Indag. Mathem., N.S., 15 (3), 383–412 September 27, 2004

Intersection cohomology of $\mathbb{S}^1$-actions on pseudomanifolds

by G. Padilla

Universidad Central de Venezuela, Escuela de Matemática, Caracas 1010, Venezuela

Communicated by Prof. J.J. Duistermaat at the meeting of October 27, 2003

ABSTRACT

For any smooth free action of the unit circle $\mathbb{S}^1$ in a manifold M; the Gysin sequence of M is a long exact sequence relating the DeRham cohomologies of M and its orbit space $M/\mathbb{S}^1$. If the action is not free then $M/\mathbb{S}^1$ is not a manifold but a stratified pseudomanifold and there is a Gysin sequence relating the DeRham cohomology of M with the intersection cohomology of $M/\mathbb{S}^1$. In this work we extend the above statements for any stratified pseudomanifold X of length 1, whenever the action of $\mathbb{S}^1$ preserves the local structure. We give a Gysin sequence relating the intersection cohomologies of X and $X/\mathbb{S}^1$ with a third term $\mathcal{G}$, the Gysin term; whose cohomology depends on basic cohomological data of two flavors: global data concerns the Euler class induced by the action, local data relates the Gysin term and the cohomology of the fixed strata with values on a locally trivial presheaf.

0. FOREWORD

A pseudomanifold is a topological space X with two features. First, there is a closed $\Sigma \subset X$ called the singular part, which is the disjoint union of smooth manifolds. The set $X - \Sigma$ is a dense smooth manifold. We call strata the connected components of Σ and $X - \Sigma$; they constitute a locally finite partition of X. The second feature is the local conical behavior of X, the model being a product $U \times c(L)$ of a smooth manifold U with the open cone of a compact smooth manifold L called the link of U. A careful reader will notice that stratified pseudomanifolds with arbitrary length

MSC: 35S35, 55N33

Key words and phrases: Intersection cohomology, Stratified pseudomanifolds

E-mail: gabrielp@euler.ciens.ucv.ve.

have a richer and more complicated topological structure; in this article we deal with stratified pseudomanifolds of length $\leqslant 1$, which we call just *pseudomanifolds*.

Between the various ways for defining the intersection (co)homology; the reader can see [6,8] for a definition in *pl*-stratified pseudomanifolds; [1,7,13] for a definition with sheaves; [15] for an approach with $\mathcal{L}^2$-cohomology; [4] for an exposition in Thom–Mather spaces.

In this article, we use the DeRham-like definition exposed in [22] where the reader will find a beautiful proof of the DeRham theorem for stratified spaces. We work with differential forms in $X - \Sigma$ and measure their behavior when approaching to Σ, trough an auxiliary construction called an *unfolding* of X. Although X may have many different unfoldings, its intersection cohomology does not depend on any particular choice. This point of view is the dual of the intersection homology defined by King [12], who works with a broader family of perversities. When $\mathbb{S}^1$ acts on X preserving the local structure then the orbit space $X/\mathbb{S}^1$ is again a pseudomanifold with an unfolding.

The well-known Gysin sequence of a smooth manifold M with a principal action of $\mathbb{S}^1$ is the long exact sequence

$$\cdots \to H^i(M) \stackrel{\oint}{\to} H^{i-1}(M/\mathbb{S}^1) \stackrel{\varepsilon}{\to} H^{i+1}(M/\mathbb{S}^1) \stackrel{\pi^*}{\to} H^{i+1}(M) \to \cdots$$

where π^* is induced by the orbit map $\pi : M \to M/\mathbb{S}^1$, which is a smooth $\mathbb{S}^1$-principal bundle. The map $\oint$ is induced by the integration along the fibers and the connecting homomorphism ε is the multiplication by the Euler class $\varepsilon \in H^2(M/\mathbb{S}^1)$.

When the action of $\mathbb{S}^1$ on M is not free then the base space is not anymore a smooth manifold, but a stratified pseudomanifold $M/\mathbb{S}^1$ whose length depends on the number of orbit types. There is a Gysin-like sequence relating the DeRham cohomology of M with the intersection cohomology of $M/\mathbb{S}^1$

$$\cdots \to H^i(M) \stackrel{\oint}{\to} H^{i-1}_{\bar{q}-\bar{2}}(M/\mathbb{S}^1) \stackrel{\varepsilon}{\to} H^{i+1}_{\bar{q}}(M/\mathbb{S}^1) \stackrel{\pi^*}{\to} H^{i+1}(M) \to \cdots$$

where $\bar{q}$, $\bar{2}$ are perversities in $M/\mathbb{S}^1$. The connecting homomorphism is again the multiplication by the Euler class $\varepsilon \in H^2_{\bar{2}}(M/\mathbb{S}^1)$. The fixed points' subspace $M^{\mathbb{S}^1}$ is naturally contained in $M/\mathbb{S}^1$. The link of a fixed stratum $S \subset M/\mathbb{S}^1$ is always a cohomological complex projective space [10,14].

In this article we extend the above situation for any pseudomanifold X and any action of $\mathbb{S}^1$ on X preserving the local structure. The orbit map $\pi : X \to X/\mathbb{S}^1$ induces a long exact sequence

$$\cdots \to H^i_{\bar{q}}(X) \to H^i(\mathcal{G}_{\bar{q}}(X/\mathbb{S}^1)) \stackrel{\partial}{\to} H^{i+1}_{\bar{q}}(X/\mathbb{S}^1) \stackrel{\pi^*}{\to} H^{i+1}_{\bar{q}}(X) \to \cdots$$

relating the intersection cohomologies of X and $X/\mathbb{S}^1$ with a third term $H^*(\mathcal{G}_{\bar{q}}(X/\mathbb{S}^1))$ whose cohomology can be given in terms of local and global basic cohomological data; we call it the *Gysin term*. The above long exact sequence is the *Gysin sequence*.

Global data concerns the Euler class $\varepsilon \in H^2_{\overline{2}}(X/\mathbb{S}^1)$. For instance, if $\varepsilon = 0$ then $H^*(\mathcal{G}_{\overline{q}}(X/\mathbb{S}^1)) = H^*_{\overline{q}-\overline{\chi}}(X/\mathbb{S}^1)$ where $\overline{\chi}$ is the perversity defined by

$$\overline{\chi}(S) = \begin{cases} 1, & S \text{ a fixed stratum,} \\ 0, & \text{else.} \end{cases}$$

The connecting homomorphism ∂ of the Gysin sequence depends on the Euler class, though it's not the multiplication. The Euler class vanishes if and only if there is a foliation on $X - \Sigma$ transverse to the orbits of the action [18,23].

Local data relates the Gysin term with the fixed strata. In general, there is a second long exact sequence

$$\cdots \to H^i_{\overline{q}-\overline{\chi}}(X/\mathbb{S}^1) \to H^i(\mathfrak{Upp}_{\overline{q}}(X/\mathbb{S}^1)) \xrightarrow{\partial'} H^{i+1}(\mathcal{G}_{\overline{q}}(X/\mathbb{S}^1)) \xrightarrow{\iota^*} H^{i+1}_{\overline{q}-\overline{\chi}}(X/\mathbb{S}^1) \to \cdots$$

the residual term satisfying

$$H^*(\mathfrak{Upp}_{\overline{q}}(X/\mathbb{S}^1)) = \prod_S H^*(S, \mathfrak{Im}(\varepsilon_L))$$

where S runs over the fixed strata and $H^*(S, \mathfrak{Im}(\varepsilon))$ is the cohomology of S with values on a locally trivial constructible presheaf [8] $\mathfrak{Im}(\varepsilon_L)$ with stalk

$$\mathcal{F} = \operatorname{Im}\{\varepsilon_L : H^{\overline{q}(S)-1}(L/\mathbb{S}^1) \to H^{\overline{q}(S)+1}(L/\mathbb{S}^1)\}$$

the image of the multiplication by the Euler class $\varepsilon_L \in H^2(L/\mathbb{S}^1)$ of the action on the Link L of S. Since L may not be a sphere, this term could not vanish.

Henceforth, when we write the word *manifold* we are talking about a smooth differential manifold of class C^∞.

1. PSEUDOMANIFOLDS

Recall the definition of unfoldable pseudomanifolds. The definitions and results of this section where taken of [1,17,22]; where the reader will find a general treatment of stratified pseudomanifolds and unfoldings.

1.1. Simple spaces. Let X be a Hausdorff, paracompact, second countable topological space. We say that X is a *simple space* if

(1) There is a closed subspace $\Sigma \subset X$, called the *singular part*; which is a disjoint union of manifolds. Its complement $X - \Sigma$ is a dense open manifold, we call it the *regular part*.
(2) A *singular* (respectively *regular*) *stratum* of X is a connected component of Σ (respectively $X - \Sigma$). The family of strata is locally finite.

For instance, every manifold is a simple space whose singular part is the empty set. If M is a manifold and X is a simple space then the product $M \times X$ is a simple space with singular part $M \times \Sigma$.

Let L be a compact manifold. The *cone of* L is the quotient space

$$c(L) = L \times [0, \infty)/L \times \{0\}.$$

We write $[p, r]$ for the equivalence class of a point (p, r). We reserve the symbol $\star$ for the *vertex* of the cone, which by definition is the equivalence class of $L \times \{0\}$. By convention we define $c(\emptyset) = \{\star\}$. The radium of the cone is the function $\rho : c(L) \to [0, \infty)$ given by $\rho[p, r] = r$. For each $\epsilon > 0$ we write $c_\epsilon(L) = \rho^{-1}[0, \epsilon)$ and $\overline{c_\epsilon}(L) = \rho^{-1}[0, \epsilon]$.

A continuous function $f : X \to X'$ between two simple spaces is a *morphism* (respectively *isomorphism*) if $f(\Sigma) \subset \Sigma'$, $f(X - \Sigma) \subset (X' - \Sigma')$ and the restriction of f to each stratum is smooth (respectively a diffeomorphism). In particular, f is an *embedding* if $f(X) \subset Y$ is an open simple space with singular part $f(X) \cap \Sigma$; and $f : X \to f(X)$ is an isomorphism.

For instance, the change of radium

$$f : c(L) \to c_\epsilon(L), \quad [p, r] \mapsto [p, \epsilon \cdot \arctan(r)/\pi]$$

is an isomorphism.

1.2. Pseudomanifolds. Let X be a simple space, S a stratum. A *chart* of S in X is an embedding

$$\alpha : U \times c(L) \to X$$

where $U \subset S$ is open in S and $\alpha(u, \star) = u$ for each $u \in U$; $c(L)$ is the cone of a compact manifold L. The singular part of $U \times c(L)$ is $U \times \{\star\}$.

We say that X is a *pseudomanifold* if for each stratum S there is a family of charts,

$$\mathcal{A}_S = \{\alpha : U_\alpha \times c(L) \to X\}_\alpha$$

such that $\{U_\alpha\}_\alpha$ is a good covering of S (cf. [3, p. 42]). Notice that the compact manifold L only depends on S, we call it a *link* of S. An *atlas* of X is the choice of such a family of charts for each stratum.

Remark that the topology of a *stratified pseudomanifold* is in general more complicated; see [8,17]. A familiarized reader will notice that we work with stratified pseudomanifolds of length 0 or 1; which we call just pseudomanifolds. Also we allow the singular strata to have codimension 1; this will be justified in the next section, when we present the definition of intersection cohomology.

For instance, any product $U \times c(L)$ of a manifold U and a cone of a compact manifold L is a pseudomanifold. Since we can adjust the size of the charts, any open subspace of a pseudomanifold is again a pseudomanifold.

1.3. Unfoldings. One way for defining the DeRham-like intersection cohomology of X is to control the behavior of differential forms on $X - \Sigma$ when approaching to Σ. This control is imposed through an *unfolding* of X; which is a manifold $\widetilde{X}$; a surjective, proper, continuous function

$$\mathcal{L}: \widetilde{X} \to X$$

and a family $\{\mathcal{L}_L : \tilde{L} \to L\}_L$ of smooth finite trivial coverings of the links of X; satisfying

(1) The open $\mathcal{L}^{-1}(X - \Sigma)$ is a union of finitely many copies of $X - \Sigma$; and the restriction of $\mathcal{L}$ to each copy is a diffeomorphism.
(2) For each singular stratum S and each $z \in \mathcal{L}^{-1}(S)$, there is an *unfoldable chart*; i.e., a commutative diagram

$$\begin{array}{ccc} U \times \tilde{L} \times \mathbb{R} & \xrightarrow{\tilde{\alpha}} & \widetilde{X} \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \mathcal{L}} \\ U \times c(L) & \xrightarrow{\alpha} & X \end{array}$$

where
(a) α is a chart.
(b) $\tilde{\alpha}$ is a diffeomorphism onto $\mathcal{L}^{-1}(\mathrm{Im}(\alpha))$.
(c) The left vertical arrow is $c(u, p, t) = (u, [\mathcal{L}_L(p), |t|])$ for each $u \in U$, $p \in \tilde{L}$, $t \in \mathbb{R}$.

We say that X is *unfoldable* when it has an unfolding.

For instance, if $\mathcal{L}: \widetilde{X} \to X$ is an unfolding then, for each link L, the covering $\mathcal{L}_L : \tilde{L} \to L$ is an unfolding of L. The product $\imath \times \mathcal{L}: M \times \widetilde{X} \to M \times X$ is an unfolding for each manifold M. The left vertical arrow in the commutative diagram (2) of Section 1.3 is an unfolding.

An *unfoldable morphism* is a commutative square

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\alpha}} & \widetilde{X}' \\ \downarrow{\scriptstyle \mathcal{L}} & & \downarrow{\scriptstyle \mathcal{L}'} \\ X & \xrightarrow{\alpha} & X' \end{array}$$

where the vertical arrows are unfoldings, α is a morphism and $\tilde{\alpha}$ is smooth.

The next result can be easily verified by using the definition of Section 1.3.

1.4. Lemma. *Let $\mathcal{L}: \widetilde{X} \to X$ be an unfolding. Then*

(1) *The restriction $\mathcal{L}: \mathcal{L}^{-1}(A) \to A$ is an unfolding for each open subset $A \subset X$.*
(2) *The restriction $\mathcal{L}: \mathcal{L}^{-1}(S) \to S$ is a smooth locally trivial fiber bundle with fiber $\tilde{L}$, for each singular stratum S with link L.*

Now we recall the definition an properties of intersection cohomology as it is exposed in [22]. Some of the results of this section were taken from [6].

2.1. Liftable forms. Fix an unfolding $\mathcal{L}: \widetilde{X} \to X$. A form $\omega \in \Omega^*(X - \Sigma)$ is *liftable* if there is a form $\tilde{\omega} \in \Omega^*(\widetilde{X})$ such that $\mathcal{L}^*(\omega) = \tilde{\omega}$ on $\mathcal{L}^{-1}(X - \Sigma)$. If $\tilde{\omega}$ does exist then it is unique by density; we call it the *lifting* of ω. If ω, η are liftable forms then $d\omega$ is also liftable and $\widetilde{d\omega} = d\tilde{\omega}$ and $\widetilde{\omega + \eta} = \tilde{\omega} + \tilde{\eta}$ and $\widetilde{\omega \wedge \eta} = \tilde{\omega} \wedge \tilde{\eta}$.

2.2. Intersection cohomology. Let $p: M \to B$ be a surjective submersion. A smooth vector field ξ in M is *vertical* if it is tangent to the fibers of p. The *perverse degree* $\|\omega\|_B$ of a differential form $\omega \in \Omega(M)$ is the first integer m such that the contraction

$$i_{\xi_0} \cdots i_{\xi_m}(\omega) = 0$$

for each vertical vector fields $\xi_0, \ldots, \xi_m$. Since contractions are antiderivatives of degree -1, for each $\omega, \nu \in \Omega(M)$

$$(1) \qquad \|\omega + \nu\|_B \leqslant \max\{\|\omega\|_B, \|\nu\|_B\}, \qquad \|\omega \wedge \nu\|_B \leqslant \|\omega\|_B + \|\nu\|_B.$$

By convention $\|0\|_B = -\infty$.

We define the DeRham-like intersection cohomology of X by means of liftable differential forms and an additional parameter which controls their behavior when approaching to Σ. This new parameter is a map $\bar{q}$ which sends each singular stratum S to an integer $\bar{q}(S) \in \mathbb{Z}$; we call it a *perversity*. For instance, given an integer $n \in \mathbb{Z}$ we denote by $\bar{n}$ the constant perversity assigning n to any singular stratum. Another example is the top perversity defined by $\bar{t}(S) = \operatorname{codim}(S) - 2$ on each singular stratum S.

Fix a perversity $\bar{q}$. A *$\bar{q}$-form* on X is a liftable form ω on $X - \Sigma$ satisfying

$$\max\{\|\omega\|_S, \|d\omega\|_S\} \leqslant \bar{q}(S) \quad \forall S \text{ singular stratum}$$

where, with a little abuse of language, we denote by $\|\omega\|_S$ the perverse degree of the restriction $\tilde{\omega}|_{\mathcal{L}^{-1}(S)}$ with respect to the submersion $\mathcal{L}: \mathcal{L}^{-1}(S) \to S$. The $\bar{q}$-forms define a differential subcomplex $\Omega^*_{\bar{q}}(X)$ whose cohomology $H^*_{\bar{q}}(X)$ is the *$\bar{q}$-intersection cohomology* of X.

2.3. Topological invariance of intersection cohomology. When Goresky and MacPherson defined the intersection (co)homology on stratified pseudomanifolds for the first time, they showed that it does not depend on the choice of a particular stratification, provided that there were no strata of codimension 1. This is quite natural since originally a perversity was a parameter depending on the dimensions of the singular strata, but not on the strata themselves.

By the other hand, the works of King and Saralegi enlarged the family of allowed perversities (see [12,22]). In Section 2.2 a perversity is an arbitrary integer-valued function defined on the family of singular strata, thus depending on the stratification. Nevertheless, if we fix a stratification of X and then we declare new artificial strata by decomposing the regular part $X - \Sigma$; then the intersection (co)homology does not change. This happens because the link of an artificial stratum is a (co)homological sphere. In general, we have the following invariance properties:

(a) $H^*_{\bar{q}}(X)$ does not depend on the particular choice of an unfolding, for any perversity $\bar{q}$.
(b) If $\bar{q} > \bar{t}$ then $H^*_{\bar{q}}(X) = H^*(X - \Sigma)$ is the DeRham cohomology of $X - \Sigma$.
(c) If $\bar{q} < \bar{0}$ then $H^*_{\bar{q}}(X) = H^*(X, \Sigma)$ is the relative cohomology of the pair.
(d) If X is a manifold then $H^*_{\bar{q}}(X)$ coincides with the DeRham cohomology $H^*(X)$, for any perversity $\bar{0} \leqslant \bar{q} \leqslant \bar{t}$.
(e) A *controlled* form is a $\bar{0}$-form. The $\bar{0}$-intersection cohomology $H_{\bar{0}}(X)$ is a differential graded algebra and $H^*_{\bar{q}}(X)$ is an $H_{\bar{0}}(X)$-module for any perversity $\bar{q}$, see Eq. (1). The $\bar{0}$-intersection cohomology $H^*_{\bar{0}}(X)$ coincides with the singular cohomology $H^*(X^N)$ of the normalization X^N of X. For a brief introduction to normalizations the reader can see [8,16]; we will give more details in Appendix A.

3. MODELLED ACTIONS

We introduce the family of modelled actions, whose main property is that the orbit spaces always remain in the category of unfoldable pseudomanifolds. Henceforth, we denote by $\mathbb{S}^1$ the unit circle. We fix a pseudomanifold X and a continuous effective action

$$\Phi : \mathbb{S}^1 \times X \to X.$$

We will write $\Phi(g, x) = gx$, $B = X/\mathbb{S}^1$ for the orbit space and $\pi : X \to B$ for the orbit map.

3.1. Modelled actions. We say that Φ is a *modelled action* whenever it satisfies conditions MA(1), MA(2), MA(3) and MA(4) stated below. First notice that $\mathbb{S}^1 \times X$ is a pseudomanifold with singular part $\mathbb{S}^1 \times \Sigma$.

MA(1). *The action $\Phi : \mathbb{S}^1 \times X \to X$ is a morphism.*

In consequence, for each $g \in \mathbb{S}^1$ the function $\Phi_g : X \to X$ is an isomorphism. Notice that each stratum is $\mathbb{S}^1$-invariant.

MA(2). *For each stratum S of X the points in S have the same isotropy subgroup H_S. In particular, the action on $X - \Sigma$ is free.*

So the restriction $\pi : S \to \pi(S)$ is a smooth locally trivial fibre bundle with fiber $\mathbb{S}^1/H_S$, for each stratum S of X.

An *equivariant unfolding* of X is an unfolding $\mathcal{L} : \widetilde{X} \to X$ such that there is a smooth free action $\widetilde{\Phi} : \mathbb{S}^1 \times \widetilde{X} \to \widetilde{X}$ and the function $\mathcal{L}$ is $\mathbb{S}^1$-equivariant.

MA(3). *There is an equivariant unfolding $\mathcal{L} : \widetilde{X} \to X$.*

When $\Sigma = \emptyset$ the above condition is trivial. For each stratum S the restriction $\mathcal{L}:\mathcal{L}^{-1}(S) \to S$ is an equivariant locally trivial fiber bundle; so there is a smooth free action of the isotropy H_S on the covering $\tilde{L}$ of the link L and an H_S-equivariant trivializing atlas of the fiber bundle.

Now we describe the local behavior of the action near the singular part. Take a singular stratum S with link L. A *modelled chart* is an unfoldable chart

$$\begin{array}{ccc} U \times \tilde{L} \times \mathbb{R} & \xrightarrow{\tilde{\alpha}} & \widetilde{X} \\ \big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle \mathcal{L}} \\ U \times c(L) & \xrightarrow{\alpha} & X \end{array}$$

satisfying

(a) The above diagram is H_S-equivariant; i.e., for each $g \in H_S$, $u \in U$, $p \in \tilde{L}$ and $t \in \mathbb{R}$

$$\mathcal{L}\tilde{\alpha}(u, gp, t) = g\alpha\big(u, [p, |t|]\big).$$

(b) For each $u \in U$, $g \in \mathbb{S}^1$; if $\Phi_g(\alpha(\{u\} \times c(L))) \cap \operatorname{Im}(\alpha) \neq \emptyset$ then the arrow

$$\alpha^{-1}\Phi_g\alpha|_u : \{u\} \times c(L) \to \{gu\} \times c(L)$$

is an isomorphism and commutes with the radium $\rho : U \times c(L) \to [0, \infty)$.

MA(4). *For each singular stratum S there is a smooth free action $\Psi_S : H_S \times L \to L$ of the isotropy of S on its link, such that S is covered by H_S-modelled charts.*

3.2. Some examples of modelled actions.

(1) If $G \times X \to X$ is a modelled action on a pseudomanifold X then, for each manifold M, the induced action on $M \times X$ which is trivial in the factor M is a modelled action.
(2) Each free smooth action $G \times L \to L$ of a compact Lie group G on a compact manifold L, induces on $c(L)$ a modelled action given by the rule

$$G \times c(L) \to c(L), \quad g[p, r] = [gp, r].$$

(3) A Thom–Mather equivariant space is a Thom–Mather space X together with a compact Lie group G and an effective action $G \times X \to X$ preserving the tubular neighborhoods; i.e., each tubular neighborhood is a G-equivariant locally trivial fiber bundle with a suitable family of cocycles (see [17,24]). Indeed, the definition of a modelled chart is inspired in the behavior of these cocycles. When X is a pseudomanifold then the action of G on X is modelled.
(4) For each manifold M and each smooth effective action $\mathbb{S}^1 \times M \to M$, the decomposition of M in orbit types induces a Thom–Mather equivariant structure

in M. When the action has locally few orbit types then M is a pseudomanifold and the action is modelled.

(5) The following is an example of an iterated modelled toric action on a manifold M (see [5]). Let $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-torus, M a manifold and $\mathbb{T} \times M \to M$ a smooth effective action with fixed points and with locally few orbit types. By the above examples this action is modelled, the restriction of the action to the first factor $\mathbb{S}^1$ is a modelled action. As we will see immediately in Section 3.3, the orbit space $X_1 = M/\mathbb{S}^1$ is a pseudomanifold. There is a natural action of the second factor $\mathbb{S}^1$ on X_1 which preserves the strata, this action is again modelled and the orbit space $X_2 = X_1/\mathbb{S}^1$ is again a pseudomanifold.

Now we use conditions MA(1), ..., MA(4) in order to describe the orbit space.

3.3. Proposition. *Let $\Phi : \mathbb{S}^1 \times X \to X$ be a modelled action. Then the orbit space $B = X/\mathbb{S}^1$ is a pseudomanifold and the induced map*;

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\mathcal{L}} & X \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \widetilde{B} = \widetilde{X}/\mathbb{S}^1 & \xrightarrow{\mathcal{L}_B} & B \end{array} \qquad \mathcal{L}_B\big(\tilde{\pi}(x)\big) = \pi\big(\mathcal{L}(x)\big)$$

is an unfolding.

Proof. The orbit map $\pi : X \to B$ is open and closed, so B is also a Hausdorff, paracompact, 2nd countable space. Conditions MA(1) and MA(2) imply that B is a simple space and π is a morphism. We proceed in two steps.

- *B is a pseudomanifold*: Take a modelled chart

$$\alpha : U \times c(L) \to X$$

whose existence is guaranteed by MA(4). Assume that $U = WV$ where $W \subset \mathbb{S}^1$ is a contractible open neighborhood of $1 \in \mathbb{S}^1$, V a slice in the stratum S containing U. Write $\pi_L : L \to L/H_S$ for the orbit map. Since α is H_S-equivariant, V is a slice and the transformations of $\mathbb{S}^1$ preserve the radium of α; the function

$$\beta : V \times c(L/H_S) \to B, \quad \beta\big(y, [\pi_L(p), r]\big) = \pi\alpha\big(y, [p, r]\big)$$

is well defined. More over

(a) *β is an homeomorphism*: Since B and L/H_S have their respective quotient topologies, β is a continuous function. But any continuous bijection from a compact space onto a Hausdorff space is an homeomorphism, so it is enough to reduce the domain of β in a convenient way.

(b) *β preserves diffeomorphically each stratum*: Since V is a slice, on the singular part the function $\beta: V \times \{\star\} \to \pi(V)$ is a diffeomorphism. Restricted to $V \times L \times \mathbb{R}^+$ we obtain the following commutative square

$$\begin{array}{ccc} V \times L \times \mathbb{R}^+ & \xrightarrow{\alpha} & S' \\ \downarrow{\scriptstyle 1\times\pi_L} & & \downarrow{\scriptstyle \pi} \\ V \times L/H_S \times \mathbb{R}^+ & \xrightarrow{\beta} & \pi(S') \end{array}$$

where S' is the stratum containing $\alpha(V \times L \times \mathbb{R}^+)$. The vertical arrows are submersions and α is smooth, then so is α'. The same argument can be applied to the inverse β^{-1}.

• *$\mathcal{L}_B : \widetilde{B} \to B$ is an unfolding*: The function $\mathcal{L}_B$ is well defined because $\mathcal{L}$ is equivariant. If $\Sigma = \emptyset$ the proof is immediate, because on each connected component of X the map $\mathcal{L}$ is a trivial covering. Assume that $\Sigma \neq \emptyset$. By the above remark, $\mathcal{L}_B$ satisfies (1) of Section 1.3. We verify the existence of unfoldable charts. Take a singular stratum S with link L. Since (again) L has no singular part, the H_S-equivariant unfolding $\mathcal{L}_L : \tilde{L} \to L$ induces an unfolding $\mathcal{L}_{L/H_S} : \tilde{L}/H_S \to L/H_S$.

Let $\tilde{\alpha} : U \times \tilde{L} \times \mathbb{R} \to \widetilde{X}$ be the unfolding of the modelled chart α given before. Define

$$\tilde{\beta} : V \times \tilde{L}/H_S \times \mathbb{R} \to \tilde{\pi}\big(\mathrm{Im}(\tilde{\alpha})\big), \quad \tilde{\beta}\big(y, \tilde{\pi}_L(\tilde{p}), t\big) = \tilde{\pi}\tilde{\alpha}(y, \tilde{p}, t).$$

Then $\tilde{\beta}$ is an unfolding of the chart $\beta : V \times c(L/H_S) \to B$ induced in the first step of this proof. We leave the details to the reader. □

4. INVARIANT FORMS

Now we display the algebraic tools involved on modelled circle actions. Some results of this section where taken of [10,14]; these references deal with smooth non-free circle actions on manifolds, but the same proofs still hold in our context. From now on, we fix a pseudomanifold X, a modelled action

$$\Phi : \mathbb{S}^1 \times X \to X$$

and an equivariant unfolding $\mathcal{L} : \widetilde{X} \to X$ with modelled charts.

4.1. Invariant cohomology. A $\bar{q}$-form ω on X is *invariant* if for each $g \in \mathbb{S}^1$ the equation $g^*(\omega) = \omega$ holds. Since

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\Phi}_g} & \widetilde{X} \\ \downarrow{\scriptstyle \mathcal{L}} & & \downarrow{\scriptstyle \mathcal{L}} \\ X & \xrightarrow{\Phi_g} & X \end{array}$$

is an unfoldable isomorphism, $g^*: \Omega^*_{\bar q}(X) \to \Omega^*_{\bar q}(X)$ is an isomorphism of differential complexes. Invariant $\bar q$-forms define a differential complex, denoted $\mathrm{I}\Omega^*_{\bar q}(X)$. The inclusion

$$\iota : \mathrm{I}\Omega^*_{\bar q}(X) \to \Omega^*_{\bar q}(X)$$

induces an isomorphism in cohomology.

Next we will study the algebraic decomposition of an invariant $\bar q$-form; this decomposition depends on the existence of a Riemannian metric and a connection form compatible with the equivariant unfolding. All this will be useful for later purposes, when we have to write the Gysin sequence of X.

The *fundamental vector field* on X is the smooth vector field C defined on $X - \Sigma$ by the rule

$$C_x = d\Phi_x\left(\frac{\partial}{\partial g}\right)\bigg|_{g=1}.$$

In other words, C is the smooth vector field tangent to the orbits of the action. It never vanishes because $X - \Sigma$ has no fixed points. The lifted action $\widetilde\Phi : \mathbb{S}^1 \times \widetilde X \to \widetilde X$ defines a fundamental vector field $\widetilde C$ on $\widetilde X$.

We establish a little convention in order to classify the strata of X: A stratum S is *mobile* (respectively *fixed*) if $H_S \neq \mathbb{S}^1$ (respectively $H_S = \mathbb{S}^1$). For a proof of the next result the reader can see [10]; although it deals with smooth effective actions on manifolds, the proof is still valid in our context.

4.2. Lemma. *There are Riemannian metrics μ, $\tilde\mu$ respectively on $X - \Sigma$, $\widetilde X$; satisfying*

(1) *μ and $\tilde\mu$ are $\mathbb{S}^1$-invariant.*
(2) *$\mathcal{L}^*(\mu) = \tilde\mu$ on $\mathcal{L}^{-1}(X - \Sigma)$.*
(3) *$\mu\langle C, C\rangle = \tilde\mu\langle \widetilde C, \widetilde C\rangle = 1$.*
(4) *For each mobile stratum S and each vertical vector field v respectively the submersion $\mathcal{L}^{-1}(S) \xrightarrow{\mathcal{L}} S$; the following equation holds: $\tilde\mu\langle \widetilde C, v\rangle = 0$.*

Let μ be an invariant Riemannian metric on $X - \Sigma$. The *fundamental form* induced by μ is the 1-form χ defined by the rule $\chi(v) = \mu\langle C, v\rangle$. By the other hand, we will say that μ is *unfoldable* if there is a Riemannian metric $\tilde\mu$ in $\widetilde X$ satisfying Lemma 4.2. In that case $\tilde\mu$ is unique by density and, by Lemma 4.2(2), the fundamental form χ on X lifts to the fundamental form $\widetilde\chi = \iota_{\widetilde C}(\tilde\mu)$ on $\widetilde X$.

4.3. Lemma. *For each unfoldable metric μ in X the fundamental form χ satisfies*

$$\|\chi\|_S = \begin{cases} 1, & S \text{ a fixed stratum,} \\ 0, & S \text{ a mobile stratum.} \end{cases}$$

Proof. The equation arises directly from (3) and (4) of Lemma 4.2. □

Henceforth, we fix an unfoldable metric μ.

4.4. Decomposition of an invariant form. A form η on $X - \Sigma$ is *basic* if one of the following equivalent statements holds:

(a) η is invariant and $\iota_C(\eta) = 0$. Notice that

$$0 = L_C(\eta) = d\iota_C(\eta) + \iota_C d(\eta) = \iota_C d(\eta)$$

where L_C is the Lie derivative with respect to the fundamental vector field.

(b) $\eta = \pi^*(\xi)$ for some differential form θ on $B - \Sigma = \pi(X - \Sigma)$. If there is such a ξ then it is unique, because π^* is injective.

For each invariant form $\omega \in \mathrm{I}\Omega^*(X - \Sigma)$ there are $\nu \in \Omega^*(B - \Sigma)$ and $\theta \in \Omega^{*-1}(B - \Sigma)$ satisfying

$$\omega = \pi^*(\nu) + \chi \wedge \pi^*(\theta).$$

The above expression is the *decomposition* of ω. The forms ν, θ are uniquely determined by the following equations

$$\pi^*(\theta) = \iota_C(\omega), \quad \pi^*(\nu) = \omega - \chi \wedge \iota_C(\omega).$$

When ω is a liftable form then

$$\tilde{\omega} = \tilde{\pi}^*(\tilde{\nu}) + \tilde{\chi} \wedge \tilde{\pi}^*(\tilde{\theta}).$$

So ν, θ lift respectively to $\tilde{\nu}$, $\tilde{\theta}$.

4.5. The Gysin sequence of a free smooth action. Assume that $\Sigma = \emptyset$. Then X is a manifold, $\Phi : \mathbb{S}^1 \times X \to X$ is a free smooth action and $\pi : X \to B$ is a smooth $\mathbb{S}^1$-principal fiber bundle. There is a morphism of integration along the orbits

$$\oint = (-1)^{i-1}\pi^{-*}\iota_C : \mathrm{I}\Omega^i(X) \to \Omega^{i-1}(B)$$

defined by

$$\oint \omega = (-1)^{i-1}\theta, \quad \omega = \pi^*(\nu) + \chi \wedge \pi^*(\theta) \in \mathrm{I}\Omega^i(X).$$

We obtain a short exact sequence

$$0 \to \Omega^*(B) \xrightarrow{\pi^*} \mathrm{I}\Omega^*(X) \xrightarrow{\oint} \Omega^{*-1}(B) \to 0.$$

The induced long exact sequence

$$(2)\qquad \cdots \to H^i(X) \xrightarrow{\oint^*} H^{i-1}(B) \xrightarrow{\varepsilon} H^{i+1}(B) \xrightarrow{\pi^*} H^{i+1}(X) \to \cdots$$

is the *Gysin sequence* of X by the action Φ. It does not depend on the particular choice of a metric μ. The form $d\chi$ is basic, so there is a unique $e \in \Omega^2(B)$ such that

$$d\chi = \pi^*(e).$$

This e is the *Euler form* induced by the action Φ and the metric μ. The cohomology class $\varepsilon = [e] \in H^2(B)$ is the *Euler class* of B. The connecting homomorphism ε of the Gysin sequence (2) is the multiplication by the Euler class $\varepsilon = [e] \in H^2(B)$. The Euler class ε does not depend on the particular choice of a metric μ. Following [9,23] we get the equivalent propositions

(a) The Euler class $\varepsilon \in H^2(B)$ vanishes.
(b) $H(X) = H(B) \otimes H(\mathbb{S}^1)$; then we say that X is a *cohomological product* in the DeRham cohomology.
(c) There is a foliation $\mathcal{F}$ on X transverse to the orbits of the action.

Now we return to the stratified case. Since μ is an unfoldable metric, the Euler form e on $B - \Sigma$ can be lifted to the Euler form $\tilde{e}$ on $\widetilde{B}$ induced by the metric $\tilde{\mu}$. Notice that $e \in \Omega^2_{\bar{2}}(B)$.

4.6. Proposition. *The Euler class $\varepsilon \in H^2_{\bar{2}}(B)$ vanishes if and only if there is a foliation $\mathcal{F}$ on $X - \Sigma$ transverse to the orbits of the action.*

Proof. By the above equivalences (a), (b) and (c) applied to $X - \Sigma$; it is enough to verify that the Euler class vanishes in $H^2_{\bar{2}}(B)$ if and only if it vanishes in $H^2(B - \Sigma)$. By (b) of Section 2.3, there is an isomorphism

$$H^2_{\bar{t}+\bar{1}}(B) \to H^2(B - \Sigma)$$

induced by the inclusion of the respective complexes. So it is enough to verify that the Euler class vanishes in $H^2_{\bar{2}}(B)$ if and only if it vanishes in $H^2_{\bar{t}+\bar{1}}(B)$.

Take a representative $e \in \Omega^2_{\bar{t}+\bar{1}}(B)$ of the Euler class and suppose that $e = d\theta$ for some $\theta \in \Omega^1_{\bar{t}+\bar{1}}(B)$. Since the perverse degree of a form is lower or equal to its usual degree, $\theta \in \Omega^1_{\bar{2}}(B)$ and e is a border in $\Omega^1_{\bar{2}}(B)$. This proves one implication, the converse is trivial. □

We finish this section with a description of the perverse degree of the invariant forms. For a proof of the following lemma the reader can see [10].

4.7. Lemma. *Take a perversity $\bar{q}$ on X, write also $\bar{q}$ for the perversity induced on B in the obvious way. Then the arrow*

$$\pi^* : \Omega^*_{\bar{q}}(B) \to \mathrm{I}\Omega^*_{\bar{q}}(X)$$

is well defined. What's more, for each invariant form $\omega = \pi^*(\nu) + \chi \wedge \pi^*(\theta)$ and each singular stratum S, we have

$$\|\omega\|_S = \max\{\|\nu\|_{\pi(S)}, \|\chi\|_S + \|\theta\|_{\pi(S)}\}.$$

Now we refine our classification of the fixed strata: A fixed stratum S is *perverse* if and only if its link is not a cohomological product.

4.8. Proposition. *There is an unfoldable metric μ such that, for each singular stratum S with link L, the Euler form satisfies*

- $\|e\|_{\pi(S)} \leqslant 0$ *if S is a mobile stratum.*
- $\|e\|_{\pi(S)} \leqslant 1$ *if S is a fixed non perverse stratum.*
- $\|e\|_{\pi(S)} = 2$ *if S is a perverse stratum.*

Proof. See Appendix A. □

5. THE GYSIN SEQUENCE

Given a modelled action $\Phi : \mathbb{S}^1 \times X \to X$ we want to know the cohomological relationship between X and B. The answer is a long exact sequence relating the intersection cohomologies of X, B with a third algebraic complex; we call it the *Gysin sequence* of X. The third complex is the *Gysin term*, whose cohomology depends on B plus some data on the perverse strata.

As we have seen, if $\Sigma = \emptyset$ then we get the Gysin sequence by integrating along the fibers. If X is a manifold and Φ is a smooth effective action with fixed points; then B is not a manifold but a stratified pseudomanifold. There is a Gysin sequence relating the DeRham cohomology of X with the intersection cohomology of B [10]. Something analogous happens for smooth actions of $\mathbb{S}^3$ and $\mathbb{T}^n$ with few local orbit types [21,20], and for Riemannian flows [19].

5.1. The Gysin sequence. Fix a modelled action $\Phi : \mathbb{S}^1 \times X \to X$. Take a perversity $\bar{q}$ in X, write also $\bar{q}$ for the perversity induced on B in the obvious way. The *$\bar{q}$-Gysin term* is the cokernel

$$0 \to \Omega^{*+1}_{\bar{q}}(B) \xrightarrow{\pi^*} \mathrm{I}\Omega^{*+1}_{\bar{q}}(X) \xrightarrow{pr} \mathcal{G}^*_{\bar{q}}(B) \to 0.$$

The induced long exact sequence

$$\cdots \to H^{i+1}_{\bar{q}}(X) \xrightarrow{pr} H^i\big(\mathcal{G}_{\bar{q}}(B)\big) \xrightarrow{\partial} H^{i+2}_{\bar{q}}(B) \xrightarrow{\pi^*} H^{i+2}_{\bar{q}}(X) \to \cdots \tag{3}$$

is the *Gysin sequence* of X. By Section 2.3, when $\bar{q} > \bar{t}$ the sequence (3) is the usual Gysin sequence (2) of $X - \Sigma$; when $\bar{q} < \bar{0}$ it is the Gysin sequence of the pair (X, Σ). Looking at the free smooth case, the reader could think that $H^*(\mathcal{G}_{\bar{q}}(B))$ coincides with the intersection cohomology of B and the connecting homomorphism is the multiplication by the Euler class. As we will see, this is just a naive conjecture. The real situation is more complicated. Define on B the *fundamental perversity*

$$\bar{\chi}(\pi(S)) = \|\chi\|_S = \begin{cases} 1, & S \text{ a fixed stratum,} \\ 0, & \text{else} \end{cases}$$

and the *Euler perversity*

$$\bar{e}(\pi(S)) = \begin{cases} 0, & S \text{ is a mobile stratum,} \\ 1, & S \text{ is a fixed non perverse stratum,} \\ 2, & S \text{ is a perverse stratum} \end{cases}$$

(cf. Lemma 4.3, Proposition 4.8). Notice that, by definition $\chi \in \Omega^1_{\bar{\chi}}(B)$ while $e \in \Omega^2_{\bar{e}}(B)$.

The Gysin term can be written by means of basic differential forms.

5.2. Lemma. *For each perversity* $\bar{0} \leqslant \bar{q} \leqslant \bar{t}$ *there is a differential isomorphism*

$$\mathcal{G}^*_{\bar{q}}(B) \cong \big\{\theta \in \Omega^*_{\bar{q}-\bar{\chi}}(B) / \exists \nu \in \Omega^*(B - \Sigma)\text{: (1) } \nu \textit{ is liftable;} \\ \text{(2) } \max\big\{\|\nu\|_S, \|d\nu + e \wedge \theta\|_S\big\} \leqslant \bar{q}(S)\ \forall S \textit{ perverse stratum}\big\}.$$

Under this identification, the connecting homomorphism is

$$\partial : H^i(\mathcal{G}_{\bar{q}}(B)) \to H^{i+2}_{\bar{q}}(B), \quad \partial[\theta] = [d\nu + e \wedge \theta].$$

Proof. By definition

$$\mathcal{G}^*_{\bar{q}}(B) = \frac{\mathrm{I}\Omega^{*+1}_{\bar{q}}(X)}{\pi^*(\Omega^{*+1}_{\bar{q}}(B))}$$

is a quotient complex with differential operator $\bar{d}(\bar{\omega}) = \overline{d\omega}$, where $\bar{\omega}$ is the equivalence class of a differential form $\omega \in \mathrm{I}\Omega^*_{\bar{q}}(X)$. Take an invariant form $\omega = \pi^*\nu + \chi \wedge \pi^*\theta$. Then

$$\bar{\omega} = \overline{\chi \wedge \pi^*(\theta)}, \quad \overline{d\omega} = \overline{\chi \wedge \pi^*(-d\theta)}.$$

The function

$$f : \mathcal{G}^*_{\bar{q}}(B) \to \Omega^*_{\bar{q}-\bar{\chi}}(B), \quad \bar{\omega} \mapsto \theta$$

is well defined and injective and linear. The right term appearing in our statement is the image $\mathrm{Im}(f)$. Since $df = -f\bar{d}$, the map f is an isomorphism (modulo a sign) and induces an isomorphism in cohomology. The connecting homomorphism arises as usual from the Snake's lemma. □

The above lemma allows us to calculate directly $H^*(\mathcal{G}_{\bar{q}}(B))$ when X has no perverse strata. In that case, the cohomology of the Gysin term is closer of our naive conjecture.

5.3. Proposition. *If X has no perverse strata then the Euler class ε is in $H^2_{\bar{e}}(B)$ and, for each perversity $\bar{0} \leqslant \bar{q} \leqslant \bar{t}$, the Gysin sequence* (3) *becomes*

$$\cdots \to H^{i+1}_{\bar{q}}(X) \stackrel{pr}{\to} H^{i}_{\bar{q}-\bar{e}}(B) \stackrel{\varepsilon}{\to} H^{i+2}_{\bar{q}}(B) \stackrel{\pi^*}{\to} H^{i+2}_{\bar{q}}(X) \to \cdots$$

where the connecting homomorphism is the multiplication by the Euler class. If additionally X has no fixed strata then $\varepsilon \in H^2_{\bar{0}}(B)$ and then the above sequence becomes

$$\cdots \to H^{i+1}_{\bar{q}}(X) \stackrel{pr}{\to} H^{i}_{\bar{q}}(B) \stackrel{\varepsilon}{\to} H^{i+2}_{\bar{q}}(B) \stackrel{\pi^*}{\to} H^{i+2}_{\bar{q}}(X) \to \cdots.$$

Proof. By Lemma 5.2, the Gysin term is an intermediate complex

$$(4) \qquad \Omega^*_{\bar{q}-\bar{e}}(B) \subset \mathcal{G}^*_{\bar{q}}(B) \subset \Omega^*_{\bar{q}-\bar{\chi}}(B).$$

If X has no perverse strata then $\bar{\chi} = \bar{e}$ and the extremes in the above inequality are equal; so $\mathcal{G}^*_{\bar{q}}(B) = \Omega^*_{\bar{q}-\bar{e}}(B)$. The remark about the connecting homomorphism is immediate and the second statements are straightforward. □

5.4. Corollary. *If the Euler class $\varepsilon \in H^2_{\bar{e}}(B)$ vanishes then, for each perversity $\bar{0} \leqslant \bar{q} \leqslant \bar{t}$;*

$$H^*_{\bar{q}}(X) = H^*_{\bar{q}}(B) \oplus H^{*-1}_{\bar{q}-\bar{e}}(B).$$

If additionally X has no fixed strata, then

$$H_{\bar{q}}(X) = H_{\bar{q}}(B) \otimes H(\mathbb{S}^1),$$

i.e., X is a cohomological product for intersection cohomology.

Proof. If $\varepsilon \in H^2_{\bar{e}}(B)$ vanishes then X has no perverse strata. □

5.5. Residual approximations. In the rest of this work we will calculate $H^*(\mathcal{G}_{\bar{q}}(B))$ for a modelled action with perverse strata. For this sake, we introduce the residual approximations of the Gysin term. These are the quotient complexes induced by the inequality (4);

$$0 \to \Omega^*_{\bar{q}-\bar{e}}(B) \to \mathcal{G}^*_{\bar{q}}(B)B \stackrel{pr}{\to} \mathfrak{Low}^*_{\bar{q}}(B) \to 0,$$
$$0 \to \mathcal{G}^*_{\bar{q}}(B)B \to \Omega^*_{\bar{q}-\bar{\chi}}(B) \stackrel{pr}{\to} \mathfrak{Upp}^*_{\bar{q}}(B) \to 0.$$

We call $\mathfrak{Low}^*_{\bar q}(B)$ (respectively $\mathfrak{Upp}^*_{\bar q}(B)$) the *lower residue* (respectively *upper residue*). The induced long exact sequences

$$(5)\qquad \cdots \to H^i_{\bar q-\bar e}(B) \to H^i\big(\mathcal{G}_{\bar q}(B)\big) \xrightarrow{pr} H^i\big(\mathfrak{Low}_{\bar q}(B)\big) \to H^{i+1}_{\bar q-\bar e}(B) \to \cdots,$$

$$(6)\qquad \cdots \to H^i\big(\mathcal{G}_{\bar q}(B)\big) \to H^i_{\bar q-\bar\chi}(B) \xrightarrow{pr} H^i\big(\mathfrak{Upp}_{\bar q}(B)\big) \to H^{i+1}\big(\mathcal{G}_{\bar q}(B)\big) \to \cdots$$

are the *residual approximations*. As we will see in Section 6, $\mathfrak{Low}^*_{\bar q}(B)$, $\mathfrak{Upp}^*_{\bar q}(B)$ are local terms. Next consider the cokernel

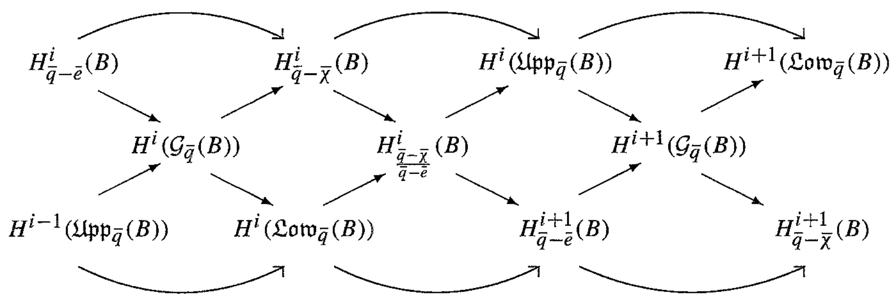
$$0 \to \Omega^*_{\bar q-\bar e}(B) \hookrightarrow \Omega^*_{\bar q-\bar\chi}(B) \xrightarrow{pr} \Omega^*_{\frac{\bar q-\bar\chi}{\bar q-\bar e}}(B) \to 0$$

its cohomology $H^*_{\frac{\bar q-\bar\chi}{\bar q-\bar e}}(B)$ is called the *step intersection cohomology* of B [11]. The residual approximations are related by the long exact sequences

$$\cdots \to H^i_{\bar q-\bar e}(B) \to H^i_{\bar q-\bar\chi}(B) \to H^i_{\frac{\bar q-\bar\chi}{\bar q-\bar e}}(B) \to H^{i+1}_{\bar q-\bar e}(B) \to \cdots,$$

$$\cdots \to H^i\big(\mathfrak{Low}_{\bar q}(B)\big) \to H^i_{\frac{\bar q-\bar\chi}{\bar q-\bar e}}(B) \to H^i\big(\mathfrak{Upp}_{\bar q}(B)\big) \to H^i\big(\mathfrak{Low}_{\bar q}(B)\big) \to \cdots.$$

These sequences can be arranged in a commutative exact diagram; called the *Gysin braid*



5.6. Remark. The cohomology complexes in this section are $H_{\bar 0}(B)$-modules. For instance, the linear action of $H^*_{\bar 0}(B)$ in $H_{\bar q}(X)$ is given by the rule

$$H^*_{\bar 0}(B) \times H_{\bar q}(X) \to H_{\bar q}(X), \quad (\theta, \omega) \mapsto \pi^*(\theta) \wedge \omega.$$

The reader is invited to check that also the arrows are morphisms of $H^*_{\bar 0}(B)$-modules.

6. LOCAL CALCULATIONS

In this section we present the local properties of the Gysin term and the residues. Some results were taken from [22].

An introduction to presheaves and the Cech cohomology can be found for instance in [3]. A presheaf $\mathcal{P}$ on X is *complete* if for any open cover $\mathcal{U} = \{X_\alpha\}_\alpha$ of X the augmented Cech differential complex

$$0 \to \mathcal{P}(X) \xrightarrow{\delta} C^0(\mathcal{U},\mathcal{P}) \xrightarrow{\delta} C^1(\mathcal{U},\mathcal{P}) \xrightarrow{\delta} C^2(\mathcal{U},\mathcal{P}) \xrightarrow{\delta} \cdots$$

is exact, where $C^j(\mathcal{U},\mathcal{P}) = \prod_{\alpha_0<\cdots<\alpha_j} \mathcal{P}(X_{\alpha_0} \cap \cdots \cap X_{\alpha_j})$ and δ is given coordinatewise by the alternating sum of the restrictions.

For each perversity $\bar{q}$, the complex of $\bar{q}$-forms $\Omega^*_{\bar{q}}(-)$ is a presheaf on X (and also on B). The complex $\mathrm{I}\Omega^*_{\bar{q}}(-)$ of invariant $\bar{q}$-forms is a presheaf on X but it is defined in the topology of invariant open sets; and $\mathcal{G}^*_{\bar{q}}(-)$, $\mathfrak{Low}^*_{\bar{q}}(-)$ and $\mathfrak{Upp}^*_{\bar{q}}(-)$ are presheaves on B.

6.1. Lemma. *The presheaves* $\Omega^*_{\bar{q}}(-)$, $\mathrm{I}\Omega^*_{\bar{q}}(-)$, $\mathcal{G}^*_{\bar{q}}(-)$, $\mathfrak{Low}^*_{\bar{q}}(-)$ *and* $\mathfrak{Upp}^*_{\bar{q}}(-)$ *are complete.*

Proof. The presheaf $\Omega^*_{\bar{q}}(-)$ (respectively $\mathrm{I}\Omega^*_{\bar{q}}(-)$) is complete because for each open cover $\mathcal{U}$ of X there is a controlled (respectively also invariant) partition of the unity subordinated to $\mathcal{U}$. A proof for can be seen in [3, p. 94]. The Gysin term and the residues are complete presheaves because they are quotients of complete presheaves. □

Now we calculate the cohomology of a product $U \times c(L)$ with values on those presheaves.

6.2. Lemma. *Let* $\Phi : \mathbb{S}^1 \times X \to X$ *be a modelled actions. Consider on* $\mathbb{R} \times X$ *the induced modelled action which is trivial in the* $\mathbb{R}$ *factor. Then the projection* $pr : \mathbb{R} \times X \to X$ *induces the following isomorphisms*

$$\begin{aligned} &H^i_{\bar{q}}(\mathbb{R} \times X) = H^i_{\bar{q}}(X), && H^i\big(\mathcal{G}_{\bar{q}}(\mathbb{R} \times B)\big) = H^i\big(\mathcal{G}_{\bar{q}}(B)\big), \\ &H^i\big(\mathfrak{Low}_{\bar{q}}(\mathbb{R} \times B)\big) = H^i\big(\mathfrak{Low}_{\bar{q}}(B)\big), && H^i\big(\mathfrak{Upp}_{\bar{q}}(\mathbb{R} \times B)\big) = H^i\big(\mathfrak{Upp}_{\bar{q}}(B)\big). \end{aligned}$$

Proof. The first isomorphism can be seen at [22], we will verify the other three. Notice that the orbit space of $\mathbb{R} \times X$ is $\mathbb{R} \times B$ and the orbit map is $1 \times \pi : \mathbb{R} \times X \to \mathbb{R} \times B$. There is a commutative diagram

$$\begin{array}{ccccccccccc} \longrightarrow & H^{i+1}_{\bar{q}}(\mathbb{R} \times B) & \longrightarrow & H^{i+1}_{\bar{q}}(\mathbb{R} \times X) & \longrightarrow & H^i(\mathcal{G}_{\bar{q}}(\mathbb{R} \times B)) & \longrightarrow & H^{i+2}_{\bar{q}}(\mathbb{R} \times B) & \longrightarrow & H^{i+2}_{\bar{q}}(\mathbb{R} \times X) & \longrightarrow \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \longrightarrow & H^{i+1}_{\bar{q}}(B) & \longrightarrow & H^{i+1}_{\bar{q}}(X) & \longrightarrow & H^i(\mathcal{G}_{\bar{q}}(B)) & \longrightarrow & H^{i+2}_{\bar{q}}(B) & \longrightarrow & H^{i+2}_{\bar{q}}(X) & \longrightarrow \end{array}$$

where the horizontal rows are Gysin sequences and the vertical arrows are induced by the projections $pr : \mathbb{R} \times X \to X$ and $pr : \mathbb{R} \times B \to B$. By the Five Lemma, $H^i(\mathcal{G}_{\bar{q}}(\mathbb{R} \times B)) = H^i(\mathcal{G}_{\bar{q}}(B))$; the same argument holds for the residues. □

6.3. Lemma. *Let $\Psi:\mathbb{S}^1\times L\to L$ be a smooth free circle action on a compact manifold. Then for each $r>0$ the inclusion $\iota_r:L\to c(L)$ defined by $x\mapsto[x,r]$ induces the following isomorphisms*

$$H^i_{\bar q}\big(c(L)\big)=\begin{cases}H^i(L), & 0\leqslant i\leqslant \bar q(\star),\\ 0, & \textit{else},\end{cases}$$

$$H^i\big(\mathcal{G}_{\bar q}\big(c\big(L/\mathbb{S}^1\big)\big)\big)=\begin{cases}H^i(L/\mathbb{S}^1), & i\leqslant \bar q(\star)-2,\\ \ker[\varepsilon:H^{\bar q(\star)-1}(L/\mathbb{S}^1)\to H^{\bar q(\star)+1}(L/\mathbb{S}^1)], & i=\bar q(\star)-1,\\ 0, & i\geqslant \bar q(\star)\end{cases}$$

where ε is the multiplication by the Euler class $\varepsilon\in H^2(L/\mathbb{S}^1)$. What's more

$$H^i\big(\mathfrak{Low}_{\bar q}\big(c\big(L/\mathbb{S}^1\big)\big)\big)=\begin{cases}\ker[\varepsilon:H^{\bar q(\star)-1}(L/\mathbb{S}^1)\to H^{\bar q(\star)+1}(L/\mathbb{S}^1)], & i=\bar q(\star)-1,\\ 0, & i\neq \bar q(\star)-1,\end{cases}$$

$$H^i\big(\mathfrak{Upp}_{\bar q}\big(c\big(L/\mathbb{S}^1\big)\big)\big)=\begin{cases}\operatorname{Im}[\varepsilon:H^{\bar q(\star)-1}(L/\mathbb{S}^1)\to H^{\bar q(\star)+1}(L/\mathbb{S}^1)], & i=\bar q(\star)-1,\\ 0, & i\neq \bar q(\star)-1.\end{cases}$$

Proof. As before, the first isomorphism can be found in [22]. For $i\leqslant \bar q(\star)-2$ there is a commutative diagram

$$\begin{array}{ccccccccccc}
\to & H^{i+1}_{\bar q}(c(L/\mathbb{S}^1)) & \to & H^{i+1}_{\bar q}(c(L)) & \to & H^i(\mathcal{G}_{\bar q}(c(L/\mathbb{S}^1))) & \to & H^{i+2}_{\bar q}(c(L/\mathbb{S}^1)) & \to & H^{i+2}_{\bar q}(c(L/\mathbb{S}^1)) & \to\\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\to & H^{i+1}(L/\mathbb{S}^1) & \to & H^{i+1}(L) & \to & H^i(\mathcal{G}_{\bar q}(L/\mathbb{S}^1)) & \to & H^{i+2}(L/\mathbb{S}^1) & \to & H^{i+2}(L) & \to
\end{array}$$

where again the horizontal rows are Gysin sequences and the vertical arrows are induced by ι_r. By the Five Lemma

$$H^i\big(\mathcal{G}_{\bar q}\big(c\big(L/\mathbb{S}^1\big)\big)\big)=H^i\big(\mathcal{G}_{\bar q}\big(L/\mathbb{S}^1\big)\big)=H^i\big(L/\mathbb{S}^1\big)$$

because L has no singular part. For $i=\bar q(\star)-1$ the upper horizontal row has two zeros in the right side. By Proposition 5.3

$$H^{\bar q(\star)}\big(\mathcal{G}_{\bar q}\big(c\big(L/\mathbb{S}^1\big)\big)\big)=\operatorname{coker}(\pi^*)=\ker\big[\varepsilon:H^{\bar q(\star)-1}\big(L/\mathbb{S}^1\big)\to H^{\bar q(\star)+1}\big(L/\mathbb{S}^1\big)\big].$$

For $i\geqslant \bar q(\star)$ the upper horizontal row has four zeros, hence $H^i(\mathcal{G}_{\bar q}(c(L/\mathbb{S}^1)))=0$.

Since $L/\mathbb{S}^1$ has no singular part the Gysin residues vanish on $L/\mathbb{S}^1$. With an analogous procedure the cohomology of the residues of $c(L/\mathbb{S}^1)$ can be easily deduced. □

7. RESIDUAL COHOMOLOGY AND THE GYSIN THEOREM

We want to calculate the cohomology of the Gysin term in the general case. For this sake we will use the upper approximation given in Section 5.5. This long exact sequence relates $H^*(\mathcal{G}_{\bar q}(B))$ with $H^i_{\bar q-\bar\chi}(B)$ and the upper residue. Trough the following two lemmas we give an explicit calculation of the residual cohomology.

7.1. Lemma. *Take a family* $\mathcal{U} = \{V_S \colon \pi(S) \subset \Sigma\}$ *of disjoint open sets separating the singular strata in* B. *Then*

$$H^i(\mathfrak{Upp}_{\bar{q}}(B)) = \prod_S H^i(\mathfrak{Upp}_{\bar{q}}(V_S))$$

where S *runs over the perverse strata.*

Proof. Apply Lemma 6.1 to the open cover $\mathcal{U} \cup \{B - \Sigma\}$. Notice that the presheaf $\mathfrak{Upp}^*_{\bar{q}}(-)$ vanishes on any open subset of $B - \Sigma$ and on V_S whenever S is a mobile stratum or its link is a cohomological product (cf. Propositions 4.8, 5.3). □

7.2. Lemma. *In the situation of Lemma* 7.1, $H^*(\mathfrak{Upp}_{\bar{q}}(V_S))$ *is the cohomology of* S *with values on the presheaf*

$$\mathfrak{Im}_{\bar{q}}(\partial)(U) = \mathrm{Im}\big[\partial : H^{\bar{q}(S)-1}_{\bar{q}}(W) \to H^{\bar{q}(S)+1}_{\bar{q}}(W)\big], \quad U = S \cap W,\ W \subset V_S$$

for each perverse stratum S. *Here* ∂ *is the connecting homomorphism of a Gysin sequence.*

Proof. We do it in two steps.

- *The restricted presheaf makes sense*: Take $U \subset S$ open. Define

$$\mathfrak{Upp}^i_{\bar{q}}(U) = \mathfrak{Upp}^i_{\bar{q}}(W)$$

for any $W \subset V_S$ open such that $U = W \cap S$. If $W' \subset V_S$ is open and $W' \cap S = W \cap S$ then $\mathfrak{Upp}^i_{\bar{q}}(W) = \mathfrak{Upp}^i_{\bar{q}}(W')$, because the residual presheaf is complete and it vanishes on each open subset of $V_S - S$.

- $H^i(\mathfrak{Upp}_{\bar{q}}(V_S)) = H^i(S, \mathfrak{Im}_{\bar{q}}(\partial))$: After a convenient adjust in the size of V_S, we can take an open cover of V_S by modelled charts

$$\mathcal{U} = \big\{\alpha : U_\alpha \times c(L/\mathbb{S}^1) \to V_\alpha \subset V_S\big\}_\alpha$$

such that $\{U_\alpha\}_\alpha$ is a good covering of S and each finite intersection of open sets in $\mathcal{U}$ admits a decomposition

$$V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} = A \cup V_\beta, \quad A \subset V_S - S,\ V_\beta \in \mathcal{U}.$$

Then

$$H^i(\mathfrak{Upp}_{\bar{q}}(V_{\alpha_1} \cap \cdots \cap V_{\alpha_n})) = \begin{cases} \mathrm{Im}[\varepsilon : H^{\bar{q}(S)-1}(L/\mathbb{S}^1) \to H^{\bar{q}(S)+1}(L/\mathbb{S}^1)], & i = \bar{q}(S) - 1, \\ 0, & i \neq \bar{q}(S) - 1. \end{cases}$$

By Lemma 6.3 the double complex $(C^j(\mathcal{U}, \mathfrak{Upp}^i_{\bar{q}}(-)), \delta, d)$ has horizontal rows exact on each level j and vertical rows exact but in level $\bar{q}(S) - 1$. Following [3, p. 130],

$$H^i(\mathfrak{Upp}_{\bar{q}}(V_S)) = H^{i-\bar{q}(S)+1}\big(\mathcal{U}, \mathcal{H}\mathfrak{Upp}^{\bar{q}(\star)-1}_{\bar{q}}\big) = H^{i-\bar{q}(S)+1}\big(S, \mathfrak{Im}_{\bar{q}}(\partial)\big). \quad \square$$

The above two lemmas show us that the cohomology of the Gysin term depends both of local and global basic information. Global information concerns the behavior of the Euler class, local information depends on the perverse strata. We summarize it in the following

7.3. Theorem (The Gysin theorem). *For each modelled circle action $\mathbb{S}^1 \times X \to X$ and each perversity $\bar{0} \leqslant \bar{q} \leqslant \bar{t}$ there are two long exact sequences relating the intersection cohomology of X and B: The Gysin sequence*

$$\cdots \to H^{i+1}_{\bar{q}}(X) \xrightarrow{q^*} H^i\big(\mathcal{G}_{\bar{q}}(B)\big) \xrightarrow{\partial} H^{i+2}_{\bar{q}}(B) \xrightarrow{\pi^*} H^{i+2}_{\bar{q}}(X) \to \cdots$$

and a second long exact sequence

$$(7) \qquad \cdots \to H^i_{\bar{q}-\bar{\chi}}(B) \to \prod_S H^i\big(S, \mathfrak{Im}_{\bar{q}}(\partial)\big) \to H^{i+1}\big(\mathcal{G}_{\bar{q}}(B)\big) \to H^{i+1}_{\bar{q}-\bar{\chi}}(B) \to \cdots$$

where S runs over the perverse strata, $H^i(S, \mathfrak{Im}_{\bar{q}}(\partial))$ is the cohomology of S with values on a locally trivial presheaf whose fiber is

$$\mathrm{Im}\big[\varepsilon : H^{\bar{q}(S)-1}\big(L/\mathbb{S}^1\big) \to H^{\bar{q}(S)+1}\big(L/\mathbb{S}^1\big)\big]$$

the image of the multiplication by the Euler class $\varepsilon \in H^2(L/\mathbb{S}^1)$.

7.4. Remark. Apply the same procedure to the lower residue $\mathfrak{Low}^*_{\bar{q}}(-)$. You will get that a third long exact sequence

$$(8) \qquad \cdots \to H^i_{\bar{q}-\bar{e}}(B) \to H^i\big(\mathcal{G}_{\bar{q}}(B)\big) \to \prod_S H^i\big(S, \mathfrak{Ker}_{\bar{q}}(\partial)\big) \to H^{i+1}_{\bar{q}-\bar{e}}(B) \to \cdots$$

relating the Gysin term with the cohomology of the perverse strata with values on a locally trivial presheaf $\mathfrak{Ker}_{\bar{q}}(\partial)$ whose fiber is

$$\ker\big[\varepsilon : H^{\bar{q}(S)-1}\big(L/\mathbb{S}^1\big) \to H^{\bar{q}(S)+1}\big(L/\mathbb{S}^1\big)\big]$$

APPENDIX A. SOME PROPERTIES OF THE EULER CLASS

A.1. The perverse degree of the Euler form. In this appendix we provide a proof of Proposition 4.8.

By Lemma 4.2(4); for any unfoldable metric μ on X and any mobile stratum S we have $\|e\|_{\pi(S)} = 0$. So we restrict our attention to the fixed strata. We must give an unfoldable metric μ such that the induced Euler form e satisfies

$$(\mathrm{A}.1) \qquad \|e\|_{\pi(S)} = 2 \quad \Leftrightarrow \quad S \text{ is a perverse stratum}$$

for each fixed stratum S in X. Any unfoldable metric μ satisfying this property will be called a *good metric*.

• *Construction of a global good metric μ from a family of local ones*:
We give an invariant open cover $\mathcal{U} = \{X_\alpha\}_\alpha$ of X, and a family $\{\mu_\alpha\}_\alpha$ of unfoldable metrics such that each μ_α is a good metric in X_α.

(a) The complement of the fixed points' set $X_0 = X - X^{\mathbb{S}^1}$ belongs to $\mathcal{U}$. We take on X_0 an unfoldable metric μ_0.
(b) For each fixed stratum S we take a family of modelled charts

$$\alpha : U_\alpha \times c(L) \to X$$

as in (2) of Section 3.1; such that $\{U_\alpha\}_\alpha$ is a good cover of S. We put $X_\alpha = \mathrm{Im}(\alpha)$ and take

$$\mu_\alpha = \alpha^{-*}\left(\mu_{U_\alpha} + \mu_L + dr^2\right)$$

where μ_{U_α} (respectively μ_L) is a Riemannian (respectively and also invariant) metric in U_α (respectively in L). So μ_α is a good metric in X_α.

Fix an invariant controlled partition of the unity $\{\rho_\alpha\}_\alpha$ subordinated to $\mathcal{U}$. Define

$$\text{(A.2)} \qquad \mu = \sum_\alpha \rho_\alpha \mu_\alpha .$$

• *Goodness of μ on a fixed stratum S*: We verify the property (A.1) on S.

($\Rightarrow$) Write χ, e (respectively χ_α, e_α) for the characteristic form and the Euler form induced by μ on X (respectively by μ_α on X_α). Notice that

$$\text{(A.3)} \qquad d\chi = \sum_\alpha (d\rho_\alpha) \wedge \chi_\alpha + \sum_\alpha \rho_\alpha d\chi_\alpha .$$

In the above expression, the first sum of the right side has perverse degree 1 (see Lemma 4.3). Recall that, by Lemma 4.7, $\|e\|_{\pi(S)} = \|d\chi\|_S$. If $\|d\chi\|_S = 2$ then, by Eq. (A.3),

$$\|d\chi_\alpha\|_{S\cap X_\alpha} = \|e_\alpha\|_{\pi(S\cap X_\alpha)} = 2$$

for some X_α intersecting S. So $\varepsilon_L \neq 0$ because μ_α is a good metric.

($\Leftarrow$) In the rest of this proof we use some local properties of intersection cohomology. In particular, we use the step cohomology of a product $U \times c(L/\mathbb{S}^1)$ as it is defined in [11]. In Section 6 the reader will find more details. Let's assume that $\|e\|_{\pi(S)} < 2$ and take some $X_\alpha = \mathrm{Im}(\alpha) \in \mathcal{U}$, the image of a modelled chart α on S. Write $B_\alpha = \pi(X_\alpha) \cong U_\alpha \times c(L/\mathbb{S}^1)$; so that $\|e|_{B_\alpha}\|_{U_\alpha} < 2$. Consider the short exact sequence of step intersection cohomology

$$0 \to \Omega^*_{\bar{1}}(B_\alpha) \xrightarrow{\iota} \Omega^*_{\bar{2}}(B_\alpha) \xrightarrow{pr} \Omega^*_{\bar{2}/\bar{1}}(B_\alpha) \to 0$$

which induces the long exact sequence

$$\cdots \to H^2_{\bar{1}}(B_\alpha) \to H^2_{\bar{2}}(B_\alpha) \xrightarrow{pr^*} H^2_{\bar{2}/\bar{1}}(B_\alpha) \xrightarrow{d} H^3_{\bar{1}}(B_\alpha) \to \cdots.$$

The inclusion $\iota_\epsilon : L/\mathbb{S}^1 \to U_\alpha \times c(L/\mathbb{S}^1)$ given by $p \mapsto (x_0, [p, \epsilon])$, induces the isomorphism

$$\iota_\epsilon^* : H^2_{\bar{2}/\bar{1}}(U_\alpha \times c(L/\mathbb{S}^1)) \xrightarrow{\cong} H^2_{\bar{2}}(L/\mathbb{S}^1)$$

where $x_0 \in U_\alpha$ and $\epsilon > 0$. Since $\|e\|_{\pi(S)} \leqslant 1$, then the double equivalence class $[\bar{e}|_{B_\alpha}]$ vanishes on $H^2_{\bar{2}/\bar{1}}(B_\alpha)$. By the above remarks, $(\alpha\iota_\epsilon)^{-*}(\varepsilon_L) = pr^*[e|_{B_\alpha}] = 0$; so $\varepsilon_L = 0$.

A.2. Chasing the Euler form in a double complex. In this appendix we show that the Euler form is cohomologous to a controlled form whenever $\bar{e} \leqslant \bar{\chi}$. In order to simplify the proof we will assume that X has a unique singular stratum S. If S is mobile there is nothing to do. If S is a fixed (non-perverse) stratum we consider the long exact sequence of step cohomology

$$\cdots \to H^2_{\bar{0}}(B) \xrightarrow{\iota^*} H^2_{\bar{1}}(B) \xrightarrow{p^*} H^2_{\bar{1}/\bar{0}}(B) \to H^3_{\bar{0}}(B) \to \cdots.$$

The Euler form e is cohomologous to a controlled form $\Leftrightarrow$ in step cohomology the double class $[\bar{e}]$ vanishes. Take an atlas $\mathcal{U}$ and a good metric μ on X as in Appendix A.1. For each $B_\alpha = \pi(X_\alpha)$ we obtain

$$\text{(A.4)} \qquad H^i_{\bar{1}/\bar{0}}(B_\alpha) = \begin{cases} H^1(L/\mathbb{S}^1), & i = 1, \\ 0, & i \neq 1. \end{cases}$$

By the above equation, the double complex

$$\left(K^{i,j} = C^j(\mathcal{U}, \Omega^i_{\bar{1}/\bar{0}}), d, \delta\right)$$

satisfies the following conditions: The horizontal rows $(K^{i,*}, \delta)$ are exact and the vertical rows $(K^{*,j}, d)$ are exact in dimension $i \neq 1$. Following [3], there are two isomorphisms

$$\text{(A.5)} \qquad H^j_\delta(\mathcal{U}, H^1_{\bar{1}/\bar{0}}) \xrightarrow{\cong} H^j_D(\mathcal{U}, \Omega^i_{\bar{1}/\bar{0}}) \xleftarrow{\cong} H^2_{\bar{1}/\bar{0}}(B)$$

where the middle term is the diagonal cohomology of the double complex, the first arrow is given by chasing in the braid of the double complex and the second arrow is induced by the restrictions to the open sets $B_\alpha = \pi(X_\alpha)$.

A double class $[[\theta]_d]_\delta \in H^j_\delta(\mathcal{U}, H^1_{\bar{1}/\bar{0}})$ is always represented by an element $\theta = (\theta_{\alpha\beta})_{\alpha\beta} \in K^{1,1}$ such that $d\theta = 0$, $\delta\theta = -d\nu$ for some $\nu \in K^{0,2}$, and $\delta\nu = 0$. Hence

$\phi = \theta + \nu \in K^2$ is a D-cocycle, where D is the diagonal differential operator. By exactness of the horizontal rows, ϕ is D-cohomologous to a D-cocycle $d\omega \in K^{2,0}$

$$\begin{array}{ccccc}
 & & \nu' & \to \nu \to 0 & \\
 & & \downarrow & \quad\downarrow & \\
\omega & \to & \theta, d\nu' & \to * & \\
\downarrow & & \downarrow & & \\
\boxed{d\omega} & \to & 0 & & \\
\downarrow & & & & \\
0 & & & &
\end{array}
\qquad
\begin{array}{ll}
\phi = \theta + \nu & D\phi = 0 \\
 & \exists \nu', \omega\colon\ \delta\nu' = \nu,\ \delta\omega = \theta - d\nu' \\
\phi' = \omega - \nu' & \\
\phi + D\phi' & = (\nu - \delta\nu') + (\theta + d\nu' + \delta\omega) + d\omega \\
 & = d\omega
\end{array}$$

This $d\omega$ is a representative element of a unique global cohomology class $[d\omega] \in H^2_{\bar{1}/\bar{0}}(B)$.

Next we show that the double class $[\bar{e}] =\in H^2_{\bar{1}/\bar{0}}(B)$ vanishes.

• *Definition of* $\theta = (\theta_{\alpha\beta})_{\alpha\beta} \in K^{1,1}$: Take $\xi_{\alpha\beta} = \chi_\beta - \chi_\alpha$ on each intersection $X_\alpha \cap X_\beta$; where $\chi_\alpha = \alpha^{-*}(\chi_L)$. These forms $\xi_{\alpha\beta}$ are basic; so $\xi_{\alpha\beta} = \pi^*(\theta_{\alpha\beta})$ for a unique form $\theta_{\alpha\beta}$ on $B_{\alpha\beta}$. Since L is a cohomological product, we can assume that $d\chi_L = 0$; so

$$\pi^*(d\theta_{\alpha\beta}) = d\pi^*(\theta_{\alpha\beta}) = d\xi_{\alpha\beta} = d\chi_\alpha - d\chi_\beta = 0$$

and

$$\pi^*(\theta_{\alpha\beta} + \theta_{\beta\gamma}) = \xi_{\alpha\beta} + \xi_{\beta\gamma} = \xi_{\alpha\gamma} = \pi^*(\theta_{\alpha\gamma}).$$

Since π^* is injective we deduce that $d\theta = \delta\theta = 0$. So θ is a representative element of a D-cocycle in the $\Omega_{\bar{1}/0}$-double complex. By exactness of the horizontal rows in the $\Omega_{\bar{1}}$-double complex, there is some $\omega = (\omega_\alpha)_\alpha$ such that $\delta\omega = \theta$. So $[[\theta]_d]_\delta = 0$.

• *Definition of* $\omega = (\omega_\alpha)_\alpha \in K^{1,0}$: By Eq. (A.3) we have

$$d\chi = \sum_\alpha (d\rho_\alpha) \wedge \xi_{\alpha\beta}$$

on X_β. Define $\omega_\beta = \sum_\alpha \rho_\alpha \theta_{\alpha\beta}$ on B_β. Then

$$e = \sum_\alpha (d\rho_\alpha) \wedge \theta_{\alpha\beta} = \sum_\alpha d(\rho_\alpha \theta_{\alpha\beta}) = d\omega_\beta.$$

This implies that the double class $[\bar{e}]$ of e in step cohomology is the image of a δd-boundary by the isomorphism (A.5).

APPENDIX B. ON THE EXISTENCE OF MODELLED ACTIONS

From now, a *pre-modelled action* is an action $\Phi : \mathbb{S}^1 \times X \to X$ of the unit circle on a pseudomanifold X satisfying conditions MA(1), MA(2), and MA(3) of Section 3.1.

The only difference between pre-modelled and modelled actions is the existence of modelled charts, we devote this section to study that problem.

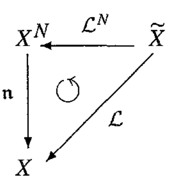
A pseudomanifold is *normal* when its links are connected. For each pseudomanifold X there is a normal pseudomanifold X^N and a morphism

$$\mathfrak{n}: X^N \to X$$

which is an isomorphism in intersection homology. The construction of X^N is functorial, thus unique; we call it a *normalization* of X. The arrow $\mathfrak{n}$ is the *normalization map*. For a detailed introduction the reader can see [8,6].

Recall that the definition of intersection cohomology given in Section 2.2 depends on the prefixed stratification, and it deals with a broader family of perversities. In [16] we show that for these perversities, the normalization still preserves the intersection homology. The proof depends on the fact that, when restricted to any stratum S in X, the arrow $\mathfrak{n}: \mathfrak{n}^{-1}(S) \to S$ is a smooth finite covering. By the DeRham Theorem, the intersection cohomology can be deduced from the intersection homology [22]. So, the normalizations also preserve the intersection cohomology. As we will see, this property can be directly deduced by topological arguments.

Let $\Phi: \mathbb{S}^1 \times X \to X$ be a pre-modelled (respectively a modelled) action. An *equivariant normalization* is a normalization $\mathfrak{n}: X^N \to X$ together with a pre-modelled (respectively a modelled) action $\Phi^N: \mathbb{S}^1 \times X^N \to X^N$ and an equivariant unfolding $\mathcal{L}^N: \widetilde{X} \to X^N$ such that the morphism $\mathfrak{n}$ is $\mathbb{S}^1$-equivariant; and the diagram



is commutative.

B.1. Proposition. *Any pre-modelled (respectively modelled) action has an equivariant normalization.*

Proof. Fix a pre-modelled action $\Phi: \mathbb{S}^1 \times X \to X$. Take a normalization $\mathfrak{n}: X^N \to X$. We proceed in two steps.

• *Definition of* $\Phi^N: \mathbb{S}^1 \times X^N \to X^N$: Since Φ is a morphism and

$$\iota \times \mathfrak{n}: \mathbb{S}^1 \times X^N \to \mathbb{S}^1 \times X$$

is a normalization, by functoriality Φ lifts to a unique morphism $\Phi^N: \mathbb{S}^1 \times X^N \to X^N$. This Φ^N is indeed an action, so MA(1) holds by construction. For each stratum S in X the restriction

$$\mathfrak{n}: \mathfrak{n}^{-1}(S) \to S$$

is a smooth finite covering. The strata of X^N contained in the preimage of S are the connected components of $\mathfrak{n}^{-1}(S)$. Since the arrow is equivariant and the isotropy of S is constant H_S, there is a smooth action of H_S in the fiber and a smooth H_S-equivariant trivializing atlas of the covering. By an argument of connectedness we get MA(2).

• *Definition of* $\mathcal{L}^N : \widetilde{X} \to X$: Take an equivariant unfolding $\mathcal{L} : \widetilde{X} \to X$. For each singular stratum S with link L; The cone $c(L)$ has an H_S-equivariant normalization

$$\mathfrak{n}_0 : c(L)^N = \bigsqcup_j c(K_j), \quad [p,r]_j \mapsto [p,r]$$

where $\{K_j\}_i$ are the connected components of L and $[p,r]_i$ is a point in $c(K_j)$. The arrow

$$c^N : \tilde{L} \times \mathbb{R} \to c(L)^N, \quad c^N(\tilde{p}, t) = \big(u, \big[\mathcal{L}_L^N(\tilde{p}), |t|\big]_j\big) \text{ if } \mathcal{L}_L^N(\tilde{p}) \in K_j$$

is an H_S-equivariant unfolding of $U \times c(L)^N$. Next we give the $\mathbb{S}^1$-equivariant unfolding of X^N as follows. Recall the restriction $\mathfrak{n} : X^N - \Sigma = \mathfrak{n}^{-1}(X - \Sigma) \to X - \Sigma$ is a diffeomorphism. Define

$$\mathcal{L}^N : \widetilde{X} \to X^N, \quad \mathcal{L}^N(z) = \begin{cases} \mathfrak{n}^{-1}\mathcal{L}(z), & \mathcal{L}(z) \in (X - \Sigma), \\ \alpha^N c^N(\tilde{\alpha})^{-1}(z), & \tilde{\alpha} \text{ an unfoldable chart and } z \in \operatorname{Im}(\tilde{\alpha}) \end{cases}$$

where α^N is the unique embedding such that the diagram

$$\text{(B.1)} \qquad \begin{array}{ccc} U \times c(L)^N & \xrightarrow{\alpha^N} & X^N \\ \Big\downarrow{\scriptstyle \iota \times \mathfrak{n}_0} & & \Big\downarrow{\scriptstyle \mathfrak{n}} \\ U \times c(L) & \xrightarrow{\alpha} & X \end{array}$$

commutes. In order to see that $\mathcal{L}^N$ is well defined take some $z \in \widetilde{X}$. If $z \in \mathcal{L}^{-1}(X - \Sigma) \cap \operatorname{Im}(\tilde{\alpha})$ for some unfoldable chart $\tilde{\alpha}$; then the above diagram implies that

$$\alpha^N c^N(\tilde{\alpha})^{-1} = \mathfrak{n}^{-1}\mathcal{L}.$$

By the other hand, if $z \in \mathcal{L}^{-1}(\Sigma)$ and $\tilde{\alpha}$, $\tilde{\beta}$ are two unfoldable charts of z; then by an argument of density

$$\alpha^N c^N(\tilde{\alpha})^{-1}(z) = \beta^N c^N(\tilde{\beta})^{-1}(z).$$

So $\mathcal{L}^N$ is well defined. The above equations imply that each unfoldable chart of $\mathcal{L} : \widetilde{X} \to X$ induces unfoldable charts of $\mathcal{L}^N : \widetilde{X} \to X^N$. In consequence, condition MA(3) also holds.

Finally, if Φ is modelled then for each modelled chart $\alpha : U \times c(L) \to X$ the induced chart α^N given in Eq. (B.1) restricted to each connected component of $U \times c(L)^N$ induces a modelled chart. This proves MA(4), we leave the details to the reader. □

B.2. Remark. Fix an equivariant normalization $\mathfrak{n}: X^N \to X$. Since for any stratum S in X the restriction $\mathfrak{n}: \mathfrak{n}^{-1}(S) \to S$ is a smooth finite covering; the induced map $\mathfrak{n}^*: \Omega^*_{\bar{q}}(X) \to \Omega^*_{\bar{q}}(X^N)$ is an isomorphism of differential complexes. Hence, for each perversity $\bar{0} \leqslant \bar{q} \leqslant \bar{t}$ in X; there is an isomorphism

$$\mathfrak{n}^*: H^*_{\bar{q}}(X) \to H^*_{\bar{q}}\left(X^N\right)$$

in intersection cohomology.

Next we want to verify the existence of modelled charts. In order to do this, we will simplify some $\mathbb{S}^1$-equivariant unfolding $\mathcal{L}: \widetilde{X} \to X$. A *bubble* on $\widetilde{X}$ is a connected component of $\mathcal{L}^{-1}(X - \Sigma)$, i.e.; a diffeomorphic copy of some regular stratum. We will say that $\widetilde{X}$ is *primary* if, for any link L of X, the unfolding $\mathcal{L}_L: \tilde{L} \to L$ is an isomorphism (and then we will write $\tilde{L} = L$).

B.3. Lemma. *Up to an equivariant normalization, each modelled action $\Phi: \mathbb{S}^1 \times X \to X$ has a equivariant unfolding $\mathcal{L}: \widetilde{X} \to X$ satisfying*:

(1) *$\widetilde{X}$ is primary.*
(2) *There is a smooth equivariant collar*

$$\Gamma: \mathcal{L}^{-1}(\Sigma) \times \mathbb{R} \to \widetilde{X}$$

such that $\mathcal{L}\Gamma(z, t) = \mathcal{L}\Gamma(z, -t)$ for each $z \in \mathcal{L}^{-1}(\Sigma)$, $t \in \mathbb{R}$.

Proof. Suppose $\widetilde{X}$ is not a primary unfolding. Assume that X is normal and connected. Since the links are connected then $X - \Sigma = R$ is a unique regular stratum R. Fix a numeration $N_0, \ldots, N_k$ of the bubbles, $k > 1$. Write

$$\phi_{i,j} = \mathcal{L}^{-1}\mathcal{L}: N_i \to N_j.$$

Take also a numeration $S_0, \ldots, S_n, \ldots$ of the singular strata and a disjoint family of open $\mathbb{S}^1$-invariant sets $A_0, \ldots, A_n, \ldots$ separating them. Write H_i (respectively L_i) for the isotropy (respectively the link) of S_i. For each i let's fix a connected component $C_i \subset \mathcal{L}^{-1}(S_i)$. By Lemma 1.4; there are integers

$$0 \leqslant a_i < b_i \leqslant k$$

such that C_i is a border of the bubbles N_{a_i}, N_{b_i}. In other words, C_i is a connected component of $\overline{N_{a_i}} \cap \overline{N_{b_i}}$. Since $\mathbb{S}^1$ is connected, the restriction $\mathcal{L}: C_i \to S_i$ is a $\mathbb{S}^1$-equivariant submersion; so it is a locally trivial fiber bundle with fiber L_i. There is a smooth free action of H_i on L_i and a trivializing atlas of $\mathcal{L}: C_i \to S_i$ consisting of H_i-equivariant trivializing charts.

(a) *Definition of a primary unfolding*: Notice that $W_i = \mathcal{L}^{-1}(A_i) \cap (N_{a_i} \cup C_i \cup N_{b_i})$ is an open $\mathbb{S}^1$-invariant neighborhood of C_i. Consider the amalgamated sum

$$\widetilde{X}' = [N_0 \sqcup N_1] \bigcup_f \Big[\bigsqcup_i W_i \Big]$$

where the collating map $f : \bigsqcup_i (W_i \cap [N_{a_i} \sqcup N_{b_i}]) \to N_0 \sqcup N_1$ is given by the change of bubbles

$$f(z) = \begin{cases} \phi_{a_i,0}(z) & \text{if } z \in W_i \cap N_{a_i}, \\ \phi_{b_i,1}(z) & \text{if } z \in W_i \cap N_{b_i}. \end{cases}$$

Write $[z]$ for the equivalence class of $z \in N_0 \sqcup N_1 \bigsqcup_i W_i$. Then $\mathbb{S}^1$ acts on $\widetilde{X}'$ by the rule $g[z] = [gz]$, this action is well defined because f is $\mathbb{S}^1$-equivariant. $\widetilde{X}'$ has a unique smooth structure such that

$$\widetilde{\mathcal{L}} : \widetilde{X}' \to X, \quad \widetilde{\mathcal{L}}[z] = \mathcal{L}(z)$$

is a primary $\mathbb{S}^1$-equivariant unfolding. The existence of unfoldable charts is left to the reader.

(b) *Existence of the equivariant collar*: Assume that $\mathcal{L} : \widetilde{X} \to X$ is the primary $\mathbb{S}^1$-equivariant unfolding given in the first step of this proof. Notice that $\widetilde{X}$ has only two bubbles N, N'. The closure $\overline{N}$ is a manifold with border and

$$N = \overline{N} - \partial\overline{N} \cong X - \Sigma, \quad \partial\overline{N} = \mathcal{L}^{-1}(\Sigma).$$

So we can obtain $\widetilde{X}$ as two copies of $\overline{N}$ glued by the border. There is a $\mathbb{S}^1$-equivariant collar $\gamma : \mathcal{L}^{-1}(\Sigma) \times [0,\infty) \to \overline{N}$. Define

$$\Gamma : \mathcal{L}^{-1}(\Sigma) \times \mathbb{R} \to \widetilde{X}, \quad \Gamma(z,t) = \begin{cases} \gamma(z,t), & t \geqslant 0, \\ \phi\gamma(z,-t), & t < 0 \end{cases}$$

where $\phi = \mathcal{L}^{-1}\mathcal{L} : N \to N'$ is the change of bubbles. Then Γ is an homeomorphism on an open set, and its restriction to $\mathcal{L}^{-1}(\Sigma) \times (\mathbb{R} - \{0\})$ is a smooth embedding. Give a new smooth structure on $\widetilde{X}$ by declaring that Γ' is smooth. Then Γ' is the desired collar. It is immediate that $\widetilde{X}$ is still an unfolding with the new smooth structure since, for each unfoldable chart $\tilde{\alpha} : U \times L \times \mathbb{R} \to \widetilde{X}$ in the old smooth structure,

$$\tilde{\alpha}'(u,p,t) = \begin{cases} \tilde{\alpha}(u,p,t), & t \geqslant 0, \\ \phi\tilde{\alpha}(u,p,-t), & t < 0 \end{cases}$$

is an unfoldable chart of $\widetilde{X}$ (up to a correction of sign) with the new smooth structure. □

B.4. Corollary. *Each unfoldable pseudomanifold X can be obtained as the quotient of a manifold with border M. Each connected component $C \subset \partial M$ is the total space of a smooth fiber bundle $p: C \to S$ with base a singular stratum S and fiber L the link of S. The quotient projection $q: M \to X$ identifies two points $z, z' \in \partial M$ if they are in the same connected component C and $p(z) = p(z')$.*

B.5. Proposition. *Up to an equivariant normalization, any pre-modelled action is modelled.*

Proof. Fix a pre-modelled action $\Phi: \mathbb{S}^1 \times X \to X$ induces on X on a normal pseudomanifold X, a primary equivariant unfolding $\mathcal{L}: \widetilde{X} \to X$ and a $\mathbb{S}^1$-equivariant collar $\Gamma: \mathcal{L}^{-1}(\Sigma) \times \mathbb{R} \to \widetilde{X}$ as in Lemma B.3. Also take a point $z \in \mathcal{L}^{-1}(S)$ in the preimage of a singular stratum S. Since the link L of S is connected, any unfoldable chart of z sends different bubbles to different bubbles. In consequence, the unfolding $L_L: \tilde{L} \to L$ is a diffeomorphism and we can write $\tilde{L} = L$.

Fix an H_S-equivariant trivialization

$$\phi: U \times L \xrightarrow{\cong} \mathcal{L}^{-1}(U)$$

of the smooth $\mathbb{S}^1$-equivariant fiber bundle $\mathcal{L}: \mathcal{L}^{-1}(S) \to S$. Consider the function

$$\tilde{\alpha}: U \times L \times \mathbb{R} \to \widetilde{X}, \quad (u, p, t) \mapsto \Gamma\big(\phi(u, p), t\big).$$

Then $\tilde{\alpha} = \Gamma \circ [\phi \times 1_{\mathbb{R}}]$ is a H_S-equivariant smooth embedding onto an open subset. Define

$$\alpha: U \times c(L) \to X, \quad \alpha\big(u, [p, r]\big) = \mathcal{L}\tilde{\alpha}(u, p, r).$$

- *α is well defined and injective*: For $r = 0$ it's trivial. For $r > 0$, since L is connected; the set $\tilde{\alpha}(U \times L \times \mathbb{R}^+)$ is contained in some bubble $N \subset \widetilde{X}$, and the restriction $\mathcal{L}|_N$ is a diffeomorphism.
- *α is an isomorphism*: This map is continuous because $\mathcal{L}\tilde{\alpha}$ is. Also α is smooth on each stratum, so it is a morphism. For seeing that α is an isomorphism one can reduce the domain of the function, because any continuous bijection from a compact space onto a Hausdorff space is an homeomorphism.
- *$\tilde{\alpha}$ is an unfoldable chart*: Because $\mathcal{L}\Gamma$ is an even function with respect to $\mathbb{R}$.
- *α is an H_S-equivariant chart*: It is straightforward.
- *$\mathbb{S}^1$ preserves the radium*: Because α is defined trough a $\mathbb{S}^1$-equivariant collar. □

B.6. Remark. We can substitute a pre-modelled action on an arbitrary pseudomanifold X by a modelled action on its normalizer X^N. Whenever it make sense, the intersection cohomology of the orbit space $B = X/\mathbb{S}^1$ is the intersection cohomology of $X^N/\mathbb{S}^1$ (which always makes sense).

B.7. Remark. All the statements in this appendix still hold if we drop $\mathbb{S}^1$ and write instead a torus $\mathbb{T}^n$.

ACKNOWLEDGEMENTS

I would like to thank for some helpful conversations with M. Saralegi and F. Dalmagro, so as the accurate comments of the journal's referee. While writing this article, I received the financial support of the CDCH, Universidad Central de Venezuela and the Math Department, Euskal Herriko Unibertsitatea, so as the hospitality of the staff in the Math Department, Université D'Artois.

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(Received May 2003)