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Möbius gyrogroups: A Clifford algebra approach

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ABSTRACT

Using the Clifford algebra formalism we study the Möbius gyrogroup of the ball of radius t of the paravector space $\mathbb{R} \oplus V$, where V is a finite-dimensional real vector space. We characterize all the gyro-subgroups of the Möbius gyrogroup and we construct left and right factorizations with respect to an arbitrary gyro-subgroup for the paravector ball. The geometric and algebraic properties of the equivalence classes are investigated. We show that the equivalence classes locate in a k-dimensional sphere, where k is the dimension of the gyro-subgroup, and the resulting quotient spaces are again Möbius gyrogroups. With the algebraic structure of the factorizations we study the sections of Möbius fiber bundles inherited by the Möbius projectors.

1. Introduction

The Möbius gyrogroup plays an important role in the theory of gyrogroups since it provides a concrete model for the abstract theory. Its study leads to a better understanding of Lorentz transformations from the special relativity theory since the Lorentz group acts on ball of all possible symmetric velocities via conformal maps [18,9]. The Möbius gyrogroup is associated with the Poincaré model of conformal geometry also known as the rapidity space [14], because the Poincaré distance from the origin of the ball to any point on the ball coincides with the rapidity of a boost. The Möbius gyrogroup has also applications in Physics for models described by equations invariant under conformal transformations [9–11,18].

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Gyrogroups are group-like structures that appeared in 1988 associated with the study of Einstein's velocity addition in the special relativity theory [15,16]. Since then gyrogroups have been intensively studied by A. Ungar (see [15–21] and the vast list of references in [17,18]) due to their interdisciplinary character, spreading from abstract algebra and non-Euclidean geometry to mathematical physics. The first known gyrogroup was the relativistic gyrogroup of the unit ball of Euclidean space $\mathbb{R}^3$ endowed with Einstein's velocity addition (see [15]), which is a non-associative and non-commutative binary operation. Another example of a gyrogroup is the complex unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$ endowed with Möbius addition defined by

$$a \oplus z = (a + z)(1 + \bar{a}z)^{-1}, \quad a, z \in \mathbb{D}. \tag{1}$$

Möbius addition on $\mathbb{D}$ is neither commutative nor associative but it is gyroassociative and gyrocommutative under gyrations defined by

$$\mathrm{gyr}[a, b]c = (1 + a\bar{b})c(1 + \bar{a}b)^{-1}, \quad a, b, c \in \mathbb{D}, \tag{2}$$

which represent rotations of the disc in turn of the origin. Employing analogies shared by complex numbers and linear transformations of vector spaces Ungar extended in [20] the Möbius addition in the complex disk to the ball of an arbitrary real inner product space. The extension of gyrations from the complex plane to a real inner product space was possible by Ungar's abstract theory on gyrogroups, through the following identity

$$\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus \big(a \oplus (b \oplus c)\big).$$

For the classical approach of describing Möbius transformations in several dimensions we refer to [1,2,13]. In [2] Ahlfors realized that the natural approach to study Möbius transformations in several dimensions is by using Clifford numbers. After Ahlfors's work Clifford algebras become the common tool for the study of Möbius transformations (see e.g. [3,22]). In [21] some connections were established between the theory of Alhfors on Möbius transformations and the hyperbolic geometry through the gyrolanguage due to Ungar. The approach of Ungar gives the right formalism for dealing with Möbius gyrogroups. However, Ungar's description of Möbius addition and gyrations in higher dimensions is very complicated, which results in his verification of some results solely by using computer algebra. For example, in [17,18] Ungar observed that the ball of radius t in the inner product space V endowed with the Möbius addition defined by

$$a \oplus b = \frac{(1 + \frac{2}{t^2}\langle a, b\rangle + \frac{1}{t^2}\|b\|^2)a + (1 - \frac{1}{t^2}\|a\|^2)b}{1 + \frac{2}{t^2}\langle a, b\rangle + \frac{1}{t^4}\|a\|^2\|b\|^2}$$

turns out to be a gyrogroup. Here $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the inner product and the norm that the ball inherits from the space V. Although it was observed by Ungar that the result can be verified by computer algebra it is natural and desirable to give a solid proof. To achieve this we combine Clifford algebra with the theory of Möbius gyrogroups. The advantage of our approach lies in the fact that Möbius gyrogroups of the ball of a real inner product space is thus analogous to the corresponding theory in the unit disc by an algebraic formalism. For example, using Clifford algebra, formulae for the Möbius addition and gyrations in the higher-dimensional case are the same as in the case of the unit disc given by (1) and (2). It allow us also to identify and to obtain a spin representation of the Möbius gyrations which makes easier the study of the structure of Möbius gyrogroups. Our approach gives the unification of the approaches by Alhfors and Ungar for the study of Möbius transformations.

In this paper we will give a comprehensive study of the algebraic structure of Möbius gyrogroups via a Clifford algebra approach. Starting from an arbitrary real inner product space of dimension n we embed it into the Clifford algebra $\mathcal{C}\ell_{0,n}$ and then we construct the paravector space $\mathbb{R} \oplus V$ of $\mathcal{C}\ell_{0,n}$, the direct sum of scalars and vectors. This paravector space will be the environment for studying the Möbius gyrogroup on the ball. Since every non-zero paravector in $\mathcal{C}\ell_{0,n}$ has an inverse we can define Möbius transformations on the paravector space $\mathbb{R} \oplus V$ as fractional linear mappings in $\mathcal{C}\ell_{0,n}$ generalizing Möbius transformations on the vector space V. Our results in paravector spaces remains true in vector spaces by restriction.

The main achievements in this paper are the characterization of all Möbius gyro-subgroups of the Möbius gyrogroup of the paravector ball in Section 6, the unique decomposition of the paravector ball with respect to an arbitrary Möbius gyro-subgroup and its orthogonal complement in Section 7.1, the geometric characterization of the equivalence classes of the quotient spaces in Section 7.3, the construction of quotient Möbius gyrogroups in Section 7.5, and the characterization of Möbius fiber bundles induced from Möbius addition in the last section.

In this paper we study the Möbius gyrogroup only from the algebraic point of view. However, we would like to point out that the theory of Möbius gyrogroups has applications in analysis and signal processing. For example, in [5] the first author used the approach of gyrogroups encoded in the conformal group of the unit sphere in $\mathbb{R}^n$, the so-called proper Lorentz group, to define spherical continuous wavelet transforms on the unit sphere via sections of a quotient Möbius gyrogroup.

2. Gyrogroups

Gyrogroups are an extension of the notion of group by introducing a gyroautomorphism to compensate the lack of associativity. If the gyroautomorphisms are all reduced to the identity map the gyrogroup becomes a group.

Definition 1. (See [17].) A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms:

(G1) There is at least one element 0 satisfying $0 \oplus a = a$, for all $a \in G$;
(G2) For each $a \in G$ there is an element $\ominus a \in G$ such that $\ominus a \oplus a = 0$;
(G3) For any $a, b, c \in G$ there exists a unique element $\mathrm{gyr}[a,b]c \in G$ such that the binary operation satisfies the *left gyroassociative law*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c; \tag{3}$$

(G4) *The map* $\mathrm{gyr}[a,b]: G \to G$ *given by* $c \mapsto \mathrm{gyr}[a,b]c$ *is an automorphism of the groupoid* $(G, \oplus)$*, that is* $\mathrm{gyr}[a,b] \in Aut(G, \oplus)$;
(G5) *The gyroautomorphism* $\mathrm{gyr}[a,b]$ *possesses the left loop property*

$$\mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b]. \tag{4}$$

Definition 2. A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation satisfies

$$a \oplus b = \mathrm{gyr}[a,b](b \oplus a), \quad \forall a, b \in G.$$

The solution of the basic equations in a gyrogroup is given by Ungar.

Proposition 3. *(See [17].) Let $(G, \oplus)$ be a gyrogroup, and let $a, b, c \in G$. The unique solution of the equation $a \oplus x = b$ in G for the unknown x is $x = \ominus a \oplus b$ and the unique solution of the equation $x \oplus a = b$ in G for the unknown x is $x = b \ominus \mathrm{gyr}[b,a]a$.*

The gyrosemidirect product is a generalization of the external semidirect product of groups and it gives rise to the construction of groups.

Proposition 4. *(See [17].) Let $(G, \oplus)$ be a gyrogroup, and let $Aut_0(G, \oplus)$ be a gyroautomorphism group of G (any subgroup of $Aut(G)$ which contains all the gyroautomorphisms $\mathrm{gyr}[a, b]$ of G, with $a, b \in G$). Then the gyrosemidirect product $G \times Aut_0(G)$ is a group, with group operation given by the gyrosemidirect product*

$$(x, X)(y, Y) = \big(x \oplus Xy, \mathrm{gyr}[x, Xy]XY\big). \tag{5}$$

3. Clifford algebras

In this section we will consider the structure of a real Clifford algebra over an inner vector space V (see e.g. [4,12]). With the embedding of V into its real Clifford algebra we will give, in the next section, a nice description of the gyrogroup structure of the paravector ball of $\mathbb{R} \oplus V$ in terms of Clifford addition operator and Möbius transformations. Moreover, the description of gyrations will be made by elements of the Spoin group, which is the double covering group of the rotation group $SO(\mathbb{R} \oplus V)$ in paravector space.

Let V be an n-dimensional real inner product space and let $\{e_j\}_{j=1}^n$ be an orthonormal basis of V. We denote the inner product and the norm on V by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. The Clifford algebra $\mathcal{C}\ell_{0,n}$ on V is the associative algebra generated by V and $\mathbb{R}$ subject to the relation

$$v^2 = -\|v\|^2, \quad \text{for all } v \in V.$$

This last relation implies

$$uv + vu = -2\langle u, v\rangle.$$

Therefore, we have the following relations: $e_je_k + e_ke_j = 0$, $j \neq k$, and $e_j^2 = -1$, $j = 1, \dots, n$. The Clifford algebra $\mathcal{C}\ell_{0,n}$ admits a basis of the form $e_\alpha = e_{\alpha_1} \dots e_{\alpha_k}$, $\alpha = \{\alpha_1, \dots, \alpha_k\}$, with $1 \leqslant \alpha_1 < \dots < \alpha_k \leqslant n$, and $e_\emptyset = 1$. Thus, an arbitrary element $x \in \mathcal{C}\ell_{0,n}$ can be written as $x = \sum x_\alpha e_\alpha$, $x_\alpha \in \mathbb{R}$. In V we can define the geometric product

$$uv = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu), \tag{6}$$

which is composed by the symmetric part $\frac{1}{2}(uv + vu) = -\langle u, v\rangle$ and the anti-symmetric part $\frac{1}{2}(uv - vu) := u \wedge v$, also known as the outer product.

In the Clifford algebra $\mathcal{C}\ell_{0,n}$ the principal automorphism satisfies $v' = -v$, $v \in V$ and $(ab)' = a'b'$, $a, b \in \mathcal{C}\ell_{0,n}$ and the reversion (or principal anti-automorphism) satisfies $v^* = v$, $v \in V$ and $(uv)^* = v^*u^*$, $u, v \in \mathcal{C}\ell_{0,n}$, and these involutions are extended by linearity to the whole Clifford algebra. Their composition is the unique anti-automorphism satisfying $\bar{u} = -u$, $u \in V$ and $\overline{uv} = \bar{v}\bar{u}$. We will denote by Δ the Clifford norm function $\Delta : \mathcal{C}\ell_{0,n} \to \mathcal{C}\ell_{0,n}$, defined by $\Delta(v) = v\bar{v}$. The norm function satisfies the properties $\Delta(ab) = \Delta(a)\Delta(b)$ if $\Delta(a) \in \mathbb{R}$ or $\Delta(b) \in \mathbb{R}$, $\Delta(a') = \Delta(a^*) = \Delta(\bar{a}) = \Delta(a)$, and $\Delta(\lambda a) = \lambda^2\Delta(a)$, $\lambda \in \mathbb{R}$, cf. [12, 5.14–5.16]. Moreover, if $\Delta(a) \neq 0$, then a is invertible and the inverse is given by $a^{-1} := (1/\Delta(a))\bar{a}$.

In this paper we will work in the subspace $\mathbb{R} \oplus V \subset \mathcal{C}\ell_{0,n}$ of paravectors. From now on we will denote $W = \mathbb{R} \oplus_{ds} V$ reserving the symbol $\oplus$ for Möbius addition and denoting by $\oplus_{ds}$ the direct sum of vector spaces. The quadratic space (W, Δ) arises naturally as an extension of $(V, \|\cdot\|)$. An element of W will be denoted by $x = x_0 + \mathbf{x}$ and it satisfies $\Delta(x) = |x_0|^2 + \Delta(\mathbf{x})$, $x_0 \in \mathbb{R}$ and $\mathbf{x} \in V$. Thus, any non-zero paravector is invertible and the inverse is given by $x^{-1} = \frac{\bar{x}}{\Delta(x)}$. To keep the same notations as in [6] we shall also denote $\|x\|^2 := \Delta(x)$, $x \in W$. The extension of the geometric product (6) to the paravector case is given by

$$x\bar{y} = \frac{1}{2}(x\bar{y} + y\bar{x}) + \frac{1}{2}(x\bar{y} - y\bar{x}). \tag{7}$$

The symmetric part of $x\bar{y}$ in (7) defines a positive bilinear form on W:

$$\langle x, y\rangle := \frac{1}{2}(x\bar{y} + y\bar{x}). \tag{8}$$

From (8) two paravectors are orthogonal if and only if $x\bar{y} = y\bar{x}$. The anti-symmetric part of $x\bar{y}$ is $\frac{1}{2}(x\bar{y} - y\bar{x})$. It represents the directed plane in paravector space that contains x and y. It is also called a biparavector. Biparavectors arise most frequently as operators in paravectors since they generate rotations in the paravector space. This anti-symmetric part allow us to characterize the parallelism between two paravectors:

$$x \parallel y \quad \Leftrightarrow \quad x\bar{y} = y\bar{x}. \tag{9}$$

The Spoin group is defined by

$$\mathrm{Spoin}(V) = \{w_1 \cdots w_k\colon\ w_i \in W,\ \Delta(w_i) = 1\}.$$

It is the double covering group of the special orthogonal group $SO(W)$ (cf. [12, 6.12]), with the double covering map given by

$$\begin{aligned} \sigma : \mathrm{Spoin}(V) &\to SO(W), \\ s &\mapsto \sigma(s) \end{aligned} \tag{10}$$

with $\sigma(s)(w) = sws^*$ for any $w \in \mathbb{R} \oplus V$.

If $\widetilde{P} \subseteq W$ is a closed subspace of W then we denote by $\mathrm{Spoin}(\widetilde{P})$ the corresponding Spoin group. It holds $\mathrm{Spoin}(\widetilde{P}) \subseteq \mathrm{Spoin}(V)$.

4. Möbius addition in the paravector ball

In [19] Ungar introduced the Möbius addition in the ball of any real inner product space. In this section we will provide basic knowledge for the Möbius gyrogroup in the framework of Clifford algebra theory. Although the results in this section are known, we prefer to give the proofs with the tool of Clifford algebra.

Starting from the paravector space W embedded into the Clifford algebra $\mathcal{C}\ell_{0,n}$ we consider the Möbius transformation on the unit ball $\mathbb{B}_1 = \{x \in W\colon\ \|x\| < 1\}$ of W defined by

$$\varphi_a(b) := (a + b)(1 + \bar{a}b)^{-1}, \quad a, b \in \mathbb{B}_1. \tag{11}$$

For more details about Möbius transformations on the unit ball see e.g. [1–3]. From now on we will always use the notation $\mathbb{B}_t$ as the open ball of radius $t > 0$ in the paravector space W, that is, $\mathbb{B}_t = \{a \in W\colon\ \|a\| < t\}$.

Definition 5. The Möbius transformation on $\mathbb{B}_t$ is defined by

$$\Psi_a(z) := t\varphi_{\frac{a}{t}}\left(\frac{z}{t}\right). \tag{12}$$

The Möbius transformation Ψ_a is a bijection on $\mathbb{B}_t$ with inverse mapping given by $\Psi_a^{-1}(z) = \Psi_{-a}(z)$. Moreover, it holds $\Psi_a(0) = a$ and $\Psi_a(-a) = 0$.

Definition 6. The Möbius addition on $\mathbb{B}_t$ is defined by

$$a \oplus b := \Psi_a(b), \quad a, b \in \mathbb{B}_t. \tag{13}$$

Proposition 7. *Möbius addition in* (13) *can be written as*

$$a \oplus b = \frac{(1 + \frac{2}{t^2}\langle a, b\rangle + \frac{1}{t^2}\|b\|^2)a + (1 - \frac{1}{t^2}\|a\|^2)b}{1 + \frac{2}{t^2}\langle a, b\rangle + \frac{1}{t^4}\|a\|^2\|b\|^2}. \tag{14}$$

Proof. By definition we have

$$a \oplus b = \Psi_a(b) = t\varphi_{\frac{a}{t}}\left(\frac{b}{t}\right) = (a+b)\left(1 + \frac{\bar{a}b}{t^2}\right)^{-1}. \tag{15}$$

First we observe that the above expression is well defined. Since $1 + \frac{\bar{a}b}{t^2} = 0$ if and only if $b = -t^2\frac{a}{\|a\|^2}$ and $\|b\| = \frac{t^2}{\|a\|} > t$ we conclude that $1 + \frac{\bar{a}b}{t^2}$ is invertible on $\mathbb{B}_t$.

Moreover, by direct computations we have

$$\begin{aligned} a\bar{b}a &= \left(\frac{1}{2}(a\bar{b} + b\bar{a}) + \frac{1}{2}(a\bar{b} - b\bar{a})\right)a \\ &= \left(\langle a, b\rangle + \frac{1}{2}(a\bar{b} - b\bar{a})\right)a \\ &= \langle a, b\rangle a + \frac{1}{2}a\bar{b}a - \frac{1}{2}\|a\|^2 b \end{aligned}$$

so that $a\bar{b}a = 2\langle a, b\rangle a - \|a\|^2 b$. Therefore, we obtain

$$\begin{aligned} a \oplus b = \Psi_a(b) = t^2\frac{(a+b)(t^2 + \bar{b}a)}{\|t^2 + \bar{b}a\|^2} &= \frac{t^2(t^2 a + a\bar{b}a + t^2 b + \|b\|^2 a)}{t^4 + 2t^2\langle a, b\rangle + \|a\|^2\|b\|^2} \\ &= \frac{t^2(t^2 + 2\langle a, b\rangle + \|b\|^2)a + t^2(t^2 - \|a\|^2)b}{t^4 + 2t^2\langle a, b\rangle + \|a\|^2\|b\|^2} \\ &= \frac{(1 + \frac{2}{t^2}\langle a, b\rangle + \frac{1}{t^2}\|b\|^2)a + (1 - \frac{1}{t^2}\|a\|^2)b}{1 + \frac{2}{t^2}\langle a, b\rangle + \frac{1}{t^4}\|a\|^2\|b\|^2}. \qquad \square \end{aligned}$$

In the limit $t \to +\infty$, the ball $\mathbb{B}_t$ expands to the whole of its space W and Möbius addition (14) reduces to vector addition in W.

Theorem 8. $(\mathbb{B}_t, \oplus)$ *is a gyrogroup.*

Proof. Axioms (G1) and (G2) of Definition 1 are easy to prove since the neutral element is 0 and each $a \in \mathbb{B}_t$ has an inverse given by $-a$. Now we will prove the left gyroassociative law, which is given in our case by

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \left(qcq^*\right), \quad \text{with } q = \frac{t^2 + a\bar{b}}{|t^2 + a\bar{b}|}. \tag{16}$$

Indeed, on the one hand, we have

$$\begin{aligned} a\oplus(b\oplus c) &= a\oplus\left((b+c)\left(1+\frac{\bar{b}c}{t^2}\right)^{-1}\right)\\ &= \left(a+(b+c)\left(1+\frac{\bar{b}c}{t^2}\right)^{-1}\right)\left(1+\frac{1}{t^2}\bar{a}(b+c)\left(1+\frac{\bar{b}c}{t^2}\right)^{-1}\right)^{-1}\\ &= \left(a\left(1+\frac{\bar{b}c}{t^2}\right)+b+c\right)\left(1+\frac{\bar{b}c}{t^2}\right)^{-1}\left(1+\frac{\bar{b}c}{t^2}\right)\left(1+\frac{\bar{b}c}{t^2}+\frac{\bar{a}b}{t^2}+\frac{\bar{a}c}{t^2}\right)^{-1}\\ &= \left(a+b+c+\frac{a\bar{b}c}{t^2}\right)\left(1+\frac{1}{t^2}(\bar{b}c+\bar{a}b+\bar{a}c)\right)^{-1}. \end{aligned}$$

On the other hand, starting from the identities

$$a\oplus b=(a+b)\left(1+\frac{\bar{a}b}{t^2}\right)^{-1}=\left(1+\frac{b\bar{a}}{t^2}\right)^{-1}(a+b) \tag{17}$$

and

$$qcq^*=\left(1+\frac{a\bar{b}}{t^2}\right)c\left(1+\frac{\bar{a}b}{t^2}\right)^{-1} \tag{18}$$

we obtain

$$\begin{aligned} (a\oplus b)\oplus\left(qcq^*\right) &= \left((a+b)\left(1+\frac{\bar{a}b}{t^2}\right)^{-1}+qcq^*\right)\left(1+\frac{1}{t^2}\overline{(a+b)\left(1+\frac{\bar{a}b}{t^2}\right)^{-1}}qcq^*\right)^{-1}\\ &= \left((a+b)\left(1+\frac{\bar{a}b}{t^2}\right)^{-1}+qcq^*\right)\left(1+\frac{1}{t^2}\overline{\left(1+\frac{b\bar{a}}{t^2}\right)^{-1}(a+b)}qcq^*\right)^{-1}\\ &= \left((a+b)\left(1+\frac{\bar{a}b}{t^2}\right)^{-1}+\left(1+\frac{a\bar{b}}{t^2}\right)c\left(1+\frac{\bar{a}b}{t^2}\right)^{-1}\right)\\ &\quad\times\left(1+\frac{1}{t^2}(\bar{a}+\bar{b})\left(1+\frac{a\bar{b}}{t^2}\right)^{-1}\left(1+\frac{a\bar{b}}{t^2}\right)c\left(1+\frac{\bar{a}b}{t^2}\right)^{-1}\right)^{-1}\\ &= \left(a+b+\left(1+\frac{a\bar{b}}{t^2}\right)c\right)\left(1+\frac{\bar{a}b}{t^2}\right)^{-1}\left(1+\frac{\bar{a}b}{t^2}\right)\left(1+\frac{\bar{a}b}{t^2}+\frac{1}{t^2}(\bar{a}+\bar{b})c\right)^{-1}\\ &= \left(a+b+c+\frac{a\bar{b}c}{t^2}\right)\left(1+\frac{1}{t^2}(\bar{b}c+\bar{a}b+\bar{a}c)\right)^{-1}. \end{aligned}$$

This proves axiom (G3).

Thus, gyrations are given by

$$\mathrm{gyr}[a,b]c=qcq^* \tag{19}$$

for all $a, b, c \in \mathbb{B}_t$, with

$$q = \frac{t^2 + a\bar{b}}{|t^2 + a\bar{b}|} = \frac{a}{|a|} \frac{t^2 a^{-1} + \bar{b}}{|t^2 a^{-1} + \bar{b}|}.$$

Therefore, q is a product of unit paravectors, which means that $q \in \mathrm{Spoin}(V)$. Hence, qcq^* is an orthogonal transformation on W and this proves axiom (G4).

It only remains to check axiom (G5). It is easy to show that

$$qcq^* = \left(1 + \frac{b\bar{a}}{t^2}\right)^{-1} c \left(1 + \frac{\bar{b}a}{t^2}\right) = \left(1 + \frac{a\bar{b}}{t^2}\right) c \left(1 + \frac{\bar{a}b}{t^2}\right)^{-1}. \tag{20}$$

From (17), (19) and (20) we obtain

$$\begin{aligned}
\mathrm{gyr}[a \oplus b, b]c &= \left(1 + \frac{1}{t^2}(a+b)\left(1 + \frac{\bar{a}b}{t^2}\right)^{-1} \bar{b}\right) c \left(1 + \frac{1}{t^2}\overline{(a+b)\left(1 + \frac{\bar{a}b}{t^2}\right)^{-1} b}\right)^{-1} \\
&= \left(1 + \frac{1}{t^2}\left(1 + \frac{b\bar{a}}{t^2}\right)^{-1} (a+b)\bar{b}\right) c \left(1 + \frac{1}{t^2}\left(1 + \frac{\bar{b}a}{t^2}\right)^{-1} (\bar{a}+\bar{b})b\right)^{-1} \\
&= \left(1 + \frac{b\bar{a}}{t^2}\right)^{-1} \left(1 + \frac{b\bar{a}}{t^2} + \frac{1}{t^2}\left(a\bar{b} + \|b\|^2\right)\right) c \left(1 + \frac{\bar{b}a}{t^2} + \frac{1}{t^2}\left(\bar{a}b + \|b\|^2\right)\right)^{-1} \left(1 + \frac{\bar{b}a}{t^2}\right) \\
&= \left(1 + \frac{b\bar{a}}{t^2}\right)^{-1} \left(1 + \frac{2}{t^2}\langle a, b\rangle + \frac{\|b\|^2}{t^2}\right) c \left(1 + \frac{2}{t^2}\langle a, b\rangle + \frac{\|b\|^2}{t^2}\right)^{-1} \left(1 + \frac{\bar{b}a}{t^2}\right) \\
&= \left(1 + \frac{b\bar{a}}{t^2}\right)^{-1} c \left(1 + \frac{\bar{b}a}{t^2}\right) \\
&= \mathrm{gyr}[a, b]c.
\end{aligned}$$

In the first step we used (17) and (20) while in the second case we used (19). □

Remark 9. From the proof of Theorem 8 we know that Möbius gyrations are given by

$$\mathrm{gyr}[a, b]c = qcq^* \quad \text{with } q = \frac{t^2 + a\bar{b}}{|t^2 + a\bar{b}|}.$$

5. Properties of the gyrogroup $(\mathbb{B}_t, \oplus)$

Although Möbius addition is non-associative and non-commutative it is crucial for applications to know when elements in Möbius gyrogroup are associative or commutative. In this section we first give the characterization of the associativity and commutativity of the elements of the Möbius gyrogroup $(\mathbb{B}_t, \oplus)$. Next, for applications, we introduce a homomorphism of $\mathrm{Spoin}(V)$ onto the Möbius gyrogroup $(\mathbb{B}_t, \oplus)$.

Lemma 10. *Let $a, b, c \in B_t$. Then*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c,$$

if and only if either $\langle a, c\rangle = \langle b, c\rangle = 0$ or $a \parallel b$.

Proof. We have to solve the equation $qcq^* = c$. Computing the left-hand side we have

$$qcq^* = \frac{t^2 + a\bar{b}}{|t^2 + a\bar{b}|} c \frac{t^2 + \bar{b}a}{|t^2 + a\bar{b}|} = \frac{t^4 c + t^2 c\bar{b}a + t^2 a\bar{b}c + a\bar{b}c\bar{b}a}{t^4 + 2t^2\langle a, b\rangle + \|a\|^2\|b\|^2}. \tag{21}$$

Since $a\bar{b} + b\bar{a} = \bar{a}b + \bar{b}a = 2\langle a, b\rangle$ we obtain

$$\begin{aligned} a\bar{b}c &= \big(2\langle a, b\rangle - b\bar{a}\big)c \\ &= 2\langle a, b\rangle c - b\bar{a}c \\ &= 2\langle a, b\rangle c - b\big(2\langle a, c\rangle - \bar{c}a\big) \\ &= 2\langle a, b\rangle c - 2\langle a, c\rangle b + b\bar{c}a \\ &= 2\langle a, b\rangle c - 2\langle a, c\rangle b + \big(2\langle b, c\rangle - c\bar{b}\big)a \\ &= 2\langle a, b\rangle c - 2\langle a, c\rangle b + 2\langle b, c\rangle a - c\bar{b}a \end{aligned}$$

and

$$\begin{aligned} a\bar{b}c\bar{b}a &= a\bar{b}\big(2\langle c, b\rangle - b\bar{c}\big)a \\ &= 2\langle c, b\rangle a\bar{b}a - \|b\|^2 a\bar{c}a \\ &= 2\langle c, b\rangle\big(2\langle a, b\rangle - b\bar{a}\big)a - \|b\|^2\big(2\langle a, c\rangle - c\bar{a}\big)a \\ &= 4\langle c, b\rangle\langle a, b\rangle a - 2\langle b, c\rangle\|a\|^2 b - 2\langle a, c\rangle\|b\|^2 a + \|a\|^2\|b\|^2 c. \end{aligned}$$

Then

$$\begin{aligned} qcq^* = {} & \frac{(t^4 + 2t^2\langle a, b\rangle + \|a\|^2\|b\|^2)c - 2(t^2\langle a, c\rangle + \langle b, c\rangle\|a\|^2)b}{t^4 + 2t^2\langle a, b\rangle + \|a\|^2\|b\|^2} \\ & + \frac{2(\langle b, c\rangle(t^2 + 2\langle a, b\rangle) - \langle a, c\rangle\|b\|^2)a}{t^4 + 2t^2\langle a, b\rangle + \|a\|^2\|b\|^2}. \end{aligned} \tag{22}$$

Thus, $qcq^* = c$ if and only if

$$\big(t^2\langle a, c\rangle + \langle b, c\rangle\|a\|^2\big)b = \big(\langle b, c\rangle\big(t^2 + 2\langle a, b\rangle\big) - \langle a, c\rangle\|b\|^2\big)a.$$

The last equality is true when $\langle c, a\rangle = 0$ and $\langle b, c\rangle = 0$ or $a = \lambda b$, for some $\lambda \in \mathbb{R}$. □

By Definition 2 the gyrogroup $(\mathbb{B}_t, \oplus)$ is gyrocommutative since it satisfies the relation

$$a \oplus b = q(b \oplus a)q^*, \quad \text{with } q = \frac{t^2 + a\bar{b}}{|t^2 + a\bar{b}|}. \tag{23}$$

For the commutativity of the elements of $(\mathbb{B}_t, \oplus)$ we have the following result.

Lemma 11. *Let $a, b \in B_t$. Then*

$$a \oplus b = b \oplus a \quad \Leftrightarrow \quad a \parallel b.$$

Proof. From (17) it is easy to see that $a \oplus b = b \oplus a$ if and only if $a\bar{b} = b\bar{a}$, which means that a and b are parallel by (9). □

Next we define a homomorphism of Spoin(V) onto the gyrogroup $(\mathbb{B}_t, \oplus)$.

Lemma 12. *For any* $s \in \mathrm{Spoin}(V)$ *and* $a, b \in \mathbb{B}_t$ *we have*

$$s(a \oplus b)s^* = \left(sas^*\right) \oplus \left(sbs^*\right). \tag{24}$$

Proof. By (15) we have

$$\begin{aligned}\left(sas^*\right) \oplus \left(sbs^*\right) &= \left(sas^* + sbs^*\right)\left(1 + \frac{\overline{sas^*}sbs^*}{t^2}\right)^{-1} \\ &= s(a+b)s^*\left(1 + \frac{\overline{s^*}\bar{a}\bar{s}sbs^*}{t^2}\right)^{-1} \\ &= s(a+b)s^*\left(s^*\right)^{-1}\left(1 + \frac{\bar{a}b}{t^2}\right)^{-1}\left(\overline{s^*}\right)^{-1} \\ &= s(a \oplus b)s^*. \qquad \square\end{aligned}$$

Remark 13. The identity (24) has two equivalent forms

(i) $$\left(sas^*\right) \oplus b = s\left(a \oplus \left(\bar{s}b\overline{s^*}\right)\right)s^*; \tag{25}$$

(ii) $$a \oplus \left(sbs^*\right) = s\left(\left(\bar{s}a\overline{s^*}\right) \oplus b\right)s^*. \tag{26}$$

Applying the left gyroassociative law (16) we can deduce the left and the right cancellation laws:

$$(-b) \oplus (b \oplus a) = a; \tag{27}$$

$$(a \oplus b) \oplus \left(q(-b)q^*\right) = a, \tag{28}$$

for all $a, b \in \mathbb{B}_t$, with $q = \frac{t^2 + a\bar{b}}{|t^2 + a\bar{b}|}$.

Since Spoin(V) is an automorphism group that contains all the gyrations (19), from Proposition 4, we obtain that Spoin$(V) \times \mathbb{B}_t$ is a group for the gyrosemidirect product given by

$$(s_1, a) \times (s_2, b) = \left(s_1 s_2 q, b \oplus \left(\overline{s_2} a \overline{s_2^*}\right)\right), \quad \text{with } q = \frac{t^2 + \overline{s_2} a \overline{s_2^*}\bar{b}}{|t^2 + \overline{s_2} a \overline{s_2^*} b|}. \tag{29}$$

6. Gyro-subgroups of $(\mathbb{B}_t, \oplus)$

A gyrogroup has sub-structures like subgroups or gyro-subgroups. In [6] we have proposed a definition for gyro-subgroups.

Definition 14. (See [6].) Let $(G, \oplus)$ be a gyrogroup and K a non-empty subset of G. K is a gyro-subgroup of $(G, \oplus)$ if it is a gyrogroup for the operation induced from G and $\mathrm{gyr}[a, b] \in Aut(K)$ for all $a, b \in K$.

The next theorem gives the characterization of all Möbius gyro-subgroups of $(\mathbb{B}_t, \oplus)$.

Theorem 15. *A non-empty subset P of $\mathbb{B}_t$ is a Möbius gyro-subgroup of $(\mathbb{B}_t, \oplus)$ if and only if $P = \widetilde{P} \cap \mathbb{B}_t$, where $\widetilde{P}$ is a closed subspace of W.*

Proof. If $\widetilde{P}$ is a closed subspace of W then by (14) we have that $a \oplus b \in \widetilde{P}$, for any $a, b \in \widetilde{P}$ and therefore, $a \oplus b \in P$, for any $a, b \in P$, where $P = \widetilde{P} \cap \mathbb{B}_t$. Moreover, by (22) we have that $\mathrm{gyr}[a, b]c = qcq^* \in P$, for any $a, b, c \in P$. Thus, we conclude that $(P, \oplus)$ is a gyro-subgroup of $(\mathbb{B}_t, \oplus)$. Conversely, if $(P, \oplus)$ is a gyro-subgroup of $(\mathbb{B}_t, \oplus)$ then its linear extension to W, $\widetilde{P}$, is a subspace of W. As in any finite-dimensional normed linear space all subspaces are closed, we conclude that $\widetilde{P}$ is a closed subspace of W. □

Corollary 16. *If $(P, \oplus)$ is a gyro-subgroup of $(\mathbb{B}_t, \oplus)$ then $(P^{\perp}, \oplus)$ is also a gyro-subgroup of $(\mathbb{B}_t, \oplus)$.*

From now on we will denote by $\widetilde{P}$ a closed subspace of W such that $P = \widetilde{P} \cap \mathbb{B}_t$ is a Möbius gyro-subgroup.

Definition 17. The dimension of the Möbius gyro-subgroup P is defined by

$$\dim P = \dim \widetilde{P}.$$

As observed in [6] the subspaces $L_\omega = \{a \in \mathbb{B}_t\colon\ a = \lambda\omega,\ |\lambda| < t\}$, where $\omega \in \mathbb{S}\colon = \{x \in W\colon \|x\| = 1\}$, give rise to subgroups. Indeed, they are the only subgroups of $(\mathbb{B}_t, \oplus)$ as shown by the next proposition.

Proposition 18. *A Möbius gyro-subgroup P of $\mathbb{B}_t$ is a subgroup of $\mathbb{B}_t$ if and only if $P = L_\omega$ for some $\omega \in \mathbb{S}$.*

Proof. If $\dim P = 1$ then $P = L_\omega$, for some $\omega \in \mathbb{S}$. Therefore, by Lemma 10 we have the associativity of the Möbius addition in L_ω.

If $\dim P > 1$ then the Möbius addition is not associative in P. Otherwise, for any $a, b, c \in P$ we would have

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c. \tag{30}$$

But, if we take non-zero and non-parallel $a, b \in P$ and $c = a$ then by Lemma 10 we obtain $\langle a, a\rangle = 0$, i.e. $a = 0$, which is a contradiction to our choice of a. □

7. Decompositions and factorizations of $(\mathbb{B}_t, \oplus)$

7.1. Möbius decompositions of B_t

From now on we assume that V is a finite-dimensional real Hilbert space. Thus, if $\widetilde{P} \subseteq W$ is a closed subspace of W then $\widetilde{P}^{\perp}$ is closed and it holds the orthogonal decomposition

$$W = \widetilde{P} \oplus_{ds} \widetilde{P}^{\perp}. \tag{31}$$

Since $(W, +)$ is the limit case of $(\mathbb{B}_t, \oplus)$ when $t \to \infty$, it is natural to consider the decompositions of $\mathbb{B}_t$ with respect to Möbius addition (14). We would like to mention that there is a decomposition theory for groups into twisted subgroups, which are related with gyrogroups [7,8]. However, there is no decomposition theory for gyrogroups into gyro-subgroups.

In this section we will give the corresponding Möbius decompositions of the paravector ball $\mathbb{B}_t$ with respect to gyro-subgroups. Our starting point will be (31). Since in the Möbius case the addition is non-commutative we will have to consider factorizations both from the left and from the right.

To keep the notation clear we will use the symbol l to denote the left case and the symbol r to the right case. The next theorem gives the decomposition of $\mathbb{B}_t$ with respect to an arbitrary Möbius gyro-subgroup. By restricting to the vector case and $\dim P = 1$ we recover the known result in [6].

Theorem 19. *Let P be a Möbius gyro-subgroup of $\mathbb{B}_t$ and $P^\perp$ its orthogonal complement in $\mathbb{B}_t$. Then for each $c \in \mathbb{B}_t$ there exist unique $b, u \in P$ and $a, v \in P^\perp$ such that $c = a \oplus b$ and $c = u \oplus v$.*

Proof. First we prove the existence of the decomposition $c = a \oplus b$. Let $\widetilde{P}$ be the linear extension of P to W and $c \in \mathbb{B}_t$ be arbitrary. Since $W = \widetilde{P} \oplus_{ds} \widetilde{P}^\perp$ then there exist unique $c_1 \in \widetilde{P}$ and $c_2 \in \widetilde{P}^\perp$ such that $c = c_1 + c_2$. Moreover, as $c \in \mathbb{B}_t$ it follows that $c_1 \in P$ and $c_2 \in P^\perp$. If $c_1 = 0$ then we can take $b = 0$ and $a = c_2$. If $c_1 \neq 0$ we take $b = \alpha_1 c_1 \in P$ and $a = \beta_1 c_2 \in P^\perp$ such that

$$c = c_1 + c_2 = (\beta_1 c_2) \oplus (\alpha_1 c_1) = \frac{(t^2 - \beta_1^2 \|c_2\|^2) t^2 \alpha_1}{t^4 + \beta_1^2 \alpha_1^2 \|c_1\|^2 \|c_2\|^2} c_1 + \frac{(t^2 + \alpha_1^2 \|c_1\|^2) t^2 \beta_1}{t^4 + \beta_1^2 \alpha_1^2 \|c_1\|^2 \|c_2\|^2} c_2. \tag{32}$$

In the last step we used formula (14) for the Möbius addition. Now we have to find β_1 and α_1 satisfying (32). The resulting system of equations has a unique solution given by

$$\alpha_1 = \frac{-t^2 + \|c\|^2 + \sqrt{(t^2 - \|c\|^2)^2 + 4t^2 \|c_1\|^2}}{2\|c_1\|^2} \tag{33}$$

and

$$\beta_1 = \frac{2t^2}{t^2 + \|c\|^2 + \sqrt{(t^2 - \|c\|^2)^2 + 4\|c_1\|^2 t^2}}. \tag{34}$$

It is easy to verify that $\|b\| = \alpha_1 \|c_1\| < t$ and $\|a\| = \beta_1 \|c_2\| < t$ since $\|c\| < t$ and $\|c_1\| \neq 0$. Thus, we proved the existence of the first decomposition.

To prove the uniqueness of the decomposition we suppose that there exist $a, d \in P^\perp$ and $b, f \in P$ such that $c = a \oplus b = d \oplus f$. Then $b = (-a) \oplus (d \oplus f)$, by (27). As $a \perp f$ and $d \perp f$ we have $b = ((-a) \oplus d) \oplus f$, by Lemma 10. Since by hypothesis $b, f \in P$ then $(-a) \oplus d$ must be an element of P. This is true if and only if $(-a) \oplus d = 0$. This implies $a = d$ and consequently $b = 0 \oplus f = f$, as we wish to prove.

To prove the decomposition $c = u \oplus v$ we have again two cases: if $c_2 = 0$ then we take $v = 0$ and $u = c_1$, otherwise we consider $u = \alpha_2 c_1 \in P$ and $v = \beta_2 c_2 \in P^\perp$ such that

$$c = c_1 + c_2 = (\alpha_2 c_1) \oplus (\beta_2 c_2) = \frac{(t^2 + \beta_2^2 \|c_2\|^2) t^2 \alpha_2}{t^4 + \alpha_2^2 \beta_2^2 \|c_1\|^2 \|c_2\|^2} c_1 + \frac{(t^2 - \alpha_2^2 \|c_1\|^2) t^2 \beta_2}{t^4 + \alpha_2^2 \beta_2^2 \|c_1\|^2 \|c_2\|^2} c_2. \tag{35}$$

Now we have to find α_2 and β_2 satisfying (35). The resulting system of equations has an unique solution given by

$$\beta_2 = \frac{-t^2 + \|c\|^2 + \sqrt{(t^2 - \|c\|^2)^2 + 4\|c_2\|^2 t^2}}{2\|c_2\|^2} \tag{36}$$

and

$$\alpha_2 = \frac{2t^2}{t^2 + \|c\|^2 + \sqrt{(t^2 - \|c\|^2)^2 + 4\|c_2\|^2 t^2}}. \tag{37}$$

Again it is easy to verify that $\|u\| = \alpha_2\|c_1\| < t$ and $\|v\| = \beta_2\|c_2\| < t$ since $\|c\| < t$ and $\|c_2\| \neq 0$. Thus, the existence of the decomposition $c = u \oplus v$ is proved. The proof of the uniqueness of this decomposition is analogous to the previous one. □

By the previous theorem $\mathbb{B}_t$ has two unique decompositions $\mathbb{B}_t = P \oplus P^\perp = P^\perp \oplus P$. The relation between them is given more precisely in the next theorem.

Theorem 20. *Let $a, b \in \mathbb{B}_t$ non-zero such that $a \perp b$. Then*

$$a \oplus b = \big(\lambda(a,b)b\big) \oplus \big(\mu(a,b)a\big) \tag{38}$$

with

$$\lambda(a,b) = \frac{2t^2(t^2 - \|a\|^2)}{\sqrt{(t^2 - \|b\|^2)^2(t^2 + \|a\|^2)^2 + 16\|a\|^2\|b\|^2t^4} + (t^2 + \|a\|^2)(t^2 + \|b\|^2)} \tag{39}$$

and

$$\mu(a,b) = \frac{(t^2 - \|a\|^2)(\|b\|^2 - t^2) + \sqrt{(t^2 - \|b\|^2)^2(t^2 + \|a\|^2)^2 + 16\|a\|^2\|b\|^2t^4}}{2\|a\|^2(t^2 + \|b\|^2)}. \tag{40}$$

Proof. Let $c \in \mathbb{B}_t$ with $c \notin P$ and $c \notin P^\perp$. By (32) and (35), we can write

$$c = \beta_1 c_2 \oplus \alpha_1 c_1 = \alpha_2 c_1 \oplus \beta_2 c_2$$

with $c_1 \in P$, $c_2 \in P^\perp$, and α_1, α_2, β_1, β_2 being given in (33), (34), (36), and (37). Now we take $a = \beta_1 c_2$ and $b = \alpha_1 c_1$. By direct computations, we have

$$\lambda(a,b)b = \alpha_2 c_1, \qquad \mu(a,b)a = \beta_2 c_2. \qquad \square$$

Remark 21. Since $(\mathbb{B}_t, \oplus)$ is a gyrocommutative group we know that

$$\begin{aligned} a \oplus b &= \mathrm{gyr}[a,b](b \oplus a) \\ &= \big(\mathrm{gyr}[a,b]b\big) \oplus \big(\mathrm{gyr}[a,b]a\big), \end{aligned}$$

and thus, the gyration operator plays the role of changing the order in Möbius addition. However it does not provide the decomposition of the form $P \oplus P^\perp$ as shown in (38).

7.2. Möbius orthogonal projectors

From (33), (34), (36), and (37) we can define Möbius orthogonal projectors for $\mathbb{B}_t$, with respect to the gyro-subgroups P and $P^\perp$. Projections to the gyro-subgroup P will be denoted by $\mathbb{P}_t^r$ and $\mathbb{P}_t^l$ and projections to $P^\perp$ will be denoted by $\mathbb{Q}_t^r$ and $\mathbb{Q}_t^l$.

In the first case, $\mathbb{B}_t = P^\perp \oplus P$, we obtain the Möbius orthogonal decomposition in $\mathbb{B}_t$

$$\mathbb{I} = \mathbb{Q}_t^l \oplus \mathbb{P}_t^r, \tag{41}$$

namely, $c = \mathbb{Q}_t^l(c) \oplus \mathbb{P}_t^r(c)$, $\forall c \in \mathbb{B}_t$. Here, the operators $\mathbb{P}_t^r : \mathbb{B}_t \to P$ and $\mathbb{Q}_t^l : \mathbb{B}_t \to P^\perp$ are defined by

$$\mathbb{P}_t^r(c) = \frac{-t^2 + \|c\|^2 + \sqrt{(t^2 - \|c\|^2)^2 + 4\|c_1\|^2 t^2}}{2\|c_1\|^2} c_1 \tag{42}$$

and

$$\mathbb{Q}_t^l(c) = \frac{2t^2}{t^2 + \|c\|^2 + \sqrt{(t^2 - \|c\|^2)^2 + 4\|c_1\|^2 t^2}} c_2, \tag{43}$$

where $c = c_1 + c_2 \in \mathbb{B}_t$, with $c_1 \in P$ and $c_2 \in P^\perp$.

Notice that when $c_1 = 0$, we have

$$\mathbb{P}_t^r(c_1 + c_2)|_{c_1=0} = \lim_{\|c_1\| \to 0} \mathbb{P}_t^r(c_1 + c_2).$$

Theorem 22. *For any $a \in P^\perp$ and $b \in P$ we have*

$$\mathbb{P}_t^r(a \oplus b) = b, \qquad \mathbb{Q}_t^l(a \oplus b) = a,$$

$$\mathbb{P}_t^r(b \oplus a) = \mu(b, a)b, \qquad \mathbb{Q}_t^l(b \oplus a) = \lambda(b, a)a,$$

where λ and μ are given by (39) *and* (40) *with the order of a and b being changed.*

In the second case, $\mathbb{B}_t = P \oplus P^\perp$ we obtain the Möbius orthogonal decomposition in $\mathbb{B}_t$

$$\mathbb{I} = \mathbb{P}_t^l \oplus \mathbb{Q}_t^r, \tag{44}$$

where

$$\mathbb{P}_t^l(c) = \frac{2t^2}{t^2 + \|c\|^2 + \sqrt{(t^2 - \|c\|)^2 + 4\|c_2\|^2 t^2}} c_1 \tag{45}$$

and

$$\mathbb{Q}_t^r(c) = \frac{-t^2 + \|c\|^2 + \sqrt{(t^2 - \|c\|^2)^2 + 4\|c_2\|^2 t^2}}{2\|c_2\|^2} c_2, \tag{46}$$

where $c = c_1 + c_2 \in \mathbb{B}_t$, with $c_1 \in P$ and $c_2 \in P^\perp$.

Here, in this case, when $c_2 = 0$ we have

$$\mathbb{Q}_t^r(c_1 + c_2)|_{c_2=0} = \lim_{\|c_2\| \to 0} \mathbb{Q}_t^r(c_1 + c_2).$$

Theorem 23. *For any $a \in P^\perp$ and $b \in P$ we have*

$$\mathbb{P}_t^l(b \oplus a) = b, \qquad \mathbb{Q}_t^r(b \oplus a) = a,$$

$$\mathbb{P}_t^l(a \oplus b) = \lambda(a, b)b, \qquad \mathbb{Q}_t^r(a \oplus b) = \mu(a, b)a,$$

where λ and μ are given by (39) *and* (40)*.*

All the operators are projectors so that the following identities hold

$$\left(\mathbb{P}_t^r\right)^2 = \mathbb{P}_t^r, \qquad \left(\mathbb{P}_t^l\right)^2 = \mathbb{P}_t^l, \qquad \left(\mathbb{Q}_t^r\right)^2 = \mathbb{Q}_t^r, \qquad \left(\mathbb{P}_t^l\right)^2 = \mathbb{P}_t^l. \tag{47}$$

In the limit case, when $t \to \infty$ we recover the Euclidean projectors, since $\mathbb{P}_\infty^l(c) = c_1$, $\mathbb{Q}_\infty^l(c) = c_2$, $\mathbb{P}_\infty^r(c) = c_1$, and $\mathbb{Q}_\infty^r(c) = c_2$. Thus, in the Euclidean case we have that $\mathbb{P}_\infty^l = \mathbb{P}_\infty^r$ and $\mathbb{Q}_\infty^l = \mathbb{Q}_\infty^r$. Therefore, (41) and (44) reduce to

$$\mathbb{I} = \mathbb{Q}_\infty^l + \mathbb{P}_\infty^r = \mathbb{P}_\infty^l + \mathbb{Q}_\infty^r \quad \text{in } W. \tag{48}$$

7.3. *Factorizations of* $(\mathbb{B}_t, \oplus)$ *by* $(P, \oplus)$

In the Euclidean case the factorization of W by an arbitrary subgroup $\widetilde{P}$ with respect to vector addition in W is defined by the equivalence relation:

$$\forall u, v \in W, \quad u \sim v \quad \Leftrightarrow \quad \exists w \in \widetilde{P}\colon u = v + w. \tag{49}$$

In the Möbius case we cannot use (49) to obtain the factorization of $\mathbb{B}_t$ by an arbitrary gyro-subgroup P due to the non-associativity of the Möbius addition. To define equivalence relations on $\mathbb{B}_t$, we adopt a constructive approach by providing convenient partitions of $\mathbb{B}_t$. We will consider first left cosets.

Lemma 24. *Let P be an arbitrary Möbius gyro-subgroup of $\mathbb{B}_t$. If $a \in P^\perp$ and $b, c \in P$ such that $a \oplus b = c$ then $a = 0$ and $b = c$.*

Proof. By (14) we have

$$c = a \oplus b = \frac{1 + \frac{1}{t^2}\|b\|^2}{1 + \frac{1}{t^4}\|a\|^2\|b\|^2} a + \frac{1 - \frac{1}{t^2}\|a\|^2}{1 + \frac{1}{t^4}\|a\|^2\|b\|^2} b.$$

Since $a \in P^\perp$ and $b, c \in P$ by assumption it follows that $a \in P^\perp \cap P = \{0\}$. Then $a = 0$ and $b = c$. □

Proposition 25. *Let P be an arbitrary Möbius gyro-subgroup of $\mathbb{B}_t$. Then the family $\{a \oplus P\colon a \in P^\perp\}$ is a disjoint partition of $\mathbb{B}_t$, i.e.*

$$\mathbb{B}_t = \bigcup_{a \in P^\perp} (a \oplus P).$$

Proof. We first prove that this family is indeed disjoint. Let $a, c \in P^\perp$ with $a \neq c$ and assume that $(a \oplus P) \cap (c \oplus P) \neq \emptyset$. Then there exists $f \in \mathbb{B}_t$ such that $f = a \oplus b$ and $f = c \oplus d$ for some $b, d \in P$. By (27) and Lemma 10 we have that

$$b = (-a) \oplus (c \oplus d) = \big((-a) \oplus c\big) \oplus d, \tag{50}$$

since $\langle -a, d\rangle = 0$ and $\langle c, d\rangle = 0$. Due to $a, c \in P^\perp$ we have $(-a) \oplus c \in P^\perp$. Then (50) and Lemma 24 imply $(-a) \oplus c = 0$, i.e. $a = c$. But this contradicts our assumption. Thus, $(a \oplus P) \cap (c \oplus P) = \emptyset$ provided $a, c \in P^\perp$ and $a \neq c$. Finally, by Theorem 19 we have that $\mathbb{B}_t = \bigcup_{a \in P^\perp}(a \oplus P)$. □

This partition induces a left equivalence relation $\sim_l$ on $\mathbb{B}_t$:

$$\forall c, d \in \mathbb{B}_t, \quad c \sim_l d \quad \Leftrightarrow \quad c \oplus P = d \oplus P. \tag{51}$$

Corollary 26. *The space $(\mathbb{B}_t/P, \sim_l)$ is a left coset space of $\mathbb{B}_t$ whose cosets are of the form $a \oplus P$ with $a \in P^\perp$.*

By Corollary 26 we have the following bijection:

$$(\mathbb{B}_t/P, \sim_l) \cong P^\perp.$$

The next proposition gives a geometric characterization of the cosets $a \oplus P$ in terms of a curve under the action of the Spoin group.

Theorem 27. *Let $a \in P^\perp$ and $c \in P$ be fixed. Then*

$$a \oplus P = \big\{\sigma(s)\gamma_{(a,c)}\colon\ s \in \operatorname{Spoin}(P)\big\},$$

where the curve

$$\gamma_{(a,c)} := \left\{a \oplus \left(\alpha \frac{c}{\|c\|}\right)\colon\ |\alpha| < t\right\}$$

is in the sphere orthogonal to the boundary of $\mathbb{B}_t$, with center at $C = \frac{t^2+\|a\|^2}{2\|a\|^2}a$ and radius $\tau = \frac{t^2-\|a\|^2}{2\|a\|}$.

Proof. Let $a \in P^\perp$ and $\omega = \frac{c}{\|c\|} \in \mathbb{S}$. Since

$$a \oplus (\alpha\omega) = \frac{(t^2+\alpha^2)t^2}{t^4+\|a\|^2\alpha^2}a + \frac{(t^2-\|a\|^2)t^2}{t^4+\|a\|^2\alpha^2}\alpha\omega \tag{52}$$

we know that $\gamma_{(a,c)}$ is a curve inside $\mathbb{B}_t$ in the $\omega\xi$-plane. For any $b \in P$ we take $\alpha = \|b\|$. Since $\|b\| = \|\alpha\omega\|$, there exists an orthogonal transformation $O \in SO(n)$ such that $b = O(\alpha\omega)$ and O leaves P invariant and takes each element of $P^\perp$ as a fixed point. This means that each $b \in P$ can be written as $b = s(\alpha\omega)s^*$ for some $s \in \operatorname{Spoin}(P)$ fixing each element of $P^\perp$. In particular $\bar{s}a\bar{s}^* = a$ since $a \in P^\perp$. By (26) we have

$$a \oplus b = a \oplus \big(s(\alpha\omega)s^*\big) = s\big(\big(\bar{s}a\bar{s}^*\big) \oplus (\alpha\omega)\big)s^* = s\big(a \oplus (\alpha\omega)\big)s^*.$$

Thus, $a \oplus P$ is given by the action of the group $\operatorname{Spoin}(P)$ on the curve $\gamma_{(a,c)}$, i.e.,

$$a \oplus P = \big\{s\big(a \oplus (\alpha\omega)\big)s^*\colon\ |\alpha| < t,\ s \in \operatorname{Spoin}(P)\big\}.$$

By (53) and (52) we have

$$\begin{aligned}
\big\|(a \oplus b) - C\big\|^2 &= \big\|\big(a \oplus \big(s(\alpha\omega)s^*\big)\big) - C\big\|^2 \\
&= \left\|\left(\frac{(t^2+\alpha^2)t^2}{t^4+\|a\|^2\alpha^2} - \frac{t^2+\|a\|^2}{2\|a\|^2}\right)a + \frac{(t^2-\|a\|^2)t^2\alpha}{t^4+\|a\|^2\alpha^2}s\omega s^*\right\|^2 \\
&= \left(\frac{(t^2+\alpha^2)t^2}{t^4+\|a\|^2\alpha^2} - \frac{t^2+\|a\|^2}{2\|a\|^2}\right)^2\|a\|^2 + \left(\frac{(t^2-\|a\|^2)t^2\alpha}{t^4+\|a\|^2\alpha^2}\right)^2 \\
&= \frac{(t^2-\|a\|^2)^2}{4\|a\|^2}
\end{aligned} \tag{53}$$

for all $|\alpha| < t$ and all $s \in \mathrm{Spoin}(P)$. Since (53) is independent of α and s we conclude that all points of $a \oplus P$ belong to the sphere centered at $C = \frac{t^2+\|a\|^2}{2\|a\|^2}a$ and radius $\tau = \frac{t^2-\|a\|^2}{2\|a\|}$. To prove the orthogonality between this sphere and the boundary of $\mathbb{B}_t$ we will use the well-known fact that two spheres S_1 and S_2, with centers A_1 and A_2 and radii τ_1 and τ_2, respectively, intersect orthogonally if and only if $\langle A_1 - y, A_2 - y\rangle = 0$ for any $y \in S_1 \cap S_2$, or equivalently, if and only if

$$\|A_1 - A_2\|^2 = \tau_1^2 + \tau_2^2. \tag{54}$$

As in our case $\|C - 0\|^2 = \tau^2 + t^2$ we conclude our result. □

Since $(\mathbb{B}_t, \oplus)$ is a gyrocommutative gyrogroup we can consider right coset spaces arising from the decomposition of $\mathbb{B}_t$ by P. The results are analogous to the left case and therefore the proofs will be omitted.

Proposition 28. *The family $\{P \oplus a\colon\ a \in P^{\perp}\}$ is a disjoint partition of $\mathbb{B}_t$, i.e.*

$$\mathbb{B}_t = \bigcup_{a \in P^{\perp}} (P \oplus a).$$

This partition induces a left equivalence relation $\sim_r$ on $\mathbb{B}_t$:

$$\forall c, d \in \mathbb{B}_t, \quad c \sim_r d \quad \Leftrightarrow \quad P \oplus c = P \oplus d. \tag{55}$$

Corollary 29. *The space $(\mathbb{B}_t/P, \sim_r)$ is a left coset space of $\mathbb{B}_t$ whose cosets are of the form $P \oplus a$ with $a \in P^{\perp}$.*

From Proposition 28 we obtain the bijection

$$(\mathbb{B}_t/P, \sim_r) \cong P^{\perp}.$$

Next we will give a geometric characterization of the cosets $P \oplus a$.

Theorem 30. *Let $a \in P^{\perp}$ and $c \in P$ be fixed. Then*

$$P \oplus a = \big\{\sigma(s)\Gamma_{(a,c)}\colon\ s \in \mathrm{Spoin}(P)\big\},$$

where the curve

$$\Gamma_{(a,c)} := \left\{\left(\alpha \frac{c}{\|c\|}\right) \oplus a\colon\ |\alpha| < t\right\}$$

is in the sphere with center at $C_1 = \frac{\|a\|^2-t^2}{2\|a\|^2}a$ and radius $\tau_1 = \frac{t^2+\|a\|^2}{2\|a\|}$.

Remark 31. Surprisingly left and right cosets have different geometric behavior. We have shown that the left cosets $a \oplus P$ are orthogonal to the boundary of $\mathbb{B}_t$, however, the right cosets $P \oplus a$ are not orthogonal to ∂B_t as shown by (54), since $\|C_1 - 0\|^2 \neq t^2 + \tau_1^2$ for $C_1 = \frac{\|a\|^2-t^2}{2\|a\|^2}a$ and $\tau_1 = \frac{t^2+\|a\|^2}{2\|a\|}$.

7.4. *Extension of the cosets $P \oplus a$ and $a \oplus P$ to the whole space W*

From (53) we observe that the restriction $|\alpha| < t$ is not used in the proof of Theorem 27. This means that we can consider the linear extension on W of P to $\widetilde{P}$ obtaining the extension of the cosets $a \oplus P$ to $a \oplus \widetilde{P}$.

Proposition 32. *The coset $a \oplus P$ is the restriction of $a \oplus \widetilde{P}$ to $\mathbb{B}_t$, i.e.,*

$$a \oplus P = (a \oplus \widetilde{P}) \cap B_t.$$

Proof. The inclusion $a \oplus P \subset (a \oplus \widetilde{P}) \cap B_t$ is obvious since $P = \widetilde{P} \cap \mathbb{B}_t$. For the converse if $c \in (a \oplus \widetilde{P}) \cap \mathbb{B}_t$ then $c = a \oplus b$, for some $b \in \widetilde{P}$. As $c \in \mathbb{B}_t$ and $a \in P^{\perp} \subset \mathbb{B}_t$ then $b \in \mathbb{B}_t$. Thus, $b \in \widetilde{P} \cap \mathbb{B}_t = P$, which proves that $c \in a \oplus P$. $\square$

Let $\widetilde{P}$ be a subspace of $\mathbb{R} \oplus V$ with $\dim \widetilde{P} = k$. Without loss of generality we will assume $1 \leqslant k \leqslant n-1$. The k-dimensional ball in $\widetilde{P}$ with center at the point C and radius τ is denoted by $B^k(C,\tau)$ while its boundary, the $(k-1)$-dimensional sphere centered at C with radius τ, will be denoted by $S^{k-1}(C,\tau)$.

We can now characterize the extended cosets $a \oplus \widetilde{P}$.

Theorem 33. *Let $a \in P^{\perp}$ and $c \in \widetilde{P}$ be fixed. Then*

$$a \oplus \widetilde{P} = \bigl\{\sigma(s)\gamma_{(a,c)}\colon\ s \in \mathrm{Spoin}(P)\bigr\} = S^k(C,\tau) \backslash \left\{\frac{t^2}{\|a\|^2}a\right\},$$

where

$$\gamma_{(a,c)} := \left\{a \oplus \left(\alpha \frac{c}{\|c\|}\right)\colon\ \alpha \in \mathbb{R}\right\}$$

is a circle and $S^k(C,\tau)$ is the k-dimensional sphere orthogonal to the boundary of $\mathbb{B}_t$, with center at $C = \frac{t^2+\|a\|^2}{2\|a\|^2}a$ and radius $\tau = \frac{t^2-\|a\|^2}{2\|a\|}$.

Proof. The first identity

$$a \oplus \widetilde{P} = \bigl\{\sigma(s)\gamma_{(a,c)}\colon\ s \in \mathrm{Spoin}(P)\bigr\}$$

and the orthogonality of the sphere $S^k(C,\tau)$ with the boundary of $\mathbb{B}_t$ are true by following the same reasonings as in the proof of Theorem 27.

For the second identity we firstly prove that

$$\sigma\bigl(\mathrm{Spoin}(\widetilde{P})\bigr)\gamma_{(a,c)} \subset \langle a, \widetilde{P}\rangle \cap S(C,\tau) \backslash \left\{\frac{t^2}{\|a\|^2}a\right\},$$

where $\langle a, \widetilde{P}\rangle$ stands for the subspace of W generated by a and $\widetilde{P}$ and $S(C,\tau)$ is the sphere in W with $C = \frac{t^2+\|a\|^2}{2\|a\|^2}a$ and $\tau = \frac{t^2-\|a\|^2}{2\|a\|}$. Since $a \perp \widetilde{P}$ and C is a multiple of a, it follows that

$$\sigma\bigl(\mathrm{Spoin}(\widetilde{P})\bigr)S(C,\tau) = S(C,\tau)$$

and

$$\sigma\big(\mathrm{Spoin}(\widetilde{P})\big)\langle a, \widetilde{P}\rangle = \langle a, \widetilde{P}\rangle.$$

As shown in the proof of Theorem 27, we have

$$\gamma_{(a,c)} \subset \langle a, \widetilde{P}\rangle \cap S(C,\tau). \tag{56}$$

Combining the above facts together we have

$$\sigma\big(\mathrm{Spoin}(\widetilde{P})\big)\gamma_{(a,c)} \subset \langle a, \widetilde{P}\rangle \cap S(C,\tau).$$

Since when $|\alpha| \to \infty$ we have that $a \oplus c = \frac{t^2}{\|a\|^2}a$ we know that $\gamma_{(a,c)} \cup \{\frac{t^2}{\|a\|^2}a\}$ is a circle. From (56) two circles coincide, i.e.,

$$\gamma_{(a,c)} \cup \left\{\frac{t^2}{\|a\|^2}a\right\} = \langle a, c\rangle \cap S(C,\tau). \tag{57}$$

Since $\mathrm{Spoin}(\widetilde{P})$ leaves a as well as the point $\frac{t^2}{\|a\|^2}a$ invariant we finally conclude that

$$\sigma\big(\mathrm{Spoin}(\widetilde{P})\big)\gamma_{(a,c)} \subset \langle a, \widetilde{P}\rangle \cap S(C,\tau)\backslash\left\{\frac{t^2}{\|a\|^2}a\right\}.$$

Now we prove the reverse inclusion. For any $b \in \langle a, \widetilde{P}\rangle \cap S(C,\tau)\backslash\{\frac{t^2}{\|a\|^2}a\}$, since $a \perp \widetilde{P}$ we can write $b = \lambda_1 a + b_1$ for some $b_1 \in \widetilde{P}$. There exists $O \in SO(\widetilde{P})$ such that Ob_1 is a multiple of $c \in \widetilde{P}$. Now we take $\widehat{b} = Ob$. Then

$$\widehat{b} = \lambda_1 a + Ob_1 \in \langle a, c\rangle \cap S(C,\tau)\backslash\left\{\frac{t^2}{\|a\|^2}a\right\}.$$

Therefore, from (57) we have

$$b = O^{-1}\widehat{b} \in \sigma\big(\mathrm{Spoin}(\widetilde{P})\big)\widehat{b} \subset \sigma\big(\mathrm{Spoin}(\widetilde{P})\big)\gamma_{(a,c)}. \qquad \square$$

By Theorem 33 and Proposition 32 we obtain a new characterization of the cosets $a \oplus P$.

Corollary 34. *We have*

$$a \oplus P = S^k(C,\tau) \cap B_t,$$

where $S^k(C,\tau)$ is the k-dimensional sphere orthogonal to the boundary of $\mathbb{B}_t$, with center at $C = \frac{t^2+\|a\|^2}{2\|a\|^2}a$ and radius $\tau = \frac{t^2-\|a\|^2}{2\|a\|}$.

Remark 35. Similar results also hold for the right cosets $P \oplus a$.

7.5. Quotient Möbius gyrogroups

In this subsection we will consider the gyrogroup structure of the quotient spaces $(\mathbb{B}_t/P, \sim_l)$ and $(\mathbb{B}_t/P, \sim_r)$ and the action of $(\mathbb{B}_t, \oplus)$ on them.

In the limit case $t \to \infty$, $\widetilde{P}$ is a normal subgroup in $\mathbb{R} \oplus V$, so that left and right cosets are equal and coincide with $W/\widetilde{P}$, which is a quotient group under the usual binary operation defined by

$$(a \oplus \widetilde{P}) + (b \oplus \widetilde{P}) = (a + d) + \widetilde{P}, \quad a, d \in \widetilde{P}^{\perp}. \tag{58}$$

Although in the Möbius gyrogroup case left and right cosets are different we can define a binary operation on the quotient spaces $(\mathbb{B}_t/P, \sim_l)$ and $(\mathbb{B}_t/P, \sim_r)$ such that the resulting structure is a gyrogroup. We will call these spaces *quotient Möbius gyrogroups*.

Proposition 36. *The quotient space $(\mathbb{B}_t/P, \sim_l)$, endowed with the binary operation $\oplus$ defined by*

$$(a \oplus P) \oplus (d \oplus P) := (a \oplus d) \oplus P, \quad \text{with } a, d \in P^{\perp}, \tag{59}$$

is a quotient Möbius gyrogroup.

Proof. The first two axioms of Definition 1 are obviously true since the coset $0 \oplus P$ is the left identity

$$(0 \oplus P) \oplus (a \oplus P) = (0 \oplus a) \oplus P = a \oplus P$$

and left inverse of the coset $a \oplus P$ is the coset $(-a) \oplus P$ because

$$\big((-a) \oplus P\big) \oplus (a \oplus P) = \big((-a) \oplus a\big)P = 0 \oplus P.$$

With respect to the gyroassociative law we have

$$\begin{aligned}(a \oplus P) \oplus \big[(b \oplus P) \oplus (c \oplus P)\big] &= \big(a \oplus (b \oplus c)\big) \oplus P \\ &= \big((a \oplus b) \oplus qcq^*\big) \oplus P, \quad \text{by (16)} \\ &= \big[(a \oplus P) \oplus (b \oplus P)\big] \oplus \big(\big(qcq^*\big) \oplus P\big)\end{aligned}$$

with $q = \frac{t^2 + a\bar{b}}{|t^2 + a\bar{b}|}$ and $a, b, c \in P^{\perp}$. Note that qcq^* is an element of $P^{\perp}$ since $a, b, c \in P^{\perp}$. Gyrations over the quotient space $(\mathbb{B}_t/P, \sim_l)$ are defined by

$$\operatorname{gyr}[a, b](c \oplus P) := \big(\operatorname{gyr}[a, b]c\big) \oplus P = \big(qcq^*\big) \oplus P, \quad a, b, c \in P^{\perp}$$

and belong to the group of automorphisms of $(\mathbb{B}_t/P, \sim_l)$. Finally the loop property (4) holds since

$$\begin{aligned}\operatorname{gyr}[a \oplus b, b](c \oplus P) &= \big(\operatorname{gyr}[a \oplus b, b]c\big) \oplus P \\ &= \big(\operatorname{gyr}[a, b]c\big) \oplus P \\ &= \operatorname{gyr}[a, b](c \oplus P). \qquad \square\end{aligned}$$

Analogous result holds true for the right quotient space $(\mathbb{B}_t/P, \sim_r)$.

Proposition 37. *The quotient space* $(\mathbb{B}_t/P, \sim_r)$ *endowed with the binary operation* $\oplus$ *defined by*

$$(P \oplus a) \oplus (P \oplus d) = P \oplus (a \oplus d) \quad \text{with } a, d \in P^{\perp} \tag{60}$$

is a quotient Möbius gyrogroup.

It is readily seen that the gyrogroup operations (59) and (60) reduce to the group operation (58) in the limit case.

To define appropriately the action of $(\mathbb{B}_t, \oplus)$ on the quotient Möbius gyrogroups $(\mathbb{B}_t/P, \sim_l)$ and $(\mathbb{B}_t/P, \sim_r)$ let us see first what happens in the limit case. For $u = v + w$ with $v \in \widetilde{P}$ and $w \in \widetilde{P}^{\perp}$, the action of W on the quotient group $(W)/\widetilde{P}$ is defined by

$$u + (a + \widetilde{P}) := (w + a) + \widetilde{P}. \tag{61}$$

From this observation, the Möbius orthogonal projectors defined in Section 7.2 can be used to define a transitive action of $\mathbb{B}_t$ on $(\mathbb{B}_t/P, \sim_l)$ and $(\mathbb{B}_t/P, \sim_r)$.

Proposition 38. *The action of the gyrogroup* $(\mathbb{B}_t, \oplus)$ *on the quotient Möbius gyrogroup* $(\mathbb{B}_t/P, \sim_l)$ *defined by*

$$c \oplus S_a^l := S^l_{\mathbb{Q}_t^l(c) \oplus a}$$

is transitive.

Proof. Let $a \oplus P$ and $d \oplus P$ be two arbitrary cosets of $(\mathbb{B}_t/P, \sim_l)$. We want to find $c \in \mathbb{B}_t$ such that $c \oplus (a \oplus P) = d \oplus P$, that is,

$$\big(\mathbb{Q}_t^l(c) \oplus a\big) \oplus P = d \oplus P.$$

This is true if and only if $\mathbb{Q}_t^l(c) \oplus a = d$. By Proposition 3 we have that

$$\mathbb{Q}_t^l(c) = d \ominus \mathrm{gyr}[d, a]a = d \oplus q(-a)q^*,$$

with $q = \frac{t^2 - d\bar{a}}{|t^2 - d\bar{a}|}$. Therefore, all the points $c \in (d \oplus q(-a)q^*) \oplus P$ are solution for our problem. □

Analogously, using the Möbius orthogonal projector $\mathbb{Q}_t^r$ we can define a transitive action of $(\mathbb{B}_t, \oplus)$ on $(\mathbb{B}_t/P, \sim_r)$.

Proposition 39. *The action of the gyrogroup* $(\mathbb{B}_t, \oplus)$ *on the quotient Möbius gyrogroup* $(\mathbb{B}_t/P, \sim_r)$ *defined by*

$$c \oplus (P \oplus a) := P \oplus \big(a \oplus \mathbb{Q}_t^r(c)\big)$$

is transitive.

8. Möbius fiber bundles

We denote $(\mathbb{B}_t, X, \pi, Y)$ as a fiber bundle with base space X, fiber Y and bundle map $\pi : \mathbb{B}_t \to X$. A global section of the fiber bundle $(\mathbb{B}_t, X, \pi, Y)$ is a continuous map $f : X \to \mathbb{B}_t$ such that $\pi(f(y)) = y$ for all $y \in X$, while a local section is a map $f : U \to \mathbb{B}_t$, where U is an open set in X and $\pi(f(x)) = x$ for all $x \in U$.

For the construction of these sections in the vector case and applications to spherical continuous wavelet transforms we refer to [6,5].

We have four different fiber bundle structures on $\mathbb{B}_t$ with fiber bundle mappings given by

$$\begin{aligned} \pi_1 : P \oplus P^\perp &\to (\mathbb{B}_t/P, \sim_r), & \pi_2 : P^\perp \oplus P &\to (\mathbb{B}_t/P, \sim_l), \\ b \oplus a &\mapsto [a] = P \oplus a, & a \oplus b &\mapsto [a] = a \oplus P, \\ \pi_3 : P^\perp \oplus P &\to (\mathbb{B}_t/P, \sim_r), & \pi_4 : P \oplus P^\perp &\to (\mathbb{B}_t/P, \sim_l), \\ a \oplus b &\mapsto [a] = P \oplus a, & b \oplus a &\mapsto [a] = a \oplus P. \end{aligned}$$

It is easy to see that the first and the second bundles are trivial ones. The first bundle is isomorphic to the trivial bundle defined by the Möbius projector

$$Q_t^r : P \oplus P^\perp \to P^\perp.$$

Hence, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{B}_t = P \oplus P^\perp & \xrightarrow{\pi_1} & (\mathbb{B}_t/P, \sim_r) \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \Phi_1 \\ P \oplus P^\perp & \xrightarrow{Q_t^r} & P^\perp \end{array}$$

where $\Phi_1(P \oplus a) = a$ for any $a \in P^\perp$. All global sections of the first fiber bundle are given by

$$f(P \oplus a) = g\big(\Phi_1(P \oplus a)\big) \oplus \Phi_1(P \oplus a) = g(a) \oplus a,$$

for any continuous map $g : P^\perp \to P$.

The second bundle is isomorphic to the trivial bundle defined by the Möbius projector

$$Q_t^l : P^\perp \oplus P \to P^\perp.$$

Indeed, the following diagram commutes

$$\begin{array}{ccc} \mathbb{B}_t = P^\perp \oplus P & \xrightarrow{\pi_2} & (\mathbb{B}_t/P, \sim_l) \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \Phi_2 \\ P^\perp \oplus P & \xrightarrow{Q_t^l} & P^\perp \end{array}$$

where $\Phi_2(a \oplus P) = a$ for any $a \in P^\perp$. All global sections of the second fiber bundle are given by

$$f(a \oplus P) = \Phi_2(a \oplus P) \oplus g\big(\Phi_2(a \oplus P)\big) = a \oplus g(a),$$

for any continuous map $g : P^{\perp} \to P$.

In the third and fourth bundles we will consider the sections of the form $b \oplus a$ or $a \oplus b$. In the third case if we consider the map $\tau_b^{(1)}$ defined by

$$\tau_b^{(1)} : (\mathbb{B}_t/P, \sim_r) \to B_t,$$
$$[a] \mapsto a \oplus b$$

for any $b \in P$ fixed and $a \in P^{\perp}$ we obtain a global section. Clearly, $\pi_3 \tau_b^{(1)}([a]) = [a]$ for any $a \in P^{\perp}$, which means that $\tau_b^{(1)}$ is a global section for any $b \in P$.

However, if we consider the map $\tau_b^{(2)}$ defined for any $b \in P \backslash \{0\}$ by

$$\tau_b^{(2)} : (\mathbb{B}_t/P, \sim_r) \to B_t,$$
$$[a] \mapsto b \oplus a$$

we obtain only a local section. By Theorem 20 we have

$$\pi_3\big(\tau_b^{(2)}\big([a]\big)\big) = \pi_3(b \oplus a) = \pi_3\big(\lambda(b,a)a \oplus \mu(b,a)b\big) = P \oplus \big(\lambda(b,a)a\big)$$

with $\lambda(b,a)$ given by (39) changing the order of b and a. Now, for each t and $b \in P \backslash \{0\}$ fixed $\lambda(b,a)$ reaches a maximum equal to $\frac{t^2-\|b\|^2}{t^2+\|b\|^2}$ in $\|a\| = 0$, i.e., $a = 0$, which is strictly less than one. Hence, for any $a \in P^{\perp}$ we have

$$\big\|\lambda(b,a)a\big\| = \lambda(b,a)\|a\| \leqslant \frac{t^2 - \|b\|^2}{t^2 + \|b\|^2} t,$$

which does not provide all the cosets of $(\mathbb{B}_t/P, \sim_r)$. Hence, the mapping $\tau_b^{(2)}$ is only a local section for the fiber bundle defined by π_3. The case $b = 0$ gives a global section since

$$\pi_3\big(\tau_0^{(2)}\big([a]\big)\big) = \pi_3(0 \oplus a) = \pi_3(a \oplus 0) = [a] \quad \text{for any } a \in P^{\perp}.$$

In the fourth case we consider, for any $b \in P$, the sections $\tau_b^{(3)}$ and $\tau_b^{(4)}$ defined by

$$\begin{aligned} \tau_b^{(3)} : (\mathbb{B}_t/P, \sim_l) &\to B_t, \\ [a] &\mapsto b \oplus a \end{aligned} \quad \text{and} \quad \begin{aligned} \tau_b^{(4)} : (\mathbb{B}_t/P, \sim_l) &\to B_t, \\ [a] &\mapsto a \oplus b. \end{aligned}$$

These are global sections for the fiber bundle defined by π_4. Indeed, as $\pi_4(\tau_b^{(3)}([a])) = [a]$ for any $a \in P^{\perp}$ then $\tau_b^{(3)}$ is a global section for π_4. With respect to $\tau_b^{(4)}$ we observe by Theorem 20 that

$$\pi_4\big(\tau_b^{(4)}\big([a]\big)\big) = \pi_4(a \oplus b) = \pi_4\big(\lambda(a,b)b \oplus \mu(a,b)a\big) = \big(\mu(a,b)a\big) \oplus P$$

with $\mu(a,b)$ given by (40). As for each $b \in P$ fixed we have that

$$0 \leqslant \big\|\mu(a,b)a\big\| = \mu(a,b)\|a\| < t,$$

with $\lim_{\|a\|=0} \mu(a,b)\|a\| = 0$ and $\lim_{\|a\|=t} \mu(a,b)\|a\| = t$ we conclude that $\tau_b^{(4)}$ is a global section for any $b \in P$.

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