Computers and Mathematics with Applications 72 (2016) 1386–1393

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On the generation of arbitrage-free stock price models using Lie symmetry analysis

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ARTICLE INFO

Article history:
Received 8 March 2016
Received in revised form 30 June 2016
Accepted 2 July 2016
Available online 26 July 2016

Keywords:
Lie symmetries
Invariant solutions
Arbitrage-free
Black–Scholes formula
European call option

ABSTRACT

In Bell and Stelljes (2009) a scheme for constructing explicitly solvable arbitrage-free models for stock prices is proposed. Under this scheme solutions of a second-order (1+1)-partial differential equation, containing a rational parameter p drawn from the interval $[1/2, 1]$, are used to generate arbitrage-free models of the stock price. In this paper Lie symmetry analysis is employed to propose candidate models for arbitrage-free stock prices. For all values of p, many solutions of the determining partial differential equation are constructed algorithmically using routines of Lie symmetry analysis. As such the present study significantly extends the work by Bell and Stelljes who found only two arbitrage-free models based on two simple solutions of the determining equation, corresponding to $p = 1/2$ and $p = 1$.

1. Introduction

An important constituent in the valuation of options and other derivatives is the stock price. In the classical Black–Scholes model [1] the stock price S_t is assumed to follow an Itô process described by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1}$$

where μ and σ are two parameters representing the drift and volatility of the stock, respectively, and W_t is a standard Wiener process. The Black–Scholes stock price model (1.1) belongs to a class of solvable arbitrage-free models, i.e. models for which the expected value of S_t at any time t is precisely the future value at time t of a risk-free bond with present value S_0. As a result of this feature (1.1) leads to the well-known Black–Scholes formula for determining the value of a European call option [1]. In fact "arbitrage-freeness" is an essential feature in stock price models. Unfortunately, it is not always inherent in alternative models of the stock price.

Bell and Stelljes [2] describe a method for constructing explicitly solvable arbitrage-free models for stock prices. The method is based on the following solvable stochastic Bernoulli equation of *Stratonovich* type

$$d\widetilde{S}_t = \mu\widetilde{S}_t + \sigma\widetilde{S}_t^p \circ dW_t, \tag{1.2}$$

where p denotes a rational number in the interval $[\frac{1}{2}, 1]$. The solution of (1.2) (see [2] and the references therein) is

$$\widetilde{S}_t = e^{rt}\left\{(1-p)\sigma\int_0^t e^{r(p-1)u}dW_u + \widetilde{S}_0^{1-p}\right\}^{1/(1-p)}. \tag{1.3}$$

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http://dx.doi.org/10.1016/j.camwa.2016.07.003

The process $\widetilde{S}_t$ does not generally satisfy the arbitrage-free condition and hence is not a feasible model for stock price. However, $S_t \equiv G(\widetilde{S}_t, t)$ does satisfy the arbitrage-free condition provided that G solves the second-order partial differential equation

$$G_t + \left(rs + \frac{p\,\sigma^2 s^{2p-1}}{2}\right) G_s + \frac{\sigma^2 s^{2p}}{2} G_{ss} = r\,G \tag{1.4}$$

and that there exists n such that for every $T > 0$

$$\sup_{0\le t\le T} |G_s(s,t)| \le C|s|^n \tag{1.5}$$

where C is a constant depending only on T.

In [2] two values of p, namely $p = \frac{1}{2}$ and $p = 1$, are identified in which cases Eq. (1.4) is tractable. The former case gives rise to a solvable version of the Cox–Ross model [3] and the latter to the Black–Scholes model [1]. In the case $p = 1$,

$$G(s,t) = s\,e^{-\sigma^2 t/2} \tag{1.6}$$

is found to solve (1.4) and to satisfy the regularity condition (1.5). Therefore $S_t = G(\widetilde{S}_t, t)$, with $\widetilde{S}_t$ defined in (1.3), furnishes an arbitrage-free stock price model for $p = 1$. Similarly for $p = \frac{1}{2}$

$$G(s,t) = s + \frac{\sigma^2}{4r} \tag{1.7}$$

solves (1.4) and satisfies the regularity condition (1.5). Accordingly, the resulting arbitrage-free stock price model $S_t = G(\widetilde{S}_t, t)$ is obtained from (1.7) and (1.3).

The aim of this paper is to investigate Eq. (1.4) for all values of p for which the equation is tractable. From the point of view of Lie symmetry analysis this coincides with values of p for which the equation admits a nontrivial symmetry Lie algebra. We have determined that for each value of p Eq. (1.4) admits a rich symmetry group akin to the group admitted by the Black–Scholes equation or the heat equation [4]. Furthermore, we have exploited the admitted one-parameter Lie point symmetries and routines of Lie symmetry analysis to construct solutions of (1.4) as invariant solutions and by transformation of known solutions.

The paper is organised as follows. In Section 2, we introduce elements of Lie symmetry analysis of differential equations. Determination of Lie point symmetries admitted by Eq. (1.4) is done in Section 3. In Section 4 we use the admitted symmetries to construct several exact solutions of (1.4) for all rational values of p. We present concluding remarks in Section 5.

2. Preliminaries of Lie symmetry analysis

Lie symmetry analysis is one of the most powerful methods for finding analytical solutions of differential equations. It has its origins in studies by the Norwegian mathematician Sophus Lie who began to investigate continuous groups of transformations that leave differential equations invariant. Accounts of the subject and its application to differential equations are covered in many books [5–12].

Central to methods of Lie symmetry analysis is invariance of a differential equation under a continuous group of transformations. Consider a one-parameter Lie group of point transformations in infinitesimal form

$$\begin{aligned} \tilde{x} &= x + \varepsilon\,\xi(x,t,u) + O(\varepsilon^2)\\ \tilde{t} &= t + \varepsilon\,\tau(x,t,u) + O(\varepsilon^2)\\ \tilde{u} &= u + \varepsilon\,\eta(x,t,u) + O(\varepsilon^2) \end{aligned} \tag{2.1}$$

depending on a continuous parameter ε. This transformation is characterised by its infinitesimal generator,

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{2.2}$$

The corresponding finite transformations are obtained by exponentiating or by solving the *Lie equations*

$$\frac{d\widetilde{x}}{d\varepsilon} = \xi(\widetilde{x},\widetilde{t},\widetilde{u}), \qquad \frac{d\widetilde{t}}{d\varepsilon} = \tau(\widetilde{x},\widetilde{t},\widetilde{u}), \qquad \frac{d\widetilde{u}}{d\varepsilon} = \eta(\widetilde{x},\widetilde{t},\widetilde{u}) \tag{2.3}$$

subject to the initial conditions

$$(\widetilde{x},\widetilde{t},\widetilde{u})|_{\varepsilon=0} = (x,t,u). \tag{2.4}$$

A general $(1+1)$-partial differential equation with one dependent variable u and two independent variables (x,t),

$$\Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0 \tag{2.5}$$

is invariant under (2.1) if and only if

$$X^{(2)}\Delta = 0 \quad \text{when } \Delta = 0, \tag{2.6}$$

where $X^{(2)}$ is the second prolongation of X given by

$$X^{(2)} = X + \eta_i^{(1)}\partial_{u_i} + \eta_{i_1i_2}^{(2)}\partial_{u_{i_1i_2}}, \quad i_1, i_2 = 1, 2, \tag{2.7}$$

with

$$\eta_i^{(1)} = D_i\eta - \left(D_i\xi^j\right)u_j, \qquad \eta_{i_1i_2}^{(2)} = D_{i_2}\eta_{i_1}^{(1)} - \left(D_{i_2}\xi^j\right)u_{i_1j}, \quad i, i_k, j = 1, 2, \tag{2.8}$$

where $u_i = \frac{\partial u}{\partial x^i}$, $u_{i_1i_2} = \frac{\partial^2 u}{\partial x^{i_1}\partial x^{i_2}}$, $i, i_j = 1, 2$, $(x^1, x^2) = (x, t)$, $(\xi^1, \xi^2) = (\xi, \tau)$ and D_i denotes the total differential operator with respect to x^i:

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + u_{ijk}\frac{\partial}{\partial u_{jk}} + \cdots. \tag{2.9}$$

The Einstein summation convention is adopted in (2.7)–(2.9). The invariance condition (2.6) yields an overdetermined system of linear partial differential equations (called determining equations) for the symmetry group of Eq. (2.5). The infinitesimals ξ, τ and η are then determined as a general solution to the determining equations. If the infinitesimals contain more than one arbitrary constant, we normally 'split' the multi-parameter infinitesimal generator into single parameter generators, which constitute a Lie algebra of (2.5) under the operations of addition and taking the Lie Bracket, namely

$$[X_i, X_j] = X_iX_j - X_jX_i.$$

There are various methods based on infinitesimal generators of the admitted symmetry groups for constructing solutions of partial differential equations. The specific methods used in this paper are introduced in the relevant sections of the paper.

3. Lie point symmetries of (1.4)

With respect to Eq. (1.4), we shall use the less cumbersome variables x and u in place of s and G, respectively. Suppose

$$X = \xi(x, t, u)\,\partial_x + \tau(x, t, u)\,\partial_t + \eta(x, t, u)\,\partial_u, \tag{3.1}$$

where ξ, τ and η are arbitrary functions, is an infinitesimal generator of a symmetry group of (1.4). Then the invariance condition dictates the following:

$$X^{(2)}\left\{u_t + \left(rx + \frac{p\,\sigma^2x^{2p-1}}{2}\right)u_x + \frac{\sigma^2x^{2p}}{2}u_{xx} - ru\right\}\Bigg|_{(1.4)} = 0, \tag{3.2}$$

where $X^{(2)}$ is the second-prolongation of X given in (2.7). With the help of `YaLie` [13] and Mathematica [14], this leads to the following system of determining equations:

$$\xi_u = 0 \tag{3.3}$$
$$\tau_u = 0 \tag{3.4}$$
$$\tau_x = 0 \tag{3.5}$$
$$\eta_{uu} = 0 \tag{3.6}$$
$$2\,\xi\,p + x\,(\tau_t - 2\,\xi_x) = 0 \tag{3.7}$$
$$\xi\left(r + \frac{1}{2}p\,(2\,p-1)\,\sigma^2\,x^{-2+2\,p}\right) - \left(r\,x + \frac{1}{2}p\,\sigma^2\,x^{2\,p-1}\right)(\xi_x - \tau_t) + x^{2\,p}\,\sigma^2\,\eta_{xu} - \xi_t - \frac{1}{2}x^{2\,p}\,\sigma^2\,\xi_{xx} = 0 \tag{3.8}$$
$$\eta_t + r\,u\,\eta_u + \left(r\,x + \frac{1}{2}p\,\sigma^2\,x^{2\,p-1}\,\sigma^2\right)\eta_x + \frac{1}{2}x^{2\,p}\,\eta_{xx} - r\,(\eta + u\,\tau_t) = 0. \tag{3.9}$$

From the first four equations, (3.3)–(3.6), we easily determine that

$$\xi = \xi(x, t) \tag{3.10}$$
$$\tau = \tau(t) \tag{3.11}$$
$$\eta = \phi(x, t) + u\,\varphi(x, t), \tag{3.12}$$

where ϕ and φ are arbitrary functions. Eqs. (3.7)–(3.9) remain unsolved, with (3.7) reducing to

$$2\,p\,\xi + x\left(\tau' - 2\,\xi_x\right) = 0 \tag{3.13}$$

Table 3.1
Commutator table for $\langle X_1, \ldots, X_6\rangle$ in the case $p \neq 1$.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$-2DX_2$	0	$-2D\sigma^2 X_2$	$-DX_1$	0
X_2	$2DX_2$	0	$-\frac{2D}{\sigma^2}X_1$	0	DX_2	0
X_3	0	$\frac{2D}{\sigma^2}X_1$	0	$4DX_5$	$-2DX_3$	0
X_4	$2D\sigma^2 X_2$	0	$-4DX_5$	0	$2DX_4$	0
X_5	DX_1	$-DX_2$	$2DX_3$	$-2DX_4$	0	0
X_6	0	0	0	0	0	0

for which the solution is

$$\xi(x,t) = x^p\,\gamma + \frac{x\,\tau'}{2(1-p)}, \quad p \neq 1, \tag{3.14}$$

where γ is an arbitrary function of t. Next, solving Eq. (3.8) for φ, we obtain

$$\varphi(x,t) = \delta + \left[2\,r\,x\,(p-1)\,\tau' - 4\,r\,x^p\,(p-1)^2\,\gamma + 4\,x^p\,\gamma' - 4\,p\,x^p\,\gamma' + x\,\tau''\right]\frac{x^{1-2p}}{4\,(1-p)^2\,\sigma^2}, \quad p \neq 1, \tag{3.15}$$

where δ is another arbitrary function of t. The linear superposition principle for Eq. (1.4) dictates that the equation admits the infinite group. This is the case only if ϕ solves (1.4), in which case Eq. (3.9) simplifies to the equation

$$\begin{aligned}&\left[u\,(p-1)^2\,\sigma^2\,\left(2\,(-3+p)\,r\,\tau' + 4\,\delta' + \tau''\right)\right]x^p + \left[u\,\left(\tau^{(3)} - 4\,r^2\,(p-1)^2\,\tau'\right)\right]x^{2-p}\\ &\quad + \left[4\,u\,(p-1)\,\left(r^2\,(p-1)^2\,\gamma - \gamma''\right)\right]x = 0, \quad p \neq 1.\end{aligned} \tag{3.16}$$

Eq. (3.16) holds identically if and only if

$$\tau = \varepsilon_1 + \frac{\varepsilon_2\,e^{2Dt} - \varepsilon_3\,e^{-2Dt}}{2D} \tag{3.17}$$

$$\gamma = \varepsilon_4\,e^{Dt} + \varepsilon_5\,e^{-Dt} \tag{3.18}$$

$$\delta = \varepsilon_6 + \frac{r\,(3-p)}{2}\,\tau - \frac{\tau'}{4}, \tag{3.19}$$

where ε_i are arbitrary constants. This completes the solution of the determining Eqs. (3.3)–(3.9), and leads to the following basis of the infinite dimensional vector space of infinitesimal symmetries of Eq. (1.4) for $p \neq 1$:

$$\left.\begin{aligned}&X_1 = e^{Dt}\,x^{1-p}\left(\sigma^2 x^{2p-1}\partial_x - 2\,r\,u\,\partial_u\right), \qquad X_2 = e^{-Dt}\,x^p\,\partial_x\\ &X_3 = e^{2Dt}\left[r\,x\,\partial_x - \partial_t + \left(D - r - \frac{2\,r^2\,x^{2(1-p)}}{\sigma^2}\right)u\,\partial_u\right]\\ &X_4 = e^{-2Dt}\left[r\,x\,\partial_x + \partial_t + r\,u\,\partial_u\right]\\ &X_5 = \partial_t + (r - D/2)\,u\,\partial_u, \qquad X_6 = u\,\partial_u, \qquad X_\phi = \phi(x,t)\,\partial_u\end{aligned}\right\} \tag{3.20}$$

where $D = (p-1)\,r$ and $\phi(x,t)$ is any solution of (1.4)

For $p = 1$ the admitted infinite dimensional vector space of infinitesimal symmetries of Eq. (1.4) is spanned by the following operators:

$$\left.\begin{aligned}&X_1 = x\,\sigma^2\,\partial_x - r\,u\,\partial_u\\ &X_2 = t\,x\,\sigma^2\,\partial_x + u\,(\ln x - r\,t)\,\partial_u\\ &X_3 = \partial_t + M\,r\,u\,\partial_u\\ &X_4 = x\,\ln x\,\partial_x + 2\,t\,\partial_t + u\left(2\,M\,r\,t - \frac{1}{2} - \frac{r}{\sigma^2}\,\ln x\right)\partial_u\\ &X_5 = t\,x\,\ln x\,\partial_x + t^2\,\partial_t + \left[M\,r\,t^2 + \frac{1}{2\,\sigma^2}\,(\ln x)^2 - t\left(\frac{1}{2} + \frac{r}{\sigma^2}\,\ln x\right)\right]u\,\partial_u\\ &X_6 = u\,\partial_u, \qquad X_\phi = \phi(x,t)\,\partial_u\end{aligned}\right\} \tag{3.21}$$

where $M = 1 + \frac{r}{2\sigma^2}$ and $\phi(x,t)$ is any solution of (1.4) when $p = 1$. The first six operators in (3.20) and (3.21) generate finite dimensional symmetry Lie algebras for Eq. (1.4), when $p \neq 1$ and when $p = 1$, respectively. The corresponding commutators are given in Tables 3.1 and 3.2, respectively.

Table 3.2
Commutator table for $\langle X_1, \dots, X_6 \rangle$ in the case $p = 1$.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$\sigma^2 X_6$	0	X_1	X_2	0
X_2	$-\sigma^2 X_6$	0	$-X_1$	$-X_2$	0	0
X_3	0	X_1	0	$2X_3$	X_4	0
X_4	X_1	X_2	$-2X_3$	0	$2X_5$	0
X_5	X_2	0	$-X_4$	$-2X_5$	0	0
X_6	0	0	0	0	0	0

The finite symmetry transformations of Eq. (1.4) corresponding to the infinitesimal generators are obtained by solving the Lie equations (2.3)–(2.4). The results corresponding to the infinitesimal generators (3.20) and (3.21) are presented in (3.22) and (3.23), respectively.

$$\left.\begin{array}{lll}
X_1 & : & \widetilde{x} = \left(x^{-\frac{D}{r}} - \dfrac{D\,e^{D\,t}\,\varepsilon_1\,\sigma^2}{r}\right)^{-\frac{D}{r}}, \qquad \widetilde{t} = t, \qquad \widetilde{u} = e^{-2\,e^{D\,t}\,r\,x^{-\frac{D}{r}}\,\varepsilon_1}\,u \\
X_2 & : & \widetilde{x} = \left(x^{-\frac{D}{r}} - \dfrac{D\,e^{-D\,t}\,\varepsilon_2}{r}\right)^{-\frac{r}{D}}, \qquad \widetilde{t} = t, \qquad \widetilde{u} = u \\
X_3 & : & \widetilde{x} = e^{e^{2\,D\,t}\,r\,\varepsilon_3}\,x, \qquad \widetilde{t} = -\dfrac{\ln\left(e^{-2\,D\,t} + 2\,D\,\varepsilon_3\right)}{2\,D}, \\
 & & \widetilde{u} = \exp\left\{e^{2\,D\,t}\,(D-r)\;\varepsilon_3 + \dfrac{r^2\,\left(e^{-2\,D\,e^{2\,D\,t}\,\varepsilon_3} - 1\right)\,x^{-\frac{2\,D}{r}}}{D\,\sigma^2}\right\} \\
X_4 & : & \widetilde{x} = e^{e^{-2\,D\,t}\,r\,\varepsilon_4}\,x, \qquad \widetilde{t} = \dfrac{\ln(e^{2\,D\,t} + 2\,D\,\varepsilon_4)}{2\,D}, \qquad \widetilde{u} = e^{e^{-2\,D\,t}\,r\,\varepsilon_4}\,u \\
X_5 & : & \widetilde{x} = x, \qquad \widetilde{t} = t + \varepsilon_5, \qquad \widetilde{u} = e^{\left(r-\frac{D}{2}\right)\,\varepsilon_5}\,u \\
X_6 & : & \widetilde{x} = x, \qquad \widetilde{t} = t, \qquad \widetilde{u} = e^{\varepsilon_6}\,u \\
X_\phi & : & \widetilde{x} = x, \qquad \widetilde{t} = t, \qquad \widetilde{u} = u + \phi(x,t)
\end{array}\right\} \tag{3.22}$$

$$\left.\begin{array}{lll}
X_1 & : & \widetilde{x} = e^{\varepsilon_1\,\sigma^2}\,x, \qquad \widetilde{t} = t, \qquad \widetilde{u} = u\,e^{-r\,\varepsilon_1} \\
X_2 & : & \widetilde{x} = x\,e^{\varepsilon_2\,\sigma^2 t}, \qquad \widetilde{t} = t, \qquad \widetilde{u} = u\,e^{\left(\frac{\varepsilon_2{}^2\,\sigma^2}{2} - r\,\varepsilon_2\right)t}\,x^{\varepsilon_2} \\
X_3 & : & \widetilde{x} = x, \qquad \widetilde{t} = t + \varepsilon_3, \qquad \widetilde{u} = u\,e^{D\,r\,\varepsilon_3} \\
X_4 & : & \widetilde{x} = x^{\varepsilon_4}, \qquad \widetilde{t} = \varepsilon_4{}^2 t, \qquad \widetilde{u} = \dfrac{u\,e^{D\,r\,(\varepsilon_4{}^2-1)\,t}\,x^{\frac{(1-\varepsilon_4)r}{\sigma^2}}}{\sqrt{\varepsilon_4}}, \quad \varepsilon_4 \neq 0 \\
X_5 & : & \widetilde{x} = x^{\frac{1}{1-t\,\varepsilon_5}}, \qquad \widetilde{t} = \dfrac{t}{1 - t\,\varepsilon_5}, \qquad \widetilde{u} = u\,e^{\frac{D\,r\,\varepsilon_5 t^2}{1-\varepsilon_5\,t}}\,x^{\frac{\varepsilon_5\,(2\,r\,t - \ln x)}{2\,(\varepsilon_5\,t-1)\,\sigma^2}}\,\sqrt{1 - \varepsilon_5\,t} \\
X_6 & : & \widetilde{x} = x, \qquad \widetilde{t} = t, \qquad \widetilde{u} = e^{\varepsilon_6} u \\
X_\phi & : & \widetilde{x} = x, \qquad \widetilde{t} = x, \qquad \widetilde{u} = u + \phi(x,t).
\end{array}\right\} \tag{3.23}$$

4. Exact solutions of Eq. (1.4)

We shall now use the admitted symmetries to construct exact solutions of Eq. (1.4). We will do this in two ways, via group transformations of known solutions and by construction of invariant solutions.

4.1. Group invariant solutions

Let X be an infinitesimal generator of a symmetry group admitted by (1.4). A function $u = \Theta(x,t)$ is an invariant solution of (1.4) arising from X if it is a solution of (1.4) and remains unchanged under the action of every transformation of the symmetry group generated by X. Such solutions are easily found. They are characterised by a necessary condition known as the *invariant surface* condition:

$$X\,(u - \Theta(x,t)) = 0 \quad \text{when } u = \Theta(x,t). \tag{4.1}$$

Construction of invariant solutions proceeds in a very algorithmic fashion. We determine two independent invariants $r(x, t, u)$ and $v(x, t, u)$ (with $v_u \neq 0$) of the group from solutions of the associated characteristic system

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{4.2}$$

The general solution of the invariant surface condition (4.1) is now written as $v = F(r)$ or $u = \Theta(x, t)$, in terms of u, x, t. Upon substitution of the general solution in (1.4) we obtain an ODE that defines Θ. The solution of this ODE completes the process.

The infinitesimal generators of symmetry groups of Eq. (1.4) can each (except "$u\,\partial_u$" and "$\phi\,\partial_u$" [4]) be used to generate invariant solutions of (1.4). In the examples that follow we use the basis infinitesimal generators in (3.20) and (3.21), corresponding to $p \neq 1$ and $p = 1$, respectively, to do so. In each case we reduce Eq. (1.4) to an ODE and solve the ODE if it admits a "simple" solution. In the interest of brevity of the exposition, we shall only report essential elements of the solution process. In each of the solutions provided below κ_i are arbitrary constants.

Example 4.1 (*Invariant Solutions of (1.4) in the Case $p \neq 1$*).

(a) $X_1 = e^{D\,t}\, x^{1-p} \left(\sigma^2 x^{2p-1} \partial_x - 2\, r\, u\, \partial_u\right)$
- $r = t, v = u\, \exp\left\{-\frac{r\, x^{2\,(1-p)}}{(p-1)\,\sigma^2}\right\}, u = \exp\left\{\frac{r\, x^{2\,(1-p)}}{(p-1)\,\sigma^2}\right\}\, y(t)$
- $y' + (p-2)\; r\, y = 0, y(t) = e^{(2-p)\, r\, t}\, \kappa_1$
- $u(x, t) = \kappa_1\, e^{r\left[(2-p)\, t - \frac{x^{2\,(1-p)}}{(1-p)\,\sigma^2}\right]}$

(b) $X_2 = e^{-D\,t}\, x^p\, \partial_x$
- $r = t, v = u\, \exp\left\{\frac{e^{(p-1)\, r\, t}\, x^{(1-p)}}{p-1}\right\}, u = \exp\left\{-\frac{e^{(p-1)\, r\, t}\, x^{(1-p)}}{p-1}\right\}\, y(t)$
- $2\, y' - \left(2\, r - e^{2\,(p-1)\, r\, t}\, \sigma^2\right)\, y = 0, y(t) = \kappa_1\, e^{r\, t - \frac{\sigma^2\, e^{2\,(p-1)\, r\, t}}{4\,(p-1)\, r}}$
- $u(x, t) = \kappa_1\, \exp\left\{r\, t - \frac{e^{(p-1)\, r\, t}\, x^{(1-p)}}{p-1} - \frac{e^{2\,(p-1)\, r\, t}\, \sigma^2}{4\,(p-1)\, r}\right\}$

(c) $X_3 = e^{2\,D\,t}\left[r\, x\, \partial_x - \partial_t + \left(D - r - \frac{2\, r^2\, x^{2(1-p)}}{\sigma^2}\right)\, u\, \partial_u\right]$
- $r = e^{r\,t}\, x, v = e^{-\frac{r\, x^{2\,(1-p)}}{(p-1)\,\sigma^2}}\, u\, x^{2-p}, u = e^{\frac{r\, x^{2\,(1-p)}}{(p-1)\,\sigma^2}}\, x^{p-2}\, y\,(\zeta), \zeta = e^{r\,t}\, x$
- $\alpha\, y + \zeta\, \left[(3\,p - 4)\; y' + \zeta\, y''\right] = 0$
  where $\alpha = 6 - 7\,p + 2\,p^2$
- $y(\zeta) = \kappa_1\, \zeta^{3-2\,p} + \kappa_2\, \zeta^{2-p}$
- $u(x, t) = \exp\left\{r\, \left(2\; (1-p)\; t - \frac{x^{2(1-p)}}{(1-p)\,\sigma^2}\right)\right\}\, \left(\kappa_2\, e^{p\, r\, t} + \kappa_1\, e^{r\, t}\, x^{1-p}\right)$

(d) $X_4 = e^{-2\,D\,t}\,[r\, x\, \partial_x + \partial_t + r\, u\, \partial_u]$
- $r = e^{-r\,t}\, x, v = u/x, u = x\, y\,(\zeta), \zeta = e^{-r\,t}\, x$
- $p\, y + \zeta\, \left[(2+p)\; y' + \zeta\, y''\right] = 0$
- $y(\zeta) = \frac{\kappa_1}{\zeta} + \frac{\kappa_2}{\zeta^p}$
- $u(x, t) = \kappa_1\, e^{r\,t} + \kappa_2\, e^{p\, r\, t}\, x^{1-p}$

(e) $X_5 = \partial_t + (r - D/2)\; u\, \partial_u$
- $r = x, v = e^{\frac{(p-3)\, r\, t}{2}}\, u, u = e^{\frac{(3-p)\, r\, t}{2}}\, y(x)$
- $r\,(1-p)\; y + \left(2\, r\, x + p\, x^{2\,p-1}\, \sigma^2\right)\; y' + x^{2\,p}\, \sigma^2\, y'' = 0.$

Example 4.2 (*Invariant Solutions of (1.4) in the Case $p = 1$*).

(a) $X_1 = \sigma^2\, x\, \partial_x - r\, u\, \partial_u$
- $r = t, v = u\, x^{\frac{r}{\sigma^2}}, u = x^{-\frac{r}{\sigma^2}}\, y(t)$
- $y' - M\, R\, y = 0, y(t) = \kappa_1\, e^{r\, M\, t}$
- $u(x, t) = \kappa_1\, e^{M\, r\, t}\, x^{-\frac{r}{\sigma^2}}$

(b) $X_2 = \sigma^2\, t\, x\, \partial_x + u\; (\ln x - r\, t)\, \partial_u$
- $r = t, v = u\, x^{\frac{r}{\sigma^2} - \frac{\ln x}{2\,t\,\sigma^2}}, u = x^{\frac{\ln x}{2\,t\,\sigma^2} - \frac{r}{\sigma^2}}\, y(t)$
- $2\, t\, \sigma^2\, y' + \left[\sigma^2 - \left(r^2 + 2\, r\, \sigma^2\right)\; t\right]\; y = 0, y(t) = \frac{\kappa_1\, e^{r\, M\, t}}{\sqrt{t}}$
- $u(x, t) = \kappa_1\, e^{M\, r\, t}\, x^{\frac{\ln x}{2\,t\,\sigma^2} - \frac{r}{\sigma^2}}$

(c) $X_3 = \partial_t + M\,r\,u\,\partial_u$

- $r = x,\ v = u\,e^{-M\,r\,t},\ u = e^{M\,r\,t}\,y(x)$
- $y + x\left[\nu\,y' + \frac{\sigma^4}{r^2}\,x\,y''\right] = 0,\ y(x) = x^{-\frac{r}{\sigma^2}}\ (\kappa_1 + \kappa_2\ \ln x)$
- $u(x,t) = e^{M\,r\,t}\,x^{-\frac{r}{\sigma^2}}\ (\kappa_1 + \kappa_2\ \ln x),\ \nu = \frac{\sigma^2\,(2\,r+\sigma^2)}{r^2}$

(d) $X_4 = x\ \ln x\,\partial_x + 2\,t\,\partial_t + u\left(2\,M\,r\,t - \frac{1}{2} - \frac{r}{\sigma^2}\ \ln x\right)\partial_u$

- $r = \frac{\ln x}{\sqrt{t}},\ v = e^{-M\,r\,t}\,u\,x^{\frac{r}{\sigma^2}}\,t^{1/4},\ u = \frac{e^{M\,r\,t}\,x^{-\frac{r}{\sigma^2}}\,y(\zeta)}{t^{1/4}},\ \zeta = \frac{\ln x}{\sqrt{t}}$
- $y + 2\,\zeta\,y' - 2\,\sigma^2\,y'' = 0$

(e) $X_5 = t\,x\ \ln x\,\partial_x + t^2\,\partial_t + \left[M\,r\,t^2 + \frac{1}{2\,\sigma^2}\ (\ln x)^2 - t\left(\frac{1}{2} + \frac{r}{\sigma^2}\ \ln x\right)\right]u\,\partial_u$

- $r = \frac{\ln x}{t},\ v = u\,x^{\frac{2\,r\,t-\ln x}{2\,t\,\sigma^2}}\ \sqrt{t},\ u = \frac{e^{M\,r\,t}\,x^{\frac{-2\,r\,t+\ln x}{2\,t\,\sigma^2}}\,y(\zeta)}{\sqrt{t}},\ \zeta = \frac{\ln x}{t}$
- $y'' = 0,\ y(\zeta) = \kappa_1 + \kappa_2\,\zeta$
- $u(x,t) = e^{M\,r\,t}\,x^{\frac{-2\,r\,t+\ln x}{2\,t\,\sigma^2}}\left(\frac{\kappa_1}{\sqrt{t}} + \frac{\kappa_2\ \ln x}{t^{3/2}}\right).$

4.2. Group transformation of known solutions

The basis of this method is the fact that a symmetry group of an equation transforms any solution of the equation into a solution of the same equation. Let X be an infinitesimal generator of a group admitted by Eq. (2.5) and $u = f(x,t)$ be any solution of the equation. Then $\widetilde{u} = f(\widetilde{x},\widetilde{t})$, where $\widetilde{x},\widetilde{t},\widetilde{u}$ are associated with x,t,u through the symmetry transformations,

$$\widetilde{x} = f(x,t,\varepsilon), \qquad \widetilde{t} = g(x,t,\varepsilon), \qquad \widetilde{u} = h(x,t,\varepsilon) \tag{4.3}$$

generated by X, defines a family of solutions of (2.5). Replacing $\widetilde{x},\widetilde{t}$ and $\widetilde{u}$ and solving for u results in a one-parameter family of (typically new) solutions of (2.5), $u = f_\varepsilon(x,t)$. Each of the symmetry groups admitted by the equation can thus be used to transform any known solution of the equation into other solution of the same equation. Let us apply this to generate new solutions of (1.4) from the simple solutions (1.6) and (1.7).

Example 4.3. (a) $p = \frac{1}{2},\ f(x,t) = x + \frac{\sigma^2}{4\,r}$

(i) $X_2 : (\widetilde{x},\widetilde{t},\widetilde{u}) = \left(\left[\sqrt{x} + \varepsilon\,e^{\frac{r\,t}{2}}/2\right]^2, t, u\right)$

$$f_\varepsilon(x,t) = \left(\sqrt{x} + \frac{\varepsilon}{2}\,e^{\frac{r\,t}{2}}\right)^2 + \frac{\sigma^2}{4\,r} \tag{4.4}$$

(ii) $X_5 : (\widetilde{x},\widetilde{t},\widetilde{u}) = \left(x, t+\varepsilon, u\,e^{\frac{5\,r\,\varepsilon}{4}}\right)$

$$f_\varepsilon(x,t) = e^{-\frac{5\,r\,\varepsilon}{4}}\left(x + \frac{\sigma^2}{4\,r}\right) \tag{4.5}$$

(b) $p = 1, f(x,t) = x\,e^{-\sigma^2 t/2}$

(i) $X_1 : (\widetilde{x},\widetilde{t},\widetilde{u}) = \left(e^{\varepsilon\,\sigma^2}\,x, t, u\,e^{-r\,\varepsilon}\right)$

$$f_\varepsilon(x,t) = \exp\left(\varepsilon\ \left(r + \sigma^2\right) - \frac{t\,\sigma^2}{2}\right)x \tag{4.6}$$

(ii) $X_3 : (\widetilde{x},\widetilde{t},\widetilde{u}) = \left(x, t+\varepsilon, e^{M\,r\,\varepsilon}\,u\right)$

$$f_\varepsilon(x,t) = \exp\left(-\frac{t\,\sigma^2}{2} - \varepsilon\left(M\,r + \frac{\sigma^2}{2}\right)\right)x. \tag{4.7}$$

4.3. New solutions from known solutions via X_ϕ

The infinite set of operators represented by "X_ϕ" in (3.20) and (3.21) can be used to generate new solutions of (1.4) from known solutions [4]. Let $u = \omega(x,t)$ be a known solution of Eq. (1.4) so that the operator X_ω is admitted by Eq. (1.4). Then, if X is any other operator admitted by Eq. (1.4), by taking the Lie bracket, one obtains

$$[X, X_\omega] = X_{\overline{\varpi}} \tag{4.8}$$

where $\overline{\varpi}$ is also a solution of Eq. (1.4), typically different from $\omega(x,t)$. The infinite set L_ω of operators of the form X_ω can therefore be used to generate a whole range of solutions of (1.4) via the relation (4.8). We will illustrate this by transforming the simple solutions (1.6) and (1.7) into new solutions of (1.4) in the cases $p = 1$ and $p = \frac{1}{2}$, respectively.

Example 4.4. (a) $p = \frac{1}{2}, \omega(x,t) = x + \frac{\sigma^2}{4r}$

(i) $X_1 = e^{-\frac{rt}{2}} \sqrt{x} \left(\sigma^2 \partial_x - 2 r u \partial_u\right)$

$$[X_1, X_\omega] = \overline{\omega}\, \partial_u, \qquad \overline{\omega}(x,t) = e^{-\frac{rt}{2}} \sqrt{x} \left(2 r x + \frac{3\sigma^2}{2}\right) \tag{4.9}$$

(ii) $X_3 = e^{-rt} \left[r x \partial_x - \partial_t - \frac{1}{2} r u \left(3 + \frac{4rx}{\sigma^2}\right) \partial_u\right]$

$$[X_3, X_\omega] = \overline{\omega}\, \partial_u, \qquad \overline{\omega}(x,t) = e^{-rt} \left(3 r x + \frac{2 r^2 x^2}{\sigma^2} + \frac{3\sigma^2}{8}\right) \tag{4.10}$$

(b) $p = 1, \omega(x,t) = x e^{-\sigma^2 t/2}$

(i) $X_2 = t x \sigma^2 \partial_x + u\, (\ln x - r t)\, \partial_u$

$$[X_2, X_\omega] = \overline{\omega}\, \partial_u, \qquad \overline{\omega}(x,t) = e^{-\frac{t\sigma^2}{2}} x \left[t \left(r + \sigma^2\right) - \ln x\right] \tag{4.11}$$

(ii) $X_4 = x \ln x \partial_x + 2 t \partial_t + u \left(2 M r t - \frac{1}{2} - \frac{r}{\sigma^2} \ln x\right) \partial_u$

$[X, X_\omega] = \overline{\omega}\, \partial_u$

$$\overline{\omega}(x,t) = e^{-\frac{t\sigma^2}{2}} \left[x \psi(t) + \frac{x \left(r + \sigma^2\right) \ln x}{\sigma^2}\right] \tag{4.12}$$

where

$$\psi(t) = \frac{1}{2} - \frac{t \left(r + \sigma^2\right)^2}{\sigma^2}.$$

5. Concluding remarks

This paper complements the work by Bell and Stelljes [2] who proposed a method for constructing explicitly solvable arbitrage-free models for the stock price. At the centre of their method is a second-order partial differential equation that contains, as a parameter, a rational number p drawn from the interval $[\frac{1}{2}, 1]$. Solutions of this equation that satisfy a prescribed regularity requirement define solvable arbitrage-free models for the stock price. Bell and Stelljes reported the challenge of finding such solutions for a general parameter p; they only found two simple solutions for $p = 1$ and $p = \frac{1}{2}$. In this connection, the present paper augments the work by Bell and Stelljes through the use of Lie symmetry analysis. We have determined Lie point symmetries admitted by the determining PDE for all values of p and established that in each case the equation admits a rich symmetry group. As a result we have been able to construct several families of solutions of the equation via routines of Lie symmetry analysis. Those solutions that satisfy the necessary regularity condition provide models for arbitrage-free stock prices.

Acknowledgement

The author would like to thank the Directorate of Research Development of Walter Sisulu University for continued financial support.

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