Backward behavior of solutions of the Kuramoto–Sivashinsky equation

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Abstract

We prove that any solution of the Kuramoto–Sivashinsky equation either belongs to the global attractor or it cannot be continued to a solution defined for all negative times. This extends a previous result of the first author who proved that solutions which do not belong to the global attractor have superexponential backward growth. A particular consequence of the result is that the global attractor can be characterized as the maximal invariant set.

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1. Introduction

In this note, we address backward behavior properties of solutions of the Kuramoto–Sivashinsky equation (KSE)

\[ u_t + u_{xxxx} + u_{xx} + uu_x = 0 \]  

(1.1)

with initial condition

\[ u(x, 0) = u_0(x), \]

where \( u_0 \) is \( L \)-periodic with mean zero, i.e., \( \int_\Omega u_0 = 0 \), where \( \Omega = [-L/2, L/2] \).

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As it is well known, the exact reversibility is not possible for parabolic equations and systems due to the smoothing effect. The primary motivation for the present note is the following question in control: Given a state of a system, can we find an initial datum which the system will drive to a proximity of this state in a given time? In other words, given $u_0$, $\epsilon > 0$, and $t > 0$, is it possible to find $v_0$ so that $S(t)v_0$ is in the $\epsilon$-neighborhood of $u_0$? Mathematically this question in control can be restated as: Is $S(t)H$ dense in $H$? The above question might be difficult for a given system; for instance, it is still open for the Navier–Stokes equations. A weaker question of density of a linear span $L(S(t)H)$ of $S(t)H$, where $H$ is the phase space and $S(t)$ is the solution semigroup, has been addressed by Bardos and Tartar in [1]. They found a simple criterion for $L(S(t)H)$ to be dense in $H$; it is easy to check that the KSE satisfies this condition, and thus $L(S(t)H)$ is dense in $H$ (see (2.1) below for the definition of the phase space $H$). There is a vast literature on methods on recovering history from initial data—cf. [14] for the method of quasi-reversibility.

The backward behavior of solutions of the 2D Navier–Stokes equations has been studied extensively in [3], where it was shown that a 2D periodic Navier–Stokes equation (NSE) has many solutions which can be continued backward in time for $t \in (0, \infty)$—we call such solutions global solutions. In fact, the set of data leading to global solutions has infinite Hausdorff dimension and has a rather rich structure. In [16], it was proven that the analogous results hold for the 2D periodic viscous Camassa–Holm equations. We also note a result of Dascaliuc [4], who proved that for the original Burgers model of turbulence there exist solution with superexponential growth, as well as solutions with subexponential growth.

While the KSE is in many regards an equation with similar properties as the 2D NSE (fluid dynamics background, low dimensional chaos, structure of the nonlinear term), it has a different behavior as far as backward behavior of solutions is concerned. The main result in the present paper is that an initial datum leads to a global solution of the KSE if and only if it belongs to the global attractor. Regarding the control question posed above, we show that this implies that the KSE is not controllable by initial data. Moreover, given any $t > 0$, not only $S(t)H$ is not dense, it is actually nowhere dense. This should not necessarily be viewed as a negative result. It shows that $\sup_{u_0} \text{dist}(S(t)u_0, A)$, a uniform upper bound for the distance to the global attractor $A$ of a trajectory $S(t)u_0$, is a function of $t$ which converges to $0$. Another consequence of the above result is that the global attractor is the largest invariant set, a fact not true for the 2D periodic NSE, for instance.

Regarding the application of the above results, it was shown in [5] that backward blowup of solutions can be explored in seeking algebraic sets approximating the global attractor. Namely, based on backward blowup (or exponential growth of solutions) and analyticity properties, we can construct polynomials on the phase space whose zeros approximate the global attractor. This is addressed in the final section of this paper.

Here we describe the idea of the proof of the main result, Theorem 2.1. The proof relies on the dissipativity nature of the term $uu_x$ in the KSE. Dissipativity has been proved for odd data by Nicolaenko, Scheurer, and Temam, while [2,9,11] independently removed the oddness assumption. The proof in [15] relies on the change of variable $v = u - \phi$; the gauge function $\phi$ is chosen as a periodic approximation of a function $\alpha x$. This transformation was then used in [13] to show that the KSE has the following property: If $u$ is a global solution not belonging to the attractor, then $\|u(\cdot, t)\|^2_{L^2}$ grows to $\infty$ as $t \to -\infty$ quicker than any
exponential. Our Theorem 2.1 is an improvement of the argument from [13]; we write the phase space as a union of dyadic spherical shells and then find an appropriate gauge functions for each shell.

2. A negative finite time blowup of solutions

We start by introducing necessary notation and state our main result. For $L > 0$, denote $\Omega = [-L/2, L/2]$. Also, denote by $L^2_{\text{per}}(\Omega)$ the set of (real-valued) $u_0 \in L^2_{\text{per}}(\mathbb{R})$ which are periodic with period $L$. We assume throughout $L \geq 1$—note namely that if $u$ is periodic with period $L$, then it is periodic with period $nL$ for any $n \in \mathbb{N}$. The space 

$$H = \dot{L}^2_{\text{per}}(\Omega) = \left\{ u_0 \in L^2_{\text{per}}(\Omega) : \int_\Omega u_0 = 0 \right\}$$

is a Hilbert space with a scalar product

$$(u_0, v_0) = \int_\Omega u_0 v_0$$

and the norm $\|u_0\| = \|u_0\|_H = (u_0, u_0)^{1/2}$. It is well known that for every $u_0 \in H$, there exists a unique solution $u(t) = u(\cdot, t) = S(t)u_0$ which is analytic in space and time variables in $\mathbb{R} \times (0, \infty)$ and such that $u \in C([0, \infty), H)$ with $u(0) = u_0$ (cf. [17]).

Also, let

$$Au = u_{xxxx}$$

with the definition domain

$$D(A) = \{ u_0 \in H : Au_0 \in H \}.$$
Moreover, multiplying $R_0$ with a constant if necessary, we can assure that for every $R_1 \geq R_0$, the inequality $\|u_0\| \leq R_1$ implies $\|S(t)u_0\| \leq R_1 \sqrt{2}$ for $t \geq 0$. By [2,9], $A$ has the global attracting property, i.e., every solution $u(t)$ converges to $A$ as $t \to \infty$ in the $H$ norm.

The following theorem establishes a negative time blowup of solutions $u$ of the KSE (1.1).

**Theorem 2.1.** Let $u : [0, \infty) \to H$ be a solution of (1.1) which does not belong to the global attractor $A$. Then $u$ cannot be continued to a solution defined for all $t \in \mathbb{R}$.

In other words, every solution not belonging to the global attractor $A$ blows up in a negative time direction.

The above theorem characterizes the global attractor $A$ as the maximal invariant set.

**Corollary 2.2.** If a set $B \subseteq H$ is invariant for the Kuramoto–Sivashinsky flow, then $B \subseteq A$.

By [3], both previous statements fail for the 2D periodic Navier–Stokes system.

**Theorem 2.3.** There exists a sufficiently large constant $C > 0$ such that

$$S \left( \frac{CL^2}{R_0^{4/5}} \right) H \subseteq B(0, R_0)$$

for all $R_0 \geq CL^{5/2}$.

**Remark 2.4.** We can restate the last corollary in the following way: For every initial datum $u_0 \in H$, we have

$$\|S(t)u_0\|_{L^2} \leq CL^{5/2} \left( \frac{1}{t^{5/4}} + 1 \right).$$

Using the approach in [2] and proceeding in a similar way as below, one can show

$$\|S(t)u_0\|_{L^2} \leq CL^{5/5} \left( \frac{1}{t} + 1 \right).$$

The following lemma was proved in [9].

**Lemma 2.5.** For every $\epsilon \in (0, L/2)$, there exists an $L$-periodic function $b_\epsilon \in C^\infty(\mathbb{R}, \mathbb{R})$ with the following properties:

(i) $b_\epsilon(x) \geq 0$ for $x \in \mathbb{R}$,
(ii) $\text{supp} \ b_\epsilon \cap [-L/2, L/2] \subseteq (-\epsilon, \epsilon),$
(iii) $\int_\Omega b_\epsilon(x) \, dx = 3L$,
(iv) $|b_\epsilon(x)| \leq CL/\epsilon$ for $x \in \mathbb{R}$,
(v) $\|b'_\epsilon\| \leq CL/\epsilon^{3/2}$,
(vi) $\int_\Omega b_\epsilon(x)u(x) \, dx = 0$ for some $u \in H$ implies $\int_\Omega b_\epsilon(x)u^2(x) \, dx \leq 9\epsilon L \int_\Omega u_\epsilon(x)^2 \, dx$. 


Existence of \( b_\epsilon \) with properties (i)–(v) is clear. (It is a mollified and then periodically extended function \( \phi(x) = \frac{3L(\epsilon - |x|)}{\epsilon^2} \) for \( |x| \leq \epsilon \) with \( \phi(x) = 0 \) otherwise.) That (i)–(v) imply (vi) is proved in [9, Proposition 1].

For every \( \alpha > 0 \) and \( \epsilon \in (0, L/2) \), define

\[
s_{\alpha, \epsilon}(x) = 6\alpha x - 2\alpha \int_0^x b_\epsilon (y) \, dy, \quad x \in \mathbb{R}.
\]

For a solution \( u = u(x, t) \) of the Kuramoto–Sivashinsky equation, denote

\[
F_{\alpha, \epsilon}(t) = \text{dist}(u(\cdot, t), S_{\alpha, \epsilon})^2,
\]

where the distance is taken in \( H \), and \( S_{\alpha, \epsilon} = \{ \tau_\xi s : \xi \in \mathbb{R} \} \), where \( \tau_\xi s(x) = s(x - \xi) \) for \( x \in \mathbb{R} \) is a translate.

As in [9], we compute for any fixed \( \xi \in \mathbb{R} \),

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( u(x, t) - s_{\alpha, \epsilon}(x - \xi) \right)^2 \, dx
\]

\[
= -\int_\Omega u_{xx}^2 - \frac{1}{2} \int_\Omega \left( s_{\alpha, \epsilon}'(x - \xi) \right)^2 u(x, t)^2 \, dx
\]

\[
+ \int_\Omega \left( s_{\alpha, \epsilon}''(x - \xi) + s_{\alpha, \epsilon}(x - \xi) \right) u_{xx}(x, t) \, dx,
\]

from where, using the interpolation and the Cauchy–Schwarz inequalities,

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( u(x, t) - s_{\alpha, \epsilon}(x - \xi) \right)^2 \, dx
\]

\[
\leq -\frac{1}{2} \int_\Omega u_{xx}^2 + \int_\Omega u^2 - \frac{1}{2} \int_\Omega \left( 6\alpha - 2\alpha b_\epsilon(x - \xi) \right) u(x, t)^2 \, dx
\]

\[
+ C \int_\Omega s_{\alpha, \epsilon}''(x - \xi)^2 \, dx + C \int_\Omega s_{\alpha, \epsilon}(x - \xi)^2 \, dx
\]

\[
= -\frac{1}{2} \int_\Omega u_{xx}^2 + \int_\Omega u^2 - \frac{1}{2} \int_\Omega \left( 6\alpha - 2\alpha b_\epsilon(x - \xi) \right) u(x, t)^2 \, dx
\]

\[
+ C \int_\Omega s_{\alpha, \epsilon}''(x)^2 \, dx + C \int_\Omega s_{\alpha, \epsilon}(x)^2 \, dx,
\]

where the symbol \( C \) stands for a sufficiently large positive constant. A fundamental observation of Goodman [9, p. 296] is that if \( \xi \in \mathbb{R} \) is chosen so that

\[
\text{dist}(u(\cdot, t), S_{\alpha, \epsilon}) = \| u(\cdot, t) - \tau_\xi s \|
\]
then
\[ \int_{\Omega} u(x, t) \tau \xi \phi'(x) dx = 0, \]
from where
\[ \int_{\Omega} u(x, t) b_\xi (x - \xi) dx = 0. \]
But then, by part (vi) in Lemma 2.5, we get with this particular \( \xi \),
\[ \int_{\Omega} b_\xi (x - \xi) u(x, t)^2 dx \leq 9 \epsilon L \int_{\Omega} u_x(x, t)^2 dx. \]
Therefore,
\[ \frac{1}{2} \frac{d}{dt} F_{\alpha, \epsilon}(t) \leq -\frac{1}{2} \int_{\Omega} u_{xx}^2 + \int_{\Omega} u^2 - 3 \alpha \int_{\Omega} u^2 + \frac{1}{2} \int_{\Omega} u_{xx}^2 + \frac{81}{2} \epsilon^2 L^2 \alpha^2 \int_{\Omega} u^2 \]
\[ + C \int_{\Omega} s''_{\alpha, \epsilon}(x)^2 dx + C \int_{\Omega} s_{\alpha, \epsilon}(x)^2 dx, \]
where
\[ \frac{d}{dt} F_{\alpha, \epsilon}(t) = \limsup_{\tau \to t+} \frac{F_{\alpha, \epsilon}(\tau) - F_{\alpha, \epsilon}(t)}{\tau - t} \]
denotes the upper right derivative. Since \( \| s_{\alpha, \epsilon} \| \leq C_0 L^{3/2} \alpha \) and \( \| s''_{\alpha, \epsilon} \| \leq C L \alpha / \epsilon^{3/2} \), we get
\[ \frac{d}{dt} F_{\alpha, \epsilon}(t) \leq (2 - 3 \alpha + 81 \epsilon^2 L^2 \alpha^2) \int_{\Omega} u^2 + C L^3 \alpha^2 + \frac{C L^2 \alpha^2}{\epsilon^3}. \]
Fix any \( \alpha \geq 2 \), and let \( \epsilon = (9 \alpha^{1/2} L)^{-1} \). We get
\[ \frac{d}{dt} F_{\alpha, \epsilon}(t) \leq -\alpha \int_{\Omega} u^2 + C_1 (L^3 \alpha^2 + L^3 \alpha^{5/2}). \]
(2.3)

**Lemma 2.6.** If \( u \) is a solution such that \( \| u(\cdot, t) \| \leq 2 R \) and
\[ \| u(\cdot, t) \| \geq R, \quad t_1 \leq t \leq t_2. \]
then
\[ t_2 - t_1 \leq \frac{15 R^2 + 6 C_1^2 L^3 \alpha^2}{\alpha R^2}, \]
provided \( \alpha \) is chosen so that \( \alpha \geq 2 \) and
\[ 2C_1 L^3 \alpha + 2C_1 L^5 \alpha^{5/2} \leq R^2. \]
(2.5)
Proof. We have

\[ F_{\alpha, \epsilon}(t_1) \leq 8R^2 + 2C_0^2L^3\alpha^2. \]

By (2.3) and (2.4),

\[ F_{\alpha, \epsilon}(t_2) \leq 8R^2 + 2C_0^2L^3\alpha^2 + (-\alpha R^2 + C_1(L^3\alpha^2 + L^5\alpha^{7/2}))(t_2 - t_1). \]

Since

\[ F_{\alpha, \epsilon}(t_2) \geq \frac{1}{2}\|u(t_2)\|^2 - C_0^2L^3\alpha^2 \geq R^2 - C_0^2L^3\alpha^2 \]

we get

\[ \frac{1}{2}R^2 - C_0^2L^3\alpha^2 \leq 8R^2 + 2C_0^2L^3\alpha^2 + (-\alpha R^2 + C_1(L^3\alpha^2 + L^5\alpha^{7/2}))(t_2 - t_1). \]  

(2.6)

Under the assumption, we have

\[ -\alpha R^2 + C_1(L^3\alpha^2 + L^5\alpha^{7/2}) \leq -\alpha R^2/2, \]

and the claim follows by solving the inequality (2.6) for \( t_2 - t_1 \). \( \square \)

Theorems 2.1 and 2.3 then follow directly from the next lemma.

**Lemma 2.7.** There exists a sufficiently large constant \( C \) such that the following holds: If 

\[ u : [t_1, t_2] \to H \]

is a solution such that

\[ \|u(\cdot, t_2)\| \geq CL^{5/2} \]

then

\[ t_2 - t_1 \leq \frac{CL^2}{\|u(\cdot, t_2)\|^{4/5}}. \]

**Proof.** The idea is to write the set \( S = \{v_0 \in H : \|v_0\| \geq R_0/2\} \), where \( R_0 = \|u(\cdot, t_2)\| \), as a union of overlapping dyadic shells and using Lemma 2.6 give an upper bound on the time \( u \) can spend in each shell.

First, observe that there exists

\[ R_0 = C_2L^{5/2} \]

such that if

\[ \|u(\cdot, t_0)\| \leq R_1 \]

for some \( t_0 \in \mathbb{R} \), where \( R_1 \geq R_0' \), then

\[ \|u(\cdot, t)\| \leq \sqrt{2}R_1, \quad t \geq t_0. \]

In order to prove this, simply use (2.3) with \( \alpha = 2 \) and \( \epsilon = (9\alpha^{1/2}L)^{-1} \).

Now, denote \( R_0 = \|u(\cdot, t_2)\| \), and assume \( R_0/4 \geq R_0' \). Choose any sequence

\[ t_1 = sN < sN - 1 < sN - 2 < \cdots < s_1 < s_0 = t_2 \]
such that
\[ \|u(\cdot, t)\| \geq 2^{(j-2)/2} R_0, \quad t \in [s_j, s_{j-1}], \]
and
\[ \|u(\cdot, s_j)\| \leq 2^{j/2} R_0 \]
for all \( j = 1, 2, \ldots, N \). Now, fix any \( j \in \{1, \ldots, N\} \). We apply Lemma 2.6 with
\[ R = 2^{(j-2)/2} R_0 \]
and
\[ \alpha = \min \left\{ \frac{2^{j-2} R_0^2}{4C_1 L^3}, \frac{4^{j/5} R_0^{4/5} L^2}{4^{j/5} C_1^{2/5} L^2} \right\} = \frac{4^{j/5} R_0^{4/5} L^2}{4^{j/5} C_1^{2/5} L^2} \]
provided \( C_1 \) and \( C_2 \) are sufficiently large. Then
\[ s_j - s_{j+1} \leq \frac{15}{\alpha} + 6C_0^2 L^3 \frac{4\alpha}{R_0^2} 2^j \leq \frac{CL^2}{R_0^{4/5}} 4^{j/5} + \frac{CL}{R_0^{6/5}} 2^{3j/5}. \]
Summing up the geometric series, we finally obtain
\[ t_2 - t_1 = \sum_{j=1}^{N} (s_j - s_{j-1}) \leq \frac{CL^2}{R_0^{4/5}} + \frac{CL}{R_0^{6/5}}. \]
Since \( R_0 \geq CL^{5/2} \), the first term dominates, and we get \( t_2 - t_1 \leq CL^2/R_0^{4/5} \).

**Corollary 2.8.** For every \( t > 0 \), the set \( S(t)H \) is nowhere dense in \( H \), and the set \( \bigcup_{t>0} S(t)H \) is meager.

**Proof.** Let \( t > 0 \) be fixed. Then \( S(t/2)H \subseteq B(0, R_0) \) for a sufficiently large \( R_0 > 0 \), and \( S(t/2)B(0, R_0) \) is compact in \( H \).

**Remark 2.9.** Theorem 2.1 depends on the dissipativity nature of \( uu_x \), and it can thus be extended rather easily to equations of the similar form. For instance, Theorem 2.1 holds for the Burgers type equation
\[ u_t = u_{xx} + uu_x - \beta u = f \]
with periodic, mean zero boundary conditions, where \( \beta \in \mathbb{R} \) and where \( f \in L^2_{\text{per}}(\Omega) \) has mean zero.

## 3. Algebraic approximation of the global attractor

In the last section, we point out an application of Theorem 2.1 to the method of approximating the global attractor from [5]. It is well known that solutions \( u(t) \) on the global attractor can be extended to an \( H \)-valued holomorphic function \( u(z) \), defined on a strip
\[ \Pi_\delta = \{ z \in \mathbb{C} : |\Im z| < \delta \} \]
with $|u(z)| \leq M$ for $z \in \Pi_\delta$ [7,12]. The quantities $\delta > 0$ and $M > 0$ are uniform in $\mathcal{A}$ and can be explicitly computed (cf. [10,12]). Now, by a conformal change of variables

$$T = \phi(t) = \frac{\exp(\pi t/2\delta) - 1}{\exp(\pi t/2\delta) + 1}$$

we obtain

$$U(T) = u(t).$$

The function $U = U_{u_0}$ depends on $u_0$, and it can be computed for any $u_0$, whether it belongs to the global attractor or not. Based on Theorem 2.1, we have the following characterization of the global attractor: A real-analytic datum $u_0 \in H$ belongs to $\mathcal{A}$ if and only if $(U_{u_0}(T), U_{u_0}'(T))$ belongs to the Hardy space $\mathcal{H}^2$ with $\mathcal{H}^2$ norm bound $M^2$. The point is that the condition $g(T) = a_0 + a_1 T + a_2 T^2 + \cdots$ with $\|g\|_{\mathcal{H}^2} \leq M^2$ can be verified by

$$|a_0|^2 + |a_1|^2 + \cdots \leq M^2.$$ 

Note that all the Taylor coefficients of $(U(T), U'(T))$ can be explicitly computed directly from the equation by a simple recursion formula from [5, Lemma 2.2]. We note that the above construction can be modified exploring the backward blowup proven in the present paper. Namely, instead of (3.1), we can define

$$U(T) = e^{-t^2} u(t)$$

with a potential advantage of mellowing down the sharp gradients occurring in the simulations for the Lorenz system [5]. We refer the reader also to [6,8] for more on algebraic approximations of attractors.

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