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MULTIPLE FEEDBACK AT A SINGLE-SERVER STATION

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A single server facility is equipped to perform a collection of operations. The service rendered to a customer is a branching process of operations. While the performance of an operation may not be interrupted before its completion, once completed, the required follow-up work may be delayed, at a cost per unit time of waiting that depends on the type and load of work being delayed. Under some probabilistic assumptions on the nature of the required service and on the stream of customers, the problem is to find service schedules that minimize expected costs. The authors generalize results of Bruno [2], Chazan, Konheim and B. Weiss [4], Harrison [8], Klimov [10], Konheim [11], and Meilijson and G. Weiss [13], using a dynamic programming approach.

Single server station, service policies, holding costs, priorities, dynamic programming.

1. Introduction

A single server is equipped to serve several types of customers. The relevant features of a customer are his type, his time of arrival, his service length, and his holding cost. The problem of scheduling the service to the customers, under various cost criteria, is discussed in the literature for a host of such problems. Of particular interest are cases where a priority ordering of types can be shown to yield the best solution. One such ordering is the "$c\mu$" priority ordering, where higher priority is assigned to customers with higher ratio of expected holding cost (c) to expected length of service $\left(E(x)=\dfrac{1}{\mu}\right)$.

For the non-preemptive service of an M/GI/1 queue with several types of customers, when the expected cost per unit time is to be minimized, the "$c\mu$" rule has been known for some time to be the best priority rule (see Conway, Maxwell and Miller [5, pp. 159–167]). Harrison [7, 8] and Meilijson and Yechiali [14] proved

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its optimality in the class of all non-preemptive service policies (not just those of a priority nature). Under a more general cost criterion that discounts holding costs and terminal rewards, Harrison showed that a static priority policy is optimal. This policy reduces to "$c\mu$" as the discount factor is made to approach 1.

As for the non-preemptive service of arrival streams other than Poisson, the "$c\mu$" rule is optimal for the service of a batch of customers (see Conway, Maxwell and Miller [5, pp. 39–46]), it is always better than the only other priority rule under a GI/GI/1 model with two types of customers (Wolff [19]), but fails to be optimal for more complex streams. In fact there need not exist a static priority rule that is optimal. Still, for general streams the top "$c\mu$" type of customer should be given preferential service, ahead of all others (Meilijson and Yechiali [14]).

When holding costs are the same for all customers and service lengths become known upon arrival, Shrage [16] showed the Shortest Remaining Processing Time discipline to minimise the actual number of customers present at the station at every single moment, for the preemptive service of a general stream of customers. This discipline is the preemptively applied "$c\mu$" rule.

Further models introduced the possibility of partial preemption by assuming that a customer may rejoin the queue upon completing service, possibly as a customer of a different type.

Klimov [10] assumes that a type i customer upon leaving the server rejoins the queue as a type j customer with some probability $Q(i,j)$, and leaves the station with the complementary probability $1-\Sigma_j Q(i,j)$ (the non-preemptive case is $Q(i,j)\equiv 0$). He defined a priority rule and proved it to be optimal under steady state for Poisson arrivals. The rule turns out not to depend on the arrival rates of the different types of customers, a property shared with the "$c\mu$" rule. Bruno [2] and Meilijson and Weiss [13] proved Klimov's policy to be optimal for the service of a batch of customers with Klimov's kind of feedback, a result that contains those of Chazan, Konheim and B. Weiss [4], Konheim [11], Bruno and Hofri [3], and Meilijson and Weiss [12]. Tcha and Pliska [18] dealt with the discounted version of this model.

In the present note we discuss the following model. Denote by r the number of types of customers. Let the random variables I, v_I, c_I, n_{Ij} $(1\leq j\leq r)$ be the type of a customer, the length of its service (assumed throughout to be non-preemptive), its holding cost per unit time, and the number of customers of type j that arrive during its service. Assume that, given the types of the customers, vectors $(v_I; c_I; n_{I1}, n_{I2}, \ldots, n_{Ir})$ corresponding to different customers are independent, those corresponding to customers of the same type being, in addition, identically distributed.

The server's problem is to find a service policy that will minimize the expected total cost on all customers during a single busy period that started with an arbitrary load of work. We will define a priority ordering of the types similar to those in [2, 10, 13] and we will prove (Theorem 1) the stationary policy it generates to provide an optimal service policy. The ordering depends on the joint distribution of

$(v_i; c_i; n_{i1}, \ldots, n_{ir})$ only through the values of $E(v_i)$, $E(c_i)$, $E(n_{i1})$, $E(n_{i2}), \ldots, E(n_{ir})$, (as observed by Bruno [2] for his model). The calculation of the priority ordering is given explicitly in Theorem 2.

Under reasonable assumptions on the independence and time-homogeneity of busy and idle periods, a stationary policy that is optimal for the above criterion per busy period will also minimize the rate of cost under steady state (see Hordijk [9, Lemma 5.2], and take i_0 as the idle state).

It should be observed that the counting variables n_{ij} do not differentiate between the arrival process and feedback requirements at the station. Such a differentiation is unnecessary.

The most important version of this model is that where customers of the various types arrive according to independent Poisson streams. This case is covered by the description that follows.

Suppose that for some non negative vector λ, $E(n_{ij}) - \lambda(j)E(v_i)$ is non negative for all (i, j). This will occur for instance when the stream n_{ij} of customers of type j that arrive during the service of a customer of type i, is composed of $n_{ij} = n'_{ij} + n''_{ij}$, where n'_{ij} is the number of arrivals in a Poisson stream with rate $\lambda(j)$, during the service of the customer of type i (whose expectation is $\lambda(j)E(v_i)$), and where n''_{ij} is the multiple feedback generated by the customer of type i. We will show that in this case, the priority ordering will be unchanged when $E(n_{ij})$ is replaced by $E(n_{ij}) - \lambda(j)E(v_i)$.

The models discussed by Klimov, Bruno, Meilijson and Weiss allow single feedback, i.e. $\sum_j n''_{ij}$ is 0 or 1. Under Klimov's model n'_{ij} emanate from Poisson streams, under the others $n'_{ij} \equiv 0$. We have thus explained what makes the same policy optimal in all these cases of single feedback, while at the same time generalizing the models a bit further to allow for multiple feedback and for multiple arrivals, where by this we mean a stream of independent and identically but otherwise arbitrarily distributed batches of customers that arrive at moments that form a Poisson point process.

2. Assumptions and results

In this section we introduce some notation and formulate assumptions and results; the proofs of these results follow in Sections 3 and 4.

(i) $v(i) = E(v_i)$ and $c(i) = E(c_i)$ are positive and finite, $n(i, j) = E(n_{ij})$ are non-negative and finite.

(ii) All eigenvalues of $N = (n(i, j))$ are less than 1 in absolute value.

(iii) The length of a busy period and the number of customers served during it possess finite second moments.

Remark. Assumption (iii) deals with variables that do not depend on service policy.

Let r be the number of types of customers and denote $R = \{1, 2, \ldots, r\}$. For a matrix

M on $R \times R$ and sets $A \subseteq B \subsetneq R$, the matrix M_A on $A \times A$ has $M_A(i,j) = M(i,j)$ and the matrix $M_{A,B}$ on $A \times B$ has $M_{A,B}(i,j) = M(i,j)(\phi(B-A))(j)$, where $\phi(K)$ is the indicator function of the set K (i.e., $\phi(K)(x) = 1$ if $x \in K$ and $= 0$ if $x \notin K$). For a vector w on R and sets $A \subseteq B \subseteq R$, the vector w_A on A has $w_A(i) = w(i)$ and the vector $w_{A,B}$ on B has $w_{A,B}(i) = w(i)(\phi(B-A))(i)$. The *direct ratio* $w_1 // w_2$ of the vectors w_1 and w_2 on R is the vector on R with $(w_1//w_2)(i) = w_1(i)/w_2(i)$ if $w_2(i) \neq 0$; $= 0$ if $w_2(i) = 0$. Denote by v and c the vectors on R with coordinates $v(i)$ and $c(i)$ respectively, denote $d = (I - N)c$, and denote by **1** the vector all of whose coordinates are 1. All vectors are column vectors, unless transposed by ($'$).

For a non-empty subset A of R, define the following vectors on A, after justifying through Seneta ([15, Theorem 1.1]) the taking of inverses.

$$d(A) = (I_A - N_A)^{-1} d_A = c_A - (I_A - N_A)^{-1} N_{A,R} c \tag{1}$$

$$\gamma(A) = (I_A - N_A)^{-1} v_A \tag{2}$$

$$H(A) = d(A) // \gamma(A). \tag{3}$$

Define a vector $\mathcal{H}$ on R by

$$\mathcal{H}(i) = \max_{A \subsetneq R} (H(A \cup \{i\})(i)). \tag{4}$$

A service policy is $\mathcal{H}$-monotone if at every decision moment it chooses almost surely to serve one of the customers whose type has the highest value of $\mathcal{H}(i)$ among those in the queue. (Remark: Those $\mathcal{H}$-monotone policies that are of a priority nature are called "modified static policies" by Harrison [7, 8]. If it turns out that $i \neq j \implies \mathcal{H}(i) \neq \mathcal{H}(j)$ then there is only one $\mathcal{H}$-monotone policy, and it is a modified static policy.)

Theorem 1. *A service policy minimizes the expected total cost during a whole busy period if and only if it is $\mathcal{H}$-monotone.*

Let the non-empty sets $R_1^*, R_2^*, \ldots, R_l^*$ be the partition of R with $i, j \in R_k^* \implies \mathcal{H}(i) = \mathcal{H}(j)$ and $i \in R_k^*$, $j \in R_{k+1}^* \implies \mathcal{H}(i) < \mathcal{H}(j)$. $(R_1^*, R_2^*, \ldots, R_l^*)$ is called the *optimal priority partition* of R. To compute it, it is not necessary to perform all the maximizations in (4). Following Klimov [10],

Theorem 2. *Define* $\tilde{R}_1^* = \{i \in R \mid (H(R))(i) = \min_{j \in R} (H(R))(j)\}$. *Let* $R_2 = R - \tilde{R}_1^*$. *If* $R_2 = \emptyset$, *let* $\tilde{l} = 1$. *Otherwise, define inductively* $\tilde{R}_k^* = \{i \in R_k \mid (H(R_k))(i) = \min_{j \in R_k} (H(R_k))(j)\}$. *Let* $R_{k+1} = R - \tilde{R}_k^*$. *If* $R_{k+1} = \emptyset$, *let* $\tilde{l} = k$. *Then* $\tilde{l} = l$ *and* $\tilde{R}_k^* = R_k^*$ *for all* $1 \leq k \leq l$.

For the next theorem, observe that if a Poisson arrival stream had rate $\lambda(i)$ for customers of type i, then the (i, j) coordinate of the matrix $v\lambda'$ would be the expected number of arrivals of type j during the service of a customer of type i. The next theorem says that whenever a part of the matrix N can be interpreted as being "time-homogeneous" (say, a Poisson process), that part is irrelevant for the computation of the optimal priority partition of R.

Theorem 3. *Assume that for some non-negative matrix M on $R \times R$ and some non-negative vector λ on R, $N = M + v\lambda'$. Then N and M yield the same optimal priority partition of R.*

A word about γ, d and H.

Express $(I_A - N_A)^{-1} = \sum_{k=0}^{\infty} N_A^k$ to infer that $(\gamma(A))(i)$ is the expected time it will take to serve customers to exhaustion, when there is originally one customer only, its type is i, and only customers whose types belong to A are provided service. At the conclusion of that time the expected number of customers of type j in the queue is the (i, j) coordinate of $(I_A - N_A)^{-1}N_{A,R}$. $(d(A))(i)$ can thus be explained as the difference between the expected total holding cost per unit time of all customers in the station, at the beginning and conclusion of the above described time. $(H(A))(i)$ is thus the ratio of a reduction in rate of cost and an expected service time, and reduces to $c(i)/v(i)$ when $n(i, j) \equiv 0$. So $\mathcal{H}$-monotone policies generalize the "$c\mu$" rule. What follows is an intuitive justification for following $\mathcal{H}$-monotone dictates.

Imagine Tom and Dick are the only customers in the queue, Tom's type is i, Dick's is j. Let $i \in A \subseteq R$, $j \in B \subseteq R$. Under the policy TD start by serving Tom, serve to exhaustion customers with types in A but do not serve Dick. Now serve Dick, then serve to exhaustion customers with types in B but do not serve those in the queue at the moment Dick's service started. Proceed in some arbitrary manner π. Define a policy DT in the naturally similar way, using the same π as before. To compare the performances of TD and DT we may disregard the common tail π. The relevant waiting costs to compare are, then,

$$c(j)(\gamma(A))(i) + ((I_A - N_A)^{-1}N_{A,R}c)(i)(\gamma(B))(j),$$

$$c(i)(\gamma(B))(j) + ((I_B - N_B)^{-1}N_{B,R}c)(j)(\gamma(A))(i).$$

In other words, $(H(B))(j)$ and $(H(A))(i)$ are to be compared. If $\mathcal{H}(i) > \mathcal{H}(j)$ then for *some* set A containing i and *all* sets B containing j, we would rather use TD than DT.

3. Proofs of Theorems 2 and 3 and some technical lemmas

Lemma 1. *For $i \in A \subseteq B \subseteq R$ and for any vector u on B,*

$$((I_B - N_B)^{-1}u)(i) = ((I_A - N_A)^{-1}u_A)(i) + ((I_A - N_A)^{-1}N_{A,B}(I_B - N_B)^{-1}u)(i) \quad (5)$$

Proof. Immediate. Use the probabilistic interpretation given in the comments about γ, d and H.

Define, for $A \subseteq B \subseteq R$, a vector $H_A(B)$ on A by

$$H_A(B) = ((d(B))_A - d(A))//((\gamma(B))_A - \gamma(A)). \tag{6}$$

Lemma 2. *For $i \in A \subseteq B \subseteq R$,*

$$(H(B))(i) = ((\gamma(A))(i)/(\gamma(B))(i))(H(A))(i) + \\ + (1 - ((\gamma(A))(i)/(\gamma(B))(i)))(H_A(B))(i). \tag{7}$$

If, in addition, $\gamma(B)(i) > \gamma(A)(i)$, then $(H_A(B))(i)$ is a convex combination of the values of $(H(B))(j)$ for $j \in B - A$.

Proof. Expression (7) follows immediately from expressions (3) and (6). Apply Lemma 1 to $u = d_B$ and to $u = v_B$ to obtain, when $\gamma(B)(i) > \gamma(A)(i)$, $(H_A(B))(i) = (((I_A - N_A)^{-1} N_{A,B} d(B))(i))/(((I_A - N_A)^{-1} N_{A,B} \gamma(B))(i))$, from which the second part of the statement follows.

Lemma 3. *Let $j \in R$. Denote by $N^{(j)}$ the matrix on $R \times R$ with $N^{(j)}(i,k) = N(i,k)(\phi(R - \{j\}))(k)$. Denote by ξ the vector on R with $\xi(i) = N(i,j)$. Then,*

(i) *For every subset B of R containing j and for every vector u on B,*

$$((I_B - N_B^{(j)})^{-1} u)(j) = ((I_B - N_B)^{-1} u)(j)/(1 + ((I_B - N_B)^{-1} \xi_B)(j)). \tag{8}$$

(ii) *For every subset B of R containing j,*

$$(H(B))(j) = \frac{(c(j)(1 - ((I_B - N_B^{(j)})^{-1} \xi_B)(j)) - ((I_B - N_B^{(j)})^{-1} N_{B,R} c)(j))}{((I_B - N_B^{(j)})^{-1} v_B)(j)}. \tag{9}$$

Proof. Let $(I_B - N_B^{(j)})^{-1} u = y$. Then $u = (I_B - N_B^{(j)}) y = (I_B - N_B) y + y(j) \xi_B$. Hence, $(I_B - N_B^{(j)})^{-1} u = y = (I_B - N_B)^{-1} u - y(j)(I_B - N_B)^{-1} \xi_B$. Taking the j's component, (8) follows. (9) is obtained by substituting in (8) $N_{B,R} c$, ξ_B and v_B for u.

Proof of Theorem 2. As a first step, we will show that if $i \in \tilde{R}_k^*$ and $i \in A \subseteq R_k$, then

$$(H(A))(i) \leq (H(R_k))(i). \tag{10}$$

By the second part of Lemma 2, either there is trivially an equality in (10) or $(H_A(R_k))(i)$ is a convex combination of the values of $(H(R_k))(j)$ for $j \in R_k - A$. By the definition of $\tilde{R}_k^*$ and the first part of Lemma 2, (10) follows.

We will now identify $\tilde{R}_1^*$ as R_1^*. By (10), if $i \in \tilde{R}_1^*$ then $(H(R))(i) = \max_{A \subseteq R} (H(\{i\} \cup A))(i)$. By the definition of $\tilde{R}_1^*$, $H(R)$ is constant on $\tilde{R}_1^*$ and $\mathcal{H}(j) \geq (H(R))(j) > (H(R))(i)$ whenever $i \in \tilde{R}_1^*$ and $j \in R_2$. It then follows that $\tilde{R}_1^* = R_1^*$.

Assume, by induction, that $\tilde{R}_k^* = R_k^*$ for $k < m$. Assume that $R_m \neq \emptyset$. Consider any $i \in \tilde{R}_m^*$. Use Lemma 2 and the induction hypothesis to obtain that if $i \in A$ and $(H(A))(i) \geq (H(R_m))(i)$ then necessarily $(H(A \cap R_m))(i) \geq (H(A))(i)$, so to maximize $(H(A))(i)$ over A we may restrict attention to $A \subseteq R_m$. Now use (10) to obtain that $\mathcal{H}(i) = (H(R_m))(i)$. By the definition of $\tilde{R}^*_m$, $H(R_m)$ is constant on $\tilde{R}^*_m$ and $\mathcal{H}(j) \geq (H(R_m)(j)) > (H(R_m))(i)$ whenever $i \in \tilde{R}^*_m$ and $j \in R_m$... It then follows that $\tilde{R}^*_m = R^*_m$ and the proof is complete.

Proof of Theorem 3. Check, possibly imitating the proof of Lemma 3, that for every vector w on $A \subseteq R$,

$$(I_A - N_A)^{-1}w = (I_A - M_A)^{-1}w + (\lambda'_A(I_A - N_A)^{-1}w) \cdot (I_A - M_A)^{-1}v_A. \quad (11)$$

Substitute d_A as w in (11) to express

$$d(A) = (I_A - N_A)^{-1}d_A = (I_A - M_A)^{-1}d_A + (\lambda'_A d(A))(I_A - M_A)^{-1}v_A. \quad (12)$$

Substitute v_A for w in (11) to express

$$\gamma(A) = (I_A - N_A)^{-1}v_A = (1 + \lambda'_A \gamma(A)) \cdot (I_A - M_A)^{-1}v_A. \quad (13)$$

The direct ratio of (12) and (13) yields, denoting by $H^{(M)}$ the H computed as if M was N,

$$H(A) = \frac{1}{1 + \lambda'_A \gamma(A)}(H^{(M)}(A) + (\lambda'_A d(A))1_A). \quad (14)$$

Since for any fixed A, there is a strictly increasing relationship between $H(A)$ and $H^{(M)}(A)$, the construction in Theorem 2 will produce under N and M the same optimal priority partition of R.

4. Optimality of $\mathcal{H}$-monotone policies

This section deals with Dynamic Programming. Some of the concepts have been borrowed from Dubins and Savage [6]. A good general reference is Blackwell [1] and Strauch [17].

A *decision moment* is a moment in which an arriving customer finds an idle server or a departing customer leaves a non-empty queue behind. The *state* at a decision moment is the vector $s = (n_1, n_2, \ldots, n_r)$ of queue lengths of the r types. Denote by Ω the collection of all states and by J the mapping from Ω to $2^R - \{\emptyset\}$ defined by $J(n_1, n_2, \ldots, n_r) = \{i \in R \mid n_i > 0\}$. A *policy* specifies at each decision moment the type of the customer to be served next, among those in the J set for the current state. The *process generated by a policy* is the (eventually terminating) sequence of consecutive states during the busy period, under the policy. For a mapping $\Gamma : \Omega \to 2^R - \{\phi\}$ for which $\Gamma(s) \subseteq J(s)$ for all $s \in \Omega$, a policy is *available in* Γ if at almost surely all states d in the process generated by the policy, it chooses to serve a type in $\Gamma(s)$. For any two policies π and π' and any stopping time T on the process generated by π, let (π^T, π') be the policy that agrees with π up to time (re: "number of operations") T and then proceeds with π', with initial state equal to the state under π at time T. For a mapping $f : \Omega \to R$ such that for all $s \in \Omega$,

$f(s) \in J(s)$, $f^{(\infty)}$ is the *stationary policy* that serves a type i customer whenever $f(s) = i$. Shorten $((f^{(\infty)})^T, \pi)$ to (f^T, π) and (f^1, π) to (f, π). For a policy π, a positive integer (finite or $+\infty$) n, and a state $s \in \Omega$, let $V_n(\pi)(s)$ be the expected total waiting cost up to the conclusion of the nth operation (or the end of the busy period, whichever comes first), starting with an initial state s and using the policy π. Shorten $V_\infty = V$. A policy π^* is *optimal* if for all $s \in \Omega$ and all policies π, $V(\pi^*)(s) \leq V(\pi)(s)$. A policy π^* is *tail-irrelevant* if for all $s \in \Omega$ and all policies π, $V(\pi^n, \pi^*)(s) \to V(\pi)(s)$ as $n \to \infty$. A policy π is *thrifty* if for all $f: \Omega \to R$, with $f(s) \in J(s)$ for all $s \in \Omega$, $V(f, \pi)(s) \geq V(\pi)(s)$ for all $s \in \Omega$.

The problem of finding an optimal policy is a Negative Dynamic Programming problem with a finite action space. Hence, by ([17, Theorems 6.5 and 9.1]), optimal policies do exist, and a tail-irrelevant policy is optimal if and only if it is thrifty. Fix any optimal policy π. For $s \in \Omega$ let

$$G(s) = \{i \in J(s) \mid \text{for some } f: \Omega \to R \text{ with } f(s) = i,\ V(f, \pi)(s) = V(\pi)(s)\}.$$

Observe that any two optimal policies define the same mapping $G: \Omega \to 2^R - \{\emptyset\}$. Define a mapping $K: \Omega \to 2^R - \{\emptyset\}$ by

$$K(s) = \{i \in J(s) \mid \mathcal{H}(i) = \max_{j \in J(s)} \mathcal{H}(j)\}.$$

The $\mathcal{H}$-monotone policies are those policies that are available in K. Lemma 4 will prove all policies to be tail-irrelevant, Lemma 5 will prove a policy to be thrifty if and only if it is available in G, and the rest of the section will show G and K to be identical.

Lemma 4. *Under assumptions* (i), (ii) *and* (iii), $\sup_\pi V(\pi)(s) < \infty$ *and* $\sup_\pi (V(\pi)(s) - V_n(\pi)(s)) \to 0$ *as* $n \to \infty$, *for each* $s \in \Omega$.

Proof. Assume first that the random holding costs c are deterministic, i.e., they are determined by the type of the customer. Then, letting L be the number of customers served during the busy period, and X the length of the busy period,

$$\sup_\pi V(\pi)(s) \leq \max_{i \in R} c(i) \cdot E_s(X \cdot L) \leq \max_{i \in R} c(i)(E_s(X^2))^{1/2}(E_s(L^2))^{1/2} < \infty \tag{15}$$

$$\sup_\pi (V(\pi)(s) - V_n(\pi)(s)) \leq \max_{i \in R} c(i) E_s(X \cdot L \cdot \phi(L \geq n)). \tag{16}$$

Since $X \cdot L$ is integrable by (15) and $\phi(L \geq n) \to 0$ a.s. as $n \to \infty$, the right hand side of (16) goes to zero.

The proof will be finished if we show that when c is replaced by $c(I)$, $V_n(\pi)(s)$ remains unchanged.

Let c_i, v_i, I_i, A_i be the holding cost, length of service, type and arrival time of the ith customer served during the busy period ($i = 1, 2, \ldots L$). Then, using conditional independence of service periods given types,

$$V_n(\pi)(s) = E\left(\sum_{i=1}^{\min(L,n)} \sum_{j<i} c_i v_j - \sum_{i=1}^{\min(L,n)} c_i A_i\right)$$

$$= \sum_{i=1}^{n} \sum_{j<i} E(\phi(L \geq i) v_j c_i) - \sum_{i=1}^{n} E(\phi(L \geq i) c_i A_i)$$

$$= \sum_{i=1}^{n} \sum_{j<i} E(\phi(L \geq i) v_j c(I_i)) - \sum_{i=1}^{n} E(\phi(L \geq i) c(I_i) A_i)$$

$$= E\left(\sum_{i=1}^{\min(L,n)} \sum_{j<i} c(I_i) v_j - \sum_{i=1}^{\min(L,n)} c(I_i) A_i\right).$$

Lemma 5. *Under assumptions* (i), (ii) *and* (iii), *a policy is optimal if and only if it is available in G.*

Proof. Let π^* be any optimal policy.

If π is available in G, then (π^1, π^*) is optimal and so, by induction, so is (π^n, π^*), for every positive integer n. Hence, for each $s \in \Omega$ and each positive integer n,

$$|V(\pi)(s) - V(\pi^*)(s)| \leq |V(\pi)(s) - V_n(\pi)(s)| + |V_n(\pi^n, \pi^*)(s) - V(\pi^n, \pi^*)(s)|$$

$$\leq 2 \sup_{\pi'} (V(\pi')(s) - V_n(\pi')(s)).$$

So, by Lemma 4, $V(\pi) = V(\pi^*)$ and π is optimal.

If π is not available in G, let $T \geq 0$ be the least of the two times: end of the busy period, or first time π dictates an action outside the G set for the current state. Non-availability of π in G implies that (π^T, π^*) strictly improves π, so π is not optimal.

Proof of Theorem 1. Each order ρ on R defines a *priority* stationary policy $f^{(\infty)}$ with $f(s) = \sup\{i \in J(s)\}$. Let ρ be any order on R making $\mathcal{H}$ monotone non-decreasing (i.e., making $f^{(\infty)}$ avalilable in K) and let $j \in R$ be arbitrary. Define $g : \Omega \to R$ by $g(s) = j$ if $j \in J(s)$ and $g(s) = f(s)$ otherwise. In the light of Lemma 5, to obtain that K and G are identical it is enough to show that for arbitrary ρ and j as above, $V(g, f^{(\infty)})(s) \geq V(f^{(\infty)})(s)$ for all $s \in \Omega$, with strict inequality holding whenever $j \in J(s) - K(s)$. We will show that for some strictly positive stopping time T on the process generated by $f^{(\infty)}$, $V(g, f^{(\infty)})(s) \geq V(f^T, (g, f^{(\infty)}))(s)$, with strict inequality holding whenever $j \in J(s) - K(s)$. The proof of this last statement is enough, since its application again and again will yield that $V(g, f^{(\infty)})(s) \geq V(f^T, (g, f^{(\infty)}))(s) \geq V(f^T, (f^T, (g, f^{(\infty)})))(s) \geq \cdots$, and, by Lemma 4, this decreasing sequence converges to $V(f^{(\infty)})(s)$.

For initial state $s \in \Omega$ with $j \in J(s)$, let T be the first time the state (call it $s' = (n'_1, n'_2, \ldots, n'_r)$) satisfies

(a) $n'_k = 0$ for $k > f(s)$,

(b) $n'_{f(s)} = n_{f(s)} - 1$. If $f(s) = j$ then $V(g, f^{(\infty)})(s) = V(f^T, (g, f^{(\infty)}))(s) = V(f^{(\infty)})(s)$ since for such an s, $(g, f^{(\infty)})$, $(f^T, (g, f^{(\infty)}))$ and $f^{(\infty)}$ generate the same process.

If $i = f(s) > j$, then, because of the nature of T, to compare $V(g, f^{(\infty)})(s)$ and $V(f^T, (g, f^{(\infty)}))(s)$, we may assume without loss of generality that $J(s) = \{i, j\}$ and $n_i = n_j = 1$. When the initial state is such an s, the policy $(g, f^{(\infty)})$ serves the customer of type j, then serves to exhaustion customers of types $i, i+1, \ldots, r$ but not the customer of type i originally in the queue—call this block 1—then serves the original customer of type i, then serves to exhaustion customers of types $i, \ldots, r$, (call this block 2), then continues in some manner (call this block 3). The policy $(f^T, (g, f^{(\infty)}))$ serves the customer of type i, then block 2, then the original customer of type j, then block 1, and finally block 3. Let $A = \{k \in R \mid k \geq i\}$ and $B = A \cup \{j\}$. Note that $H(A)(i) = \mathscr{H}(i)$. Using the notations and results of Lemma 3,

$$
\begin{aligned}
&V(g, f^{(\infty)})(s) - V(f^T, (g, f^{(\infty)}))(s) = \\
&= c(i)((I_B - N_B^{(j)})^{-1} v_B)(j) + (((I_B - N_B^{(j)})^{-1} N_{B,R} c)(j) + \\
&\quad + c(j)((I_B - N_B^{(j)})^{-1} \xi_B)(j))((I_A - N_A)^{-1} v_A)(i) \\
&\quad - c(j)(I_A - N_A)^{-1} v_A)(i) - ((I_A - N_A)^{-1} N_{A,R} c)(i)((I_B - N_B^{(j)})^{-1} v_B)(j) \\
&= ((I_B - N_B^{(j)})^{-1} v_B)(j)((I_A - N_A)^{-1} v_A)(i)(H(A)(i) \\
&\quad - (((I_B - N_B^{(j)})^{-1} v_B)(j))^{-1} (c(j)(1 - ((I_B - N_B^{(j)})^{-1} \xi_B)(j)) - ((I_B - N_B^{(j)})^{-1} N_{B,R} c)(j))) \\
&= ((I_B - N_B^{(j)})^{-1} v_B)(j)((I_A - N_A)^{-1} v_A)(i)(H(A)(i) - H(B)(j)) \\
&\geq ((I_B - N_B^{(j)})^{-1} v_B)(j)((I_A - N_A)^{-1} v_A)(i)(\mathscr{H}(i) - \mathscr{H}(j)).
\end{aligned}
$$

The result follows.

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