\varepsilon\big(t,\psi^\varepsilon(t,x)\big),x\big) + \alpha^\varepsilon(t,x);$$

is a solution of

$$d\tilde v^\varepsilon = \tilde L^\varepsilon\big(x,\tilde v^\varepsilon,D\tilde v^\varepsilon,D^2\tilde v^\varepsilon\big)\,d\xi^\varepsilon, \quad \tilde v^\varepsilon(0,\cdot)\equiv u_0(\cdot)$$

(for details of the transform, etc. we refer to [11]). Here $\tilde L^\varepsilon$ is a linear operator with coefficients determined by the characteristics of the PDE $\partial w = \Lambda(t,x,w^\varepsilon,Dw^\varepsilon)\,dz^\varepsilon$ and $\psi^\varepsilon,\phi^\varepsilon,\alpha^\alpha$ are ODE flows converging to RDE flows $\psi^{\mathbf{z}},\phi^{\mathbf{z}},\alpha^{\mathbf{z}}$ which depend on $\mathbf{z}$ (but not the approximating sequence of (z^ε)). Further a comparison principle applies to $\partial_t - \tilde L^\varepsilon = 0$. The assumptions of Proposition 11 are then fulfilled, $\tilde L^\varepsilon \to \tilde L$ locally uniformly and using the method of semi-relaxed limits,

$$\tilde v^\varepsilon \to \tilde v \quad \text{locally uniformly.}$$

Unwrapping the transformation, that is, setting

$$u^{\mathbf{z}}(t,x) := \phi^{\mathbf{z}}\big(t,\tilde v\big(t,\big(\psi^{\mathbf{z}}\big)^{-1}(t,x)\big) - \alpha^{\mathbf{z}}\big(t,\big(\psi^{\mathbf{z}}\big)^{-1}(t,x)\big)\big), \tag{21}$$

finishes the proof since stability of the RPDE follows directly from the representation (21). □

5. Applications to stochastic PDEs

The typical applications to SPDEs are path-by-path, i.e. by taking $\mathbf{z}$ to be a realization of a continuous semimartingale Y and its stochastic area, say $\mathbf{Y}(\omega) = (Y, A)$; the most prominent example being Brownian motion and Lévy's area. Taking the linear case as an example, the stability result of Proposition 17 allows to identify

$$du = L\big(t, x, u, Du, D^2 u\big)\,dt + \Lambda(t, x, u, Du)\,d\mathbf{z}, \quad u(0,\cdot) \equiv u_0(\cdot) \tag{22}$$

with $\mathbf{z} = \mathbf{Y}(\omega)$ as *Stratonovich solution* to the SPDE

$$du = L\big(t, x, u, Du, D^2 u\big)\,dt + \Lambda(t, x, u, Du) \circ dY, \quad u(0,\cdot) = u_0(\cdot). \tag{23}$$

Indeed, under the stated assumptions, the "Wong–Zakai solutions" (the solutions of (23) when $Y(\omega)$ is replaced by its piecewise linear approximation between the points $\{0, \frac{T}{n}, \frac{2T}{n}, \dots, T\}$) converge (locally uniformly on $[0,T] \times \mathbb{R}^n$) to the solution of

$$du = L\big(t, x, u, Du, D^2 u\big)\,dt + \Lambda(t, x, u, Du)\,d\mathbf{Y}, \quad u(0,\cdot) = u_0(\cdot),$$

as constructed in Proposition 17. In view of well-known Wong–Zakai approximation results for SPDEs, ranging from [1,31] to [14,15], the rough PDE solution is then identified as Stratonovich solution. (At least for L uniformly elliptic: the (Stratonovich) integral interpretations can break down in degenerate situations; as example, consider non-differentiable initial data u_0 and the (one-dimensional) random transport equation $du = u_x \circ dB$ with explicit "Stratonovich" solution $u_0(x + B_t)$. A similar situation occurs for the classical transport equation $\dot{u} = u_x$, of course.) Motivated by this, if $u^{\mathbf{z}}$ is an RPDE solution of

$$du = F\big(t, x, u, Du, D^2 u\big)\,dt + \Lambda(t, x, u, Du)\,d\mathbf{z}, \quad u(0,\cdot) = u_0(\cdot),$$

then we call $u^{\mathbf{z}}$ with $\mathbf{z} = \mathbf{Y}(\omega)$ a *Stratonovich solution* and write

$$du = F\big(t, x, u, Du, D^2 u\big)\,dt + \Lambda(t, x, u, Du) \circ dY, \quad u(0,\cdot) = u_0(\cdot).$$

The following example was suggested in [23] and carefully worked out in [3,4].

Example 18 *(Pathwise stochastic control).* Consider

$$dX = b(X;\alpha)\,dt + W(X;\alpha) \circ d\tilde{B} + V(X) \circ dB,$$

where b, W, V are (collections of) sufficiently nice vector fields (with b, W dependent on a suitable control $\alpha = \alpha(t) \in \mathcal{A}$, applied at time t) and $\tilde{B}, B$ are multi-dimensional (independent) Brownian motions. Define[12]

$$v(x,t;B) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\left(g\big(X_T^{x,t}\big) + \int_t^T f\big(X_s^{x,t}, \alpha_s\big)\,ds\right)\middle| B\right]$$

[12] Remark that any optimal control $\alpha(\cdot)$ here will depend on knowledge of the entire path of B. Such anticipative control problems and their link to classical stochastic control problems were discussed early on by Davis and Burnstein [8].

where $X^{x,t}$ denotes the solution process to the above SDE started at $X(t)=x$. Then, at least by a formal computation,

$$dv+\inf_{\alpha\in\mathcal{A}}\big[b(x,\alpha)Dv+L_a v+f(x,\alpha)\big]dt+Dv\cdot\nu(x)\circ dB=0$$

with terminal data $v(\cdot,T)\equiv g$, and $L_\alpha=\sum W_i^2$ in Hörmander form. Setting $u(x,t)=v(x,T-t)$ turns this into the initial value (Cauchy) problem,

$$du=\inf_{\alpha\in\mathcal{A}}\big[b(x,\alpha)Du+L_a u+f(x,\alpha)\big]dt+Du\cdot V(x)\circ dB_{T-\cdot}$$

with initial data $u(\cdot,0)\equiv g$; and hence of a form which is covered by Theorem 16. (The rough driving signal in Proposition 16 is taken as $\mathbf{z}_t:=\mathbf{B}_{T-t}(\omega)$ where $\mathbf{B}$ is Brownian motion enhanced with its Lévy area.)

Using Theorem 15 we immediately get a splitting result:

Example 19 *(Splitting HJB-equations).* Let B be a standard d-dimensional Brownian motion. Then the SPDE

$$\begin{aligned}du&=\inf_{\alpha\in\mathcal{A}}\big\{\mathrm{Tr}\big[\sigma(x;\alpha)\sigma(x;\alpha)^T D^2u\big]+b(x;\alpha)\cdot Du+f(x;\alpha)\big\}dt+\big(Du\cdot\nu(x)\big)\circ dB,\\ u(\omega;0,x)&=u_0(x),\end{aligned}\tag{24}$$

has a unique solution u if $\sigma:\mathbb{R}^e\times\mathcal{A}\to\mathbb{R}^{e\times e'}$ and $b:\mathbb{R}^e\times\mathcal{A}\to\mathbb{R}^e$ are Lipschitz continuous in $x\in\mathbb{R}^e$, uniformly in $\alpha\in\mathcal{A}$, $\nu=(\nu_1,\dots,\nu_d)\subset\mathrm{Lip}^\gamma(\mathbb{R}^e;\mathbb{R}^e)$ with $\gamma>4$. Denote $\{\mathbf{P}_t,\ t\geqslant 0\}$ the solution operator[13] of

$$du=\inf_{\gamma\in\mathcal{A}}\big\{\mathrm{Tr}\big[\sigma(x;\alpha)\sigma(x;\alpha)^T D^2u\big]+b(x;a)\cdot Du+f(x;\alpha)\big\}dt,\tag{25}$$

i.e. $\mathbf{P}_t u_0(\cdot)=u(t,\cdot)$, and $\{\mathbf{Q}_{s,t},\ 0\leqslant s\leqslant t\leqslant T\}$ the solution operator defined as $\mathbf{Q}_{s,t}\varphi(\cdot)=\varphi(\pi_{-V}(s,\cdot;B)_t)$, where $\pi_{-V}(s,x;B)_t$ is the SDE solution of

$$dy=-V(y)\circ dB,\quad y_s=x\in\mathbb{R}^e.\tag{26}$$

Then[14] for all $(t,x)\in(0,T]\times\mathbb{R}^e$,

$$u^{n;\mathrm{Split}}(t,x):=\prod_{i=0}^{\lfloor t/n\rfloor-1}[\mathbf{Q}_{i/n,i/n+1/n}\circ\mathbf{P}_{1/n}]\big(u_0(\cdot)\big)(x)\to u(t,x)\quad\text{as } n\to\infty$$

and the convergence also holds locally uniformly.

Thus, Eq. (24) can be approximated by solutions of a standard HJB equation (24) and by solutions of the RDE (26) (for numerical schemes for HJB, see [12]).

[13] Note that in the case $\xi=t$ one can use a one-parameter semigroup since $\dot{\xi}^n=2$ on the time interval on which the approximation evolves.

[14] A priori the leftmost two terms would have to be $\mathbf{Q}_{\lfloor t/n\rfloor,t}\circ\mathbf{P}_{t-\lfloor t/n\rfloor n}$. However, the claimed convergence follows from Lyons' limit theorem.

Example 20 *(Linear SPDEs, filtering).* Let L and Λ be as in Proposition 17. Then there exists a unique solution to

$$du = L\big(x, u, Du, D^2u\big)\,dt + \sum_{k=1}^{d} \Lambda_k(t, x, u, Du) \circ dB^k, u(0, \cdot) = u_0(\cdot).$$

Denote by $\{\mathbf{P}_u,\ 0 \leqslant u \leqslant T\}$ the solution operator

$$\varphi \mapsto v \quad \text{with } v \text{ solution of} \quad \partial v = L\big(x, v, Dv, D^2v\big)\,dt,\ v(0, \cdot) = \varphi(\cdot),$$

and by $\{\mathbf{Q}_{s,t},\ 0 \leqslant s \leqslant t\}$ the solution operator

$$\varphi \mapsto y \quad \text{with } y \text{ solution of} \quad dy = \Lambda(t, x, u, Du) \circ dB,\ v(0, \cdot) = \varphi(\cdot).$$

Then for all $(t, x) \in (0, T] \times \mathbb{R}^e$,

$$u^{n;\mathrm{Split}}(t, x) := \prod_{i=0}^{\lfloor t/n \rfloor - 1} [\mathbf{Q}_{i/n, i/n+1/n} \circ \mathbf{P}_{1/n}]\big(u_0(\cdot)\big)(x) \to u(t, x) \quad \text{as } n \to \infty$$

and the convergence also holds locally uniformly.

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Appendix A. Time-dependent *F*

To deal with time-dependent F we need the additional assumption of uniform bounds of the derivatives of the approximating sequence (ξ^{ε}).

Proposition 21. *Let* $(\xi^{\varepsilon})_{\varepsilon} \subset C_0^{1,+}([0, T], \mathbb{R})$, $\sup_{\varepsilon} |\dot{\xi}^{\varepsilon}|_{\infty;[0,T]} < \infty$, *converge uniformly to some* $\xi \in C_0^{1\text{-var};+}([0, T], \mathbb{R})$ *as* $\varepsilon \to 0$. *Assume* $(v^{\varepsilon})_{\varepsilon} \subset \mathrm{BUC}([0, T] \times \mathbb{R}^e, \mathbb{R})$ *are locally uniformly bounded viscosity solutions of*

$$\partial_t v^{\varepsilon} = F^{\varepsilon}\big(t, x, v^{\varepsilon}, Dv^{\varepsilon}, D^2v^{\varepsilon}\big)\dot{\xi}_t^{\varepsilon}, \quad v^{\varepsilon}(0, x) = v_0(x)$$

with $F^{\varepsilon} : [0, T] \times \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e \to \mathbb{R}$ *a continuous and degenerate elliptic function. Further, assume that* F^{ε} *converges locally uniformly to a continuous, degenerate elliptic function* F *and that a comparison result holds for* $\partial_t - F^{\varepsilon} = 0$ *and* $\partial_t - F = 0$. *Then there exists a* v *such that*

$$v^{\varepsilon} \to v \quad \text{locally uniformly as } \varepsilon \to 0.$$

Further, v *does not depend on the choice of the sequence approximating* ξ *and we also write* $v \equiv v^{\xi}$ *to emphasize the dependence on* ξ *and say that* v *solves*

$$dv = F\big(t, x, v, Dv, D^2v\big)\,d\xi_t, \quad v(0, x) = v_0(x).$$

Proof. Set $w^\varepsilon(t,x) := v^\varepsilon((\xi^\varepsilon)^{-1}(t),x)$ and divide $[0,T]$ into intervals on which ξ is strictly increasing resp. constant, i.e. $0 = s_1 \leqslant t_1 \leqslant s_2 \leqslant \cdots \leqslant t_n = T$, ξ strictly increasing on $[s_i,t_i]$, constant on $[t_i,s_{i+1}]$. By Lemma 12 on intervals $[s_i,t_i]$

$$v^\varepsilon \text{ is a solution of } \partial_t - F\dot{\xi}^\varepsilon = 0 \quad \text{iff} \quad w^\varepsilon \text{ is a solution of } \partial_t - \tilde{F}^\varepsilon = 0$$

where $\tilde{F}^\varepsilon(t,x,r,p,X) = F((\xi^\varepsilon)^{-1}(t),x,r,p,X)$. Let

$$\overline{w} := \limsup_{\varepsilon}{}^* w^\varepsilon \quad \text{and} \quad \underline{w} := \liminf_{\varepsilon}{}_* w^\varepsilon$$

and note that on intervals $[s_i,t_i]$

$$\tilde{F}^\varepsilon(t,x,r,p,X) \to F\big(\xi^{-1}(t),x,r,p,X\big) =: \tilde{F}(t,x,r,p,X) \quad \text{locally uniformly.}$$

Standard viscosity theory tells us that on intervals $[s_i,t_i]$, $\overline{w}$ and $\underline{w}$ are sub- resp. supersolutions of $\partial_t - \tilde{F} = 0$. Using the method of semi-relaxed limits (by the definition, $\overline{w} \geqslant \underline{w}$ and the reversed inequality follows from comparison) we conclude that $w := \overline{w} = \underline{w}$ and that for every compact set $K \subset \mathbb{R}^n$ (by using a Dini-type argument),

$$\big|w^\varepsilon - w\big|_{\infty;[s_i,t_i]\times K} = \sup_{t\in[s_i,t_i],\, x\in K}\big|w^\varepsilon(t,x) - w(t,x)\big| \to 0 \quad \text{as } \varepsilon \to 0.$$

Now define

$$v(t,x) := \begin{cases} w(\xi(t),x), & \xi^{-1}(s_i) \leqslant t \leqslant \xi^{-1}(t_i), \\ w(\xi(t_i),x), & \xi^{-1}(t_i) < t < \xi^{-1}(s_{i+1}). \end{cases}$$

We get the claimed convergence $v^\varepsilon \to v$ on intervals $[\xi^{-1}(s_i),\xi^{-1}(t_i)]$. However, on $[\xi^{-1}(t_i), \xi^{-1}(s_{i+1})]$, v^ε is by the definition viscosity solution of $\partial_t - F\dot{\xi}^\varepsilon = 0$ with initial condition $v^\varepsilon(\xi^{-1}(t_i),\cdot)$ and $F\dot{\xi}^\varepsilon \to_\varepsilon 0$ locally uniformly since $\sup_t|\dot{\xi}^\varepsilon(t)|$ is uniformly bounded in ε by the assumption. Hence the standard stability result of viscosity theory applies and v^ε converges locally uniformly on $[\xi^{-1}(t_i),\xi^{-1}(s_{i+1})]$ against the constant-in-time function $v^\varepsilon(\xi^{-1}(t_i),\cdot)$, the only solution to $\partial_t = 0$ with initial condition $v^\varepsilon(\xi^{-1}(t_i),\cdot)$. This proves the claimed convergence. Further, note that v is given as the unique viscosity solution of $\partial_t - \tilde{F} = 0$, hence every other sequence approximating ξ will lead to the same limit. □

The proof of the main theorem (Theorem 15) and applications to examples adapt now in a straightforward way to time-dependent F.

Appendix B. Generalized viscosity solutions

Section 3.3 and Appendix A extend the notion of viscosity solutions to equations of the form

$$du = F\big(t,x,u,Du,D^2u\big)\,d\xi(t), \quad u(0,x) = u_0(x) \tag{27}$$

with $\xi \in C^{1\text{-var};+}([0,T],\mathbb{R})$. Generalizations of viscosity solutions go back to [17,20] and for the parabolic case [28]. Let us recall the definition given in [28].

Condition 22. $F(\cdot,x,r,p,X) \in L^1((0,T),\mathbb{R})$ for all $(x,r,p,X) \in \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e$ and F is continuous on $\mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e$ for almost all $t \in (0,T)$.

Condition 23. $F(t,x,\cdot,p,X)$ is non-increasing on $\mathbb{R}$ for all $t\in(0,T)$ and for all $(x,p,X)\in\mathbb{R}^e\times\mathbb{R}^e\times\mathbb{S}^e$.

Definition 24. Let F satisfy Conditions 22 and 23. A locally bounded upper semicontinuous function $u:(0,T)\times\mathbb{R}^e\to\mathbb{R}$ is called a generalized subsolution of

$$du=F\big(t,x,u,Du,D^2u\big)\,dt \tag{28}$$

if for any $(\hat t,\hat x)\in(0,T)\times\mathbb{R}^e$, $b\in L^1((0,T),\mathbb{R})$, $\phi\in C^2(\mathbb{R}^e,\mathbb{R})$, $G:(0,T)\times\mathbb{R}^e\times\mathbb{R}\times\mathbb{R}^e\times\mathbb{S}^e\to\mathbb{R}$ continuous and degenerate parabolic, such that if

$$u(t,x)+\int_0^t b(r)\,dr-\phi(t,x)\quad\text{attains a local maximum at }(\hat t,\hat x),$$

and

$$b(t)+G(t,x,r,p,X)\leqslant F(t,x,r,p,X)\quad\text{for a.e. }t\in B_\delta(\hat t)\quad\text{and}$$
$$\text{for all }(x,r,p,X)\in B_\delta\big(\hat x,u(\hat t,\hat x),D\phi|_{\hat t,\hat x},D^2\phi|_{\hat t,\hat x}\big)\text{ for some }\delta>0$$

it follows that

$$b(\hat t)+G\big(\hat x,u(\hat t,\hat x),D\phi|_{\hat t,\hat x},D^2\phi|_{\hat t,\hat x}\big)\leqslant 0.$$

A locally bounded uniformly lower semicontinuous function is called a supersolution if the above estimates hold when one replaces maximum by minimum and reverses the inequality sign.

Note that Eq. (28) is covered by this definition. However, it is quite cumbersome to derive existence, comparison and stability results in this very general setting and in the case of interest to us, the time-discontinuouity only appears multiplicatively.

Proposition 25. *Under the assumptions of Proposition* 21 *and additionally $\xi\in W^{1,1}$ the function $u=u^\xi$ is a viscosity solution of*

$$du=F\big(t,x,u,Du,D^2u\big)\,d\xi_t,\quad u(0,x)=u_0(x)$$

in the sense of Definition 24.

Proof. We partition $[0,T]$ into $0\leqslant s_1\leqslant t_1\leqslant\cdots\leqslant s_n\leqslant t_n\leqslant T$ such that ξ is increasing on $[s_i,t_i]$, constant on $[t_i,s_{i+1}]$. Say $u(t,x)+\int_0^t b(r)\,dr-\phi(t,x)$ attains a local maximum at $(\hat t,\hat x)$. If $\hat t\in[s_i,t_i]$ by construction $u(t,x)\equiv w(t,\xi_t)$ with w a viscosity subsolution of $\partial_t-\tilde F=0$, $\tilde F(t,r,x,p,X)=F(\xi^{-1}(t),r,x,p,X)$, hence also a generalized subsolution and using that ξ is invertible on $[s_i,t_i]$ one sees by a change of variable that also u is a generalized subsolution on $[s_i,t_i]$ of $\partial_t-\tilde F=0$.

If $\hat t\in[t_i,s_{i+1}]$, then ξ is constant, hence $\dot\xi=0$ a.s. and so $F(t,x,r,p,X)\dot\xi(t)=0$ for a.e. $t\in B_\delta(\hat t)$ and u is a generalized subsolution on that interval. This shows that u is a generalized subsolution and the same argument shows that u is a generalized supersolution. $\square$

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