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Uniquely and Faithfully Embeddable Projective-Planar Triangulations

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Let G be a 5-connected graph not isomorphic to the complete graph K_6 with 6 vertices and triangularly embedded in a projective plane P^2. Then it will be shown that for any embedding $f: G \to P^2$, there is a homeomorphism $h: P^2 \to P^2$ such that $h|G = f$. In our terminology, the result states that every 5-connected projective-planar triangulation is uniquely and faithfully embeddable in a projective plane unless it is isomorphic to K_6.

1. INTRODUCTION

We shall deal with only simple finite graphs and discuss when a graph can be embedded into a closed surface in only one way or in such a way that its symmetry is realized by translations on the surface. Let G be a graph, regarded as a topological space with the structure of a 1-complex, and let f_1, $f_2: G \to F^2$ be two embeddings of G into a surface F^2. They are said to be *equivalent* if there are a homeomorphism $h: F^2 \to F^2$ and an automorphism $\sigma: G \to G$ such that $h \cdot f_1 = f_2 \cdot \sigma$. A graph G is *uniquely embeddable* in a surface F^2 if there is only one equivalence class of its embeddings into F^2. An embedding $f: G \to F^2$ is *faithful* if for any automorphism $\sigma: G \to G$, there is a homeomorphism $h: F^2 \to F^2$ such that $h \cdot f = f \cdot \sigma$. A graph G is *faithfully embeddable* in a surface F^2 if it admits a faithful embedding into F^2.

Classically, Whitney showed in [5] that every 3-connected planar graph is uniquely and faithfully embeddable in a sphere, which is well known as the uniqueness of a dual of a 3-connected planar graph. The author proved in [2] that every 6-connected toroidal graph is uniquely and faithfully embeddable in a torus with precisely three exceptions. Recently, he has shown in [3] that if a 5-connected projective-planar graph contains a subdivision of the complete graph K_6 with 6 vertices as its proper subgraph then it is uniquely and faithfully embeddable in a projective plane. In this paper, we

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shall give another sufficient condition for a projective-planar graph to be uniquely and faithfully embeddable in a projective plane.

THEOREM 1. *Every 5-connected projective-planar triangulation, not isomorphic to K_6, is uniquely and faithfully embeddable in a projective plane.*

By a *triangulation* of a closed surface F^2, we mean a (simple) graph which can be *triangularly embedded in* F^2, that is, which is embeddable in F^2 so as to divide F^2 into three-edged faces. Note that all embeddings of a triangulation of F^2 into the same surface are triangular.

Hereafter, we shall assume that a graph G has already been enbedded in a closed surface F^2. Then G is uniquely and faithfully embeddable in F^2 if and only if for any embedding $f: G \to F^2$, there is a homeomorphism $h: F^2 \to F^2$ such that $h|G = f$. First, we shall give a general argument on the uniqueness and faithfulness of triangulations and next prove our theorem observing a special phenomenon in case of a projective plane.

2. Skew Vertices in Triangulations

This section presents a sufficient condition for a triangulation to be uniquely and faithfully embeddable in a surface in general cases.

Let G be a graph embedded in a closed surface F^2, and $f: G \to F^2$ another embedding of G into F^2. A face A of G is said to be *extendable* for f if there is an embedding $h: G \cup A \to F^2$ such that $h|G = f$.

PROPOSITION 2. *If all faces of G are extendable for f then there is a homeomorphism $h: F^2 \to F^2$ such that $h|G = f$.*

Let G be a triangulation embedded in a closed surface F^2 and let the *star neighborhood* $\mathrm{st}(v)$ of a vertex v be the closure of the union of triangular faces meeting v, which is homeomorphic to a 2-cell, and let the *link* $\mathrm{lk}(v)$ of v be the boundary cycle of $\mathrm{st}(v)$. Then $\mathrm{lk}(v)$ is a hamiltonian cycle in the subgraph induced by the neighbors of v in G, denoted by $\langle N(v) \rangle$. A vertex v of G is *skew* if there is another hamiltonian cycle in $\langle N(v) \rangle$, that is, if $\langle N(v) \rangle$ contains at least two hamiltonian cycles. A triangle uvw in G, a cycle of length 3, is *skew* if all u, v, w are skew vertices.

PROPOSITION 3. *If the boundary triangle of a face A of a triangulation G in a closed surface F^2 is not skew then A is extendable for any embedding $f: G \to F^2$.*

Proof. Let uvw be the boundary triangle of a face A. Suppose that the vertex v is not skew, then the edge uw lies in the unique hamiltonian cycle

$\mathrm{lk}(v)$ of $\langle N(v)\rangle$ and the triangle uvw bounds the face A in $\mathrm{st}(v)$. Let $f: G \to F^2$ be another embedding of G into F^2. Since f induces the isomorphism between G and $f(G)$, the vertex $f(v)$ also is not skew, and f carries $\mathrm{lk}(v)$ onto $\mathrm{lk}(f(v))$. Then $f(uvw)$ bounds a face A' in $\mathrm{st}(f(v))$ and hence f extends to an embedding $h: G \cup A \to F^2$ such that $h|G = f$ and $h(A) = A'$. ∎

From Proposition 3, it follows that if a triangulation of a closed surface contains no skew vertex, then it is uniquely and faithfully embeddable in the surface. This criterion can be found in [2]. We shall give a slightly weak condition for the uniqueness and faithfulness of triangular embedding.

THEOREM 4. *Let G be a triangulation of a closed surface F^2. If G has at most four skew vertices, then G is uniquely and faithfully embeddable in F^2.*

Proof. Consider the worst case; in the other cases, we can get the conclusion easily. Suppose that G has four skew vertices u, v, w, x and that two faces A_1, A_2 bounded by skew triangles uvw and vwx, respectively. Let $\mathrm{st}(v)$ consist of faces $A_3, ..., A_n$ together with A_1 and A_2, where $n = \deg(v)$, and let $f: G \to F^2$ be any embedding of G into F^2. Since $A_3, ..., A_n$ are extendable for f, the path between u and x in $\mathrm{lk}(v)$ missing w, that is, $\mathrm{lk}(v) - \{w\}$ is sent to $\mathrm{lk}(f(v)) - \{f(w)\}$ by f. Then a hamiltonian cycle in $\langle N(f(v))\rangle$ which contains $\mathrm{lk}(f(v)) - \{f(w)\}$ coincides with $\mathrm{lk}(f(v))$, although there are two or more hamiltonian cycles in $\langle N(f(v))\rangle$, and hence $f(\mathrm{lk}(v)) = \mathrm{lk}(f(v))$. It follows from this, similarly to the proof of Proposition 3, that A_1 and A_2, and hence all faces of G are extendable for f. Therefore, G is uniquely and faithfully embeddable in F^2. ∎

3. CASE OF A PROJECTIVE PLANE

In this section, we shall prove Theorem 1. We have already seen that if a triangulation has a few skew vertices then it is uniquely and faithfully embeddable in a closed surface. There seems to be an upper bound of the number of skew vertices in a triangulation for a closed surface. In fact, we shall observe that the upper bounds for a projective plane is equal to 6, as follows:

LEMMA 5. *Let G be a 5-connected triangulation embedded in a projective plane. Then the subgraph of G induced by the skew vertices is complete.*

Proof. If a vertex v of G is skew then there are four vertices a, b, c, d of G lying on $\mathrm{lk}(v)$ in order with edges ac and bd, and triangles vac and vbd

bound no 2-cell in the projective plane P^2. The hexagonal 2-cell in Fig. 1 shows P^2 cut open along the triangle vac. Let u be another skew vertex of G and suppose that u is not adjacent to v. Then u lies in one of the interior of squares $bcad$ and $abdc$. Since two triangles bounding no 2-cell must cross each other at the skew vertex u as well as v, we conclude that u had to be adjacent to all a, b, c, d. If the horizontal path between b and c contained at least one vertex in its interior, then the removal of v, b, u, and c would disconnect G. Thus this path consists of a single edge, and so does the path between d and a similarly. This implies that the removal of a, b, c, and d would disconnect G, again, contrary to G being 5-connected. Therefore, any two skew vertices of G are adjacent. ∎

Proof of Theorem 1. Note that the complete graph K_7 with 7 vertices cannot be embedded in a projective plane P^2. (See [4].) From this and Lemma 5, it follows that the number of skew vertices of G does not exceed 6. If the number is equal to 6, then G is isomorphic to K_6, since K_6 itself is a triangulation of P^2 and since G is 5-connected. In this case, G is uniquely but not faithfully embeddable in P^2, as is shown in [3]. The theorem, however, excludes K_6 as the unique exception. When G has at most four skew vertices, we get the conclusion from Theorem 4. We should analyze only the case that the skew vertices of G form the complete graph K_5 with 5 vertices.

First classify the embeddings of K_5 into P^2. It is not difficult to see that an embedding of K_5 into P^2 is equivalent to one of embeddings given in Fig. 2, which can be found in [1]. However, the embedding in Fig. 2b does not arise in the situation where K_5 is contained in a 5-connected graph G as its subgraph; if it did, then the four vertices lying the boundary of a four-edged face of K_5 would disconnect G.

Let $f: G \to P^2$ be any embedding of G into P^2, then f embeds K_5 into P^2 in the same way as Fig. 2a. Since the interior of the pentagon 12543 in Fig. 2a contains no skew triangle, the pentagonal part of G is extendable for f and

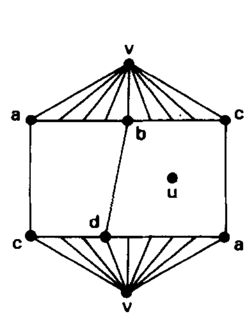

FIGURE 1

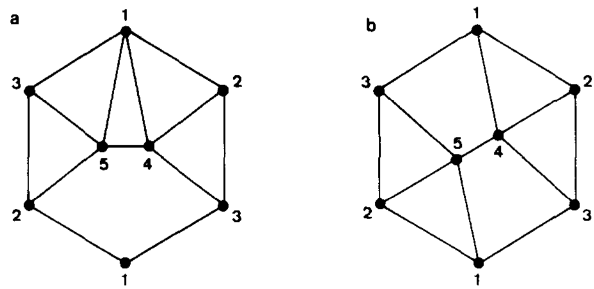

FIGURE 2

the cycle 12543 of G is sent by f onto that of $f(G)$. The permutation of labels 1, 2, 5, 4, 3 induced by f belongs to the dihedral group generated by (12543) and (23) (45). When f corresponds to each generator, it is easy to see that five skew triangles 135, 145, 124, 234, 235 are extendable for f. Thus they are always extendable for any embedding f, and G is uniquely and faithfully embeddable in P^2. ∎

As is shown in [3], the complete graph K_6 is uniquely but not faithfully embeddable in a projective plane, and there is a 4-connected projective-planar triangulation which is not uniquely embeddable in a projective plane. So our theorem is best possible in a sense. The two results in this paper and [3] seem to cover almost cases of 5-connected projective-planar graphs, but the complete answer to the following question is still unknown; Is every 5-connected projective-planar graph, possibly with at most finitely many exceptions, uniquely and faithfully embeddable in a projective plane?

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