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THE NONLINEAR FILTERING PROBLEM FOR THE UNBOUNDED CASE

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The finitely additive nonlinear filtering problem for the model $y_t = h_t(X_t) + e_t$ is solved when the function h is unbounded and satisfies no growth conditions whatever.

nonlinear filtering * finitely additive version of Bayes formula * fundamental solution * PDE

1. Introduction

In [6], we began a systematic study of nonlinear filtering theory with Gaussian white noise (on a finitely additive probability space) replacing the differential of Brownian motion as noise in the conventional model for nonlinear filtering based on stochastic calculus. The finitely additive approach has several advantages over the conventional approach. First, we do not have to enlarge the sample space of observations, but instead, we work with the natural sample space. Secondly, the equations for the optimal filter and the conditional densities turn out to be partial differential equations rather than stochastic partial differential equations. Thus, we are able to derive these equations and characterize the optimum filter (or conditional densities) as the unique solution to these equations. See [6] for complete formulation and various definitions. In [6], X_t was $\mathbb{R}^d$-valued and it was assumed that the function h (in the model (1.1)) is bounded. In [7], we considered the case when the state space of the signal process (X_t) is infinite dimensional and characterized the optimum filter as the unique solution of certain measure valued equations. We now return to the $\mathbb{R}^d$-valued signal process and solve the problem of existence of the unnormalized conditional density and its characterization as the unique solution to a 'Zakai' type equation *without imposing any growth restrictions on h.*

The preparatory results are derived in Section 2. A principal tool, besides the finitely additive version of the Bayes formula, is Theorem 2.1 in which we obtain a 'dual' Feynman–Kac type formula. The work of this paper is based on recent papers of Aronson and Besala, Besala, and Bodanko [1, 4, 5] on the existence and uniqueness of solutions of parabolic equations with unbounded coefficients. These

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results are presented in a form suitable for our purpose in Theorem 2.2. An application of Theorem 2.1 and the Bayes formula yields our main result (Theorem 3.1) on the solution of the nonlinear filtering problem in which the function $h_t(X_t)$ in (1.1) is unbounded and has no growth conditions imposed on it. The conditions of linear growth imposed in Theorem 3.1 on the coefficients of the diffusion equation for the signal process (X_t) can be improved upon using [5] as is indicated in the Remark at the end of Section 3. However, in our view, this is a matter of secondary importance, our main concern being to achieve the maximum generality for the function h and hence for the filtering model.

There have been many recent papers devoted to the case of unbounded h in the nonlinear filtering model using the Ito stochastic calculus (Paradoux [8]; Baras, Blankenship and Hopkins [2]; Baras, Blankenship and Mitter [3]). A detailed discussion of the relationship of this work with the approach of the present paper is given in Section 4.

In the remainder of this section, we briefly describe the model and state the Bayes formula.

Let $H = L^2([0, T], \mathbb{R}^m)$ with the inner product

$$\langle f_1, f_2 \rangle = \int_0^T (f_1(s), f_2(s))\, ds.$$

Let $\mathscr{C}$ be the field of cylinder sets in H and let μ be the canonical Gauss measure on $\mathscr{C}$. Let $e = (e_s)$ be the identity map from H into itself. Then the finite dimensional distributions of the process $\int_0^t e_s\, ds$ on $(H, \mathscr{C}, \mu)$ are the same as that of m-dimensional standard Brownian motion. In this sense (e_s) is the derivative of a 'Brownian Motion' on a finitely additive probability space and thus can be called Gaussian white noise. In [6], we had studied nonlinear filtering theory with (e_s) as the noise. The model we considered was

$$y_s = h_s(X_s) + e_s \tag{1.1}$$

where the signal (X_s) is a $\mathbb{R}^d$-valued process on a countably additive probability space $(\Omega, \mathscr{A}, \Pi)$, (X_s) and (e_s) are independent and $h: [0, T] \times \mathbb{R}^d \to \mathbb{R}^m$ is a measurable function such that

$$\int_0^T |h_s(X_s)|^2\, ds < \infty \quad \text{a.s. } \Pi. \tag{1.2}$$

The following Bayes formula—which is an analogue of the Kallianpur–Striebel formula—is the starting point of our study of nonlinear filtering theory with Gaussian white noise. The formula is given in terms of conditional expectation in the finitely additive set up. See [6] for the definitions and proof.

Theorem 1.1 (Bayes Formula). *Let f be a Borel function on $\mathbb{R}^d$ such that*

$$E|f(X_t)| < \infty.$$

Then

$$E(f(X_t)|y_s: 0 \leq s \leq t) = \frac{\sigma_t(f, y)}{\sigma_t(1, y)} \tag{1.3}$$

where

$$\sigma_t(f, y) = \int f(X_t) \exp\left(\int_0^t (h_s(X_s), y_s)\, ds - \tfrac{1}{2}\int_0^t |h_s(X_s)|^2\, ds\right) d\Pi. \tag{1.4}$$

2. Auxiliary results

Let (X_t), $0 \leq t \leq T$ be an $\mathbb{R}^d$-valued diffusion process on $(\Omega, \mathcal{A}, \Pi)$ with initial probability density ϕ and infinitesimal generator $\mathcal{L}_t$ given by

$$(\mathcal{L}_t g)(x) = \sum_{i,j=1}^{d} a_{ij}(t, x)\left(\frac{\partial^2 g}{\partial x_i\, \partial x_j}\right)(x) + \sum_{i=1}^{d} b_i(t, x)\left(\frac{\partial g}{\partial x_i}\right)(x) \tag{2.1}$$

where $g \in C^2(\mathbb{R}^d)$ and a, b satisfy the following conditions:

$$\sum_{i,j=1}^{d} a_{ij}(t, x)\lambda_i\lambda_j \geq K_1 \sum_{i=1}^{d} \lambda_i^2 \tag{2.2}$$

for some $K_1 > 0$, and all $(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$;

$$a_{ij}, \qquad \frac{\partial}{\partial x_i} a_{ij}, \qquad \frac{\partial^2}{\partial x_i\, \partial x_j} a_{ij}, \qquad b_i, \qquad \frac{\partial}{\partial x_i} b_i \tag{2.3}$$

are locally Hölder continuous functions satisfying the growth condition

$$|g(t, x)| \leq K_2(1 + |x|^2)^{1/2}. \tag{2.4}$$

It may be observed that given a, b satisfying (2.2), (2.3) and a density ϕ, such a process (X_t) exists and can be constructed as a solution to a martingale problem or as a solution to a stochastic differential equation (see [9]).

Let $\mathcal{L}_t^*$ be the adjoint of $\mathcal{L}_t$ given by

$$\begin{aligned}(\mathcal{L}_t^* g)(x) = &\sum_{i,j=1}^{d} a_{ij}(t, x)\left(\frac{\partial^2 g}{\partial x_i\, \partial x_j}\right)(x) \\ &+ \sum_{i=1}^{d} b_i^*(t, x)\left(\frac{\partial g}{\partial x_i}\right)(x) + c^*(t, x) g(x)\end{aligned} \tag{2.5}$$

where

$$b_i^*(t, x) = -b_i(t, x) + 2\sum_{j=1}^{d} \frac{\partial a_{ij}}{\partial x_j}(t, x) \tag{2.6}$$

and

$$c^*(t, x) = -\sum_{i=1}^{d} \frac{\partial b_i(t, x)}{\partial x_i} + \sum_{i,j=1}^{d} \frac{\partial^2 a_{ij}}{\partial x_i\, \partial x_j}(t, x). \tag{2.7}$$

It can be easily checked that (a, b^*) satisfy (2.3) and c^* is also locally Hölder continuous satisfying (2.4).

The main result of this section is

Theorem 2.1. *Let the initial density satisfy*

$$|\phi(x)| \leq \exp(K_3(1+|x|^2)^{(1/2)-\varepsilon}) \tag{2.8}$$

for some $K_3 < \infty$ and $\varepsilon > 0$. Let $c:[0, T]\times\mathbb{R}^d \to \mathbb{R}$ be a locally Hölder continuous function, bounded above. Then

(i) *the PDE*

$$\frac{\partial u}{\partial t} = \mathcal{L}_t^* u(t,\cdot) + c(t,\cdot)u(t,\cdot), \qquad u(0,x) = \phi(x) \tag{2.9}$$

has a unique classical solution in the class $\mathcal{G}$, where $\mathcal{G}$ is the class of $C^{1,2}([0, T]\times\mathbb{R}^d)$ functions g satisfying

$$|g(t,x)| \leq \exp(K_4(1+|x|^2)^{1/2}) \tag{2.10}$$

for some constant K_4.

(ii) *For all bounded Borel measurable functions f, and $0 \leq t_0 \leq T$, we have*

$$\int f(x)u(t_0,x)\,\mathrm{d}x = E_{\Pi}\left[f(X_{t_0})\exp\left(\int_0^{t_0} c(s, X_s)\,\mathrm{d}s\right)\right]. \tag{2.11}$$

We will prove a couple of auxiliary results before proving Theorem 2.1. First observe that it suffices to prove (2.11) for $f \in C_0^\infty(\mathbb{R}^d)$. Thus, fix $f \in C_0^\infty(\mathbb{R}^d)$ and $0 \leq t_0 \leq T$.

Let Lg, L^*g for $g \in C^{1,2}([0, T]\times\mathbb{R}^d)$ be defined by

$$Lg = \left(\mathcal{L}_t^* - \frac{\partial}{\partial t}\right)g + c(t,\cdot)g \tag{2.12}$$

and

$$L^*g = \left(\mathcal{L}_t + \frac{\partial}{\partial t}\right)g + c(t,\cdot)g. \tag{2.13}$$

Then we have

Theorem 2.2. (i) *The equation*

$$Lu = 0 \ \text{ on } \ (0, T)\times\mathbb{R}^d, \qquad u(0,x) = \phi(x) \tag{2.9$'$}$$

has a unique solution in the class $\mathcal{G}$.

(ii) *The equation*

$$L^*v = 0 \ \text{ on } \ (0, t_0)\times\mathbb{R}^d, \qquad v(t_0, x) = f(x) \tag{2.14}$$

has a unique solution in the class of $C^{1,2}([0, t_0]\times\mathbb{R}^d)$ functions satisfying (2.10). Furthermore, the solution v is bounded.

(iii) *Denoting by u, v the solutions to* (2.9)′ *and* (2.14) *respectively, we have*

$$\int f(x)u(t_0, x)\,\mathrm{d}x = \int \phi(z)v(0, z)\,\mathrm{d}z. \tag{2.15}$$

Proof of Theorem 2.2. Let

$$H(t, x) = \exp(K_5(1+|x|^2)^{1/2}\,\mathrm{e}^{\beta t}) \tag{2.16}$$

where K_5 and β are positive constants chosen such that

$$L(H) \le 0, \qquad L^*(H^{-1}) \le 0. \tag{2.17}$$

Such a choice is possible in view of our assumptions on a, b, c. For explicit calculations for the first inequality, see Bodanko [5] and the second inequality can be handled similarly. Thus the conditions (i), (ii), (iii) in Besala [4] are satisfied for L. Let $\Gamma(t, x, \tau, z)$ be the fundamental solution for L given in Theorem 1 in [4]. Then by Theorem 2 in [4], Γ^* defined by

$$\Gamma^*(\tau, z, t, x) = \Gamma(t, x, \tau, z), \quad t > \tau, \tag{2.18}$$

is a fundamental solution for L^*. Observe that (2.8) implies that for some choice of $K_6 < \infty$ we have

$$|\phi(x)| \le K_6 H(0, x) \tag{2.19}$$

and hence by Theorem 3 in [4], u defined by

$$u(t, x) = \int \Gamma(t, x, 0, z)\phi(z)\,\mathrm{d}z \tag{2.20}$$

is a solution to (2.9)′ and $u \in \mathscr{G}$.

By a theorem of Bodanko [5] (see Theorem B, Aronson and Besala [1]) u is the only solution of (2.9)′ in the class $\mathscr{G}$. Also, $f \in C_0^\infty(\mathbb{R}^d)$ implies that, for some $K_7 < \infty$,

$$|f(x)| \le K_7 H^{-1}(t_0, x) \tag{2.21}$$

and hence it follows that v defined by

$$v(\tau, z) = \int \Gamma^*(\tau, z, t_0, x)f(x)\,\mathrm{d}x \equiv \int \Gamma(t_0, x, \tau, z)f(x)\,\mathrm{d}x \quad (t_0 > \tau) \tag{2.22}$$

is the unique solution of (2.14) in the class $\mathscr{G}$. Further, (2.21), (2.22) and the estimate (3.7) in [4] on Γ^* implies that

$$|v(\tau, z)| \le K_7 H^{-1}(\tau, z) \tag{2.23}$$

and hence v is bounded by K_7. Finally, the estimate (3.2) in [4] on Γ implies that

$$\int \left[\int |\Gamma(t_0, x, 0, z)|\,|\phi(z)|\,\mathrm{d}z\right] |f(x)|\,\mathrm{d}x < \infty \tag{2.24}$$

and hence by Fubini's theorem, we have

$$\int f(x)u(t_0,x)\,\mathrm{d}x = \int\left[\int \Gamma(t_0,x,0,z)\phi(z)\,\mathrm{d}z\right]f(x)\,\mathrm{d}x$$

$$= \int\left[\int \Gamma(t_0,x,0,z)f(x)\,\mathrm{d}x\right]\phi(z)\,\mathrm{d}z$$

$$= \int v(0,z)\phi(z)\,\mathrm{d}z.$$

This completes the proof of Theorem 2.2.

Lemma 2.3. *Let $(M(t),\mathscr{F}_t)$ be a continuous local martingale and let $A(t)$ be a continuous $\mathscr{F}_t$-adapted process such that $A(0)$ is integrable and for all w, $t\to A(t,w)$ is of bounded variation on bounded intervals. Then*

$$N(t) = M(t)A(t) - \int_0^t M(s)\,\mathrm{d}A(s)$$

is also a local martingale.

Proof. Choose stopping times τ_n increasing to ∞ such that for all n, $M(t\wedge\tau_n)-M(0)$, $|A|(t\wedge\tau_n)$ are bounded where $|A|(t,w)$ is the total variation of the map $s\to A(s,w)$ on $[0,t]$. Then by integration by parts formula for martingales (Theorem 1.2.8 in Stroock and Varadhan [9]), it follows that $N(t\wedge\tau_n)$ is a martingale and hence the result follows.

Lemma 2.4

$$\int v(0,z)\phi(z)\,\mathrm{d}z = E_{II}\left[f(X_{t_0})\exp\left(\int_0^{t_0} c(\tau,x(\tau))\,\mathrm{d}\tau\right)\right]. \tag{2.25}$$

Proof. Since (X_t) is a diffusion process with generator $\mathscr{L}_t$, it follows that for all $g\in C^{1,2}([0,T]\times\mathbb{R}^d)$ with compact support,

$$g(t,x_t) - \int_0^t\left(\frac{\partial}{\partial\tau}+\mathscr{L}_\tau\right)g(\tau,X_\tau)\,d\tau$$

is a martingale. Thus, by using an obvious stopping time argument, it follows that

$$M(t) = v(t,X_t) - \int_0^t\left(\frac{\partial}{\partial\tau}+\mathscr{L}_\tau\right)v(\tau,X_\tau)\,\mathrm{d}\tau$$

is a local martingale. Let $A(t)$ be the process defined by

$$A(t) = \exp\left(\int_0^t c(s,X_s)\,\mathrm{d}s\right).$$

Then it can be checked that

$$\begin{aligned} N(t) &= M(t)A(t) - \int_0^t M(s)\,\mathrm{d}A(s) \\ &= v(t, X_t)A(t) - \int_0^t \left(\frac{\partial}{\partial\tau} + \mathcal{L}_\tau + c(\tau, X_\tau)\right) v(\tau, X_\tau)A(\tau)\,\mathrm{d}\tau \\ &= v(t, X_t)\exp\left(\int_0^t c(s, X_s)\,\mathrm{d}s\right), \end{aligned}$$

since v is a solution of $L^*v=0$. Hence by Lemma 2.3, $N(t)$ is a local martingale. Since v is bounded and c is bounded above, it follows that $N(t)$ is bounded and hence N is a martingale. Equating its expectations at $t=0$ and $t=t_0$, we get the required identity (2.25). □

Proof of Theorem 2.1. Part (i) follows from (i) of Theorem 2.2. Part (ii) follows from (2.15) and (2.25). □

3. The solution of the nonlinear filtering problem

We now return to the filtering model (1.1). Let the signal process (X_t) be as in Section 2. Let

$$h:[0, T]\times\mathbb{R}^d \to \mathbb{R}^m \tag{3.1}$$

be a locally Hölder continuous function. Since the paths of (X_t) are continuous, we have

$$\int_0^T |h_s(X_s)|^2\,\mathrm{d}s < \infty \quad \text{a.s. } \Pi \tag{3.2}$$

so that we can use the version of the Bayes formula given by equations (1.3) and (1.4). The Bayes formula and Theorem 2.1 yield our main result given below.

Theorem 3.1. *Let (X_t) be a $\mathbb{R}^d$-valued diffusion process with generator $(\mathcal{L}_t)$ satisfying (2.1)–(2.4) and initial density ϕ satisfying (2.8). Let*

$$H_0 = \{y \in H:\ y_t \text{ is Hölder continuous}\}.$$

Then, for all $y \in H_0$, the PDE

$$\frac{\partial p_t(x, y)}{\partial t} = \mathcal{L}_t^* p_t(x, y) + [(h_t(x), y_t) - \tfrac{1}{2}|h_t(x)|^2]p_t(x, y) \tag{3.4}$$

has a unique solution in the class $\mathcal{G}$.

Furthermore, for $y \in H_0$, *the solution* $p_t(x, y)$ *of* (3.4) *is the unnormalized conditional density of* (X_t) *given* $\{y_s : 0 \le s \le t\}$, *i.e. for all Borel measurable bounded* f *and* $y \in H_0$,

$$\sigma_t(f, y) = \int f(x) p_t(x, y)\, \mathrm{d}x. \tag{3.5}$$

Proof. Fix $y \in H_0$. Let

$$c(t, x) = (h_t(x), y_t) - \tfrac{1}{2}|h_t(x)|^2. \tag{3.6}$$

Then $c(t, x)$ is locally Hölder continuous and

$$c(t, x) \le \tfrac{1}{2}|y_t|^2 \le K \tag{3.7}$$

for some $K < \infty$. Now the required assertions follow from Theorem 2.1.

Remark 1. It is easy to see that $y \to \sigma_t(f, y)$ is a continuous function of y for f, t fixed. Since H_0 is dense in H, Theorem 3.1 gives a complete solution to the nonlinear filtering problem.

Remark 2. The growth conditions (2.4) on the diffusion and drift coefficients can be weakened. All that is required is the existence of a function H satisfying (2.17). One such set of conditions is given below (see [5]).

There exists $0 \le \alpha \le 1$ such that, for all i, j, for some $K > 0$,

$$|a_{ij}(t, x)| \le K(1 + |x|^2)^{1-\alpha}, \qquad \left| \frac{\partial}{\partial x_i} a_{ij}(t, x) \right| \le K(1 + |x|^2)^{1/2},$$

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} a_{ij}(t, x) \right| \le K(1 + |x|^2)^{\alpha},$$

$$|b_i(t, x)| \le K(1 + |x|^2)^{1/2}, \qquad \left| \frac{\partial}{\partial x_i} b_i(t, x) \right| \le K(1 + |x|^2)^{\alpha}.$$

For $0 < \alpha \le 1$, for a suitable choice of K_1, β,

$$H(t, x) = \exp(K_1(1 + |x|^2)^{\alpha}\, \mathrm{e}^{\beta t})$$

satisfies (2.17). For $\alpha = 0$, the corresponding choice is

$$H(t, x) = \mathrm{e}^{\beta t}(1 + |x|^2).$$

See Bodanko [5] for the computations in these cases.

4. Concluding remarks

In the conventional approach to nonlinear filtering theory, the canonical model is

$$Y_t = \int_0^t h_s(X_s)\, \mathrm{d}s + \beta_t \tag{4.1}$$

where (X_s) as before is the signal process, assumed to be an $\mathbb{R}^d$-valued diffusion process and the noise β_t is m-dimensional standard Brownian motion. In this case it can be shown that formally, the unnormalized conditional density $p_t(x, Y)$ satisfies the Zakai equation

$$\mathrm{d}p_t(\cdot, Y) = \mathscr{L}_t^* p_t(\cdot, Y)\,\mathrm{d}t + p_t(\cdot, Y)h_t(\cdot)\,\mathrm{d}Y_t. \tag{4.2}$$

If we let

$$\psi_t(x, Y) = \mathrm{e}^{-(h_t(x), Y_t)} p_t(x, Y), \tag{4.3}$$

then it follows that formally, for each $Y \in C([0, T], \mathbb{R}^m) - \psi$ satisfies

$$\frac{\partial}{\partial t}\psi_t(x, Y) = \mathrm{e}^{-(h_t(x), Y_t)} \mathscr{L}_t^* \,\mathrm{e}^{(h_t(x), Y_t)} \psi_t(x, Y) - \left(\frac{\partial h_t}{\partial t}(x), Y_t\right) \cdot \psi_t(x, Y). \tag{4.4}$$

Assuming that spatial derivatives of h of first and second orders exist, this reduces to a partial differential equation, also called the 'robust' form of the Zakai equation.

In most of the treatments of nonlinear filtering theory, h has been assumed to be bounded. Recently under the assumption that h has at most linear growth and imposing other growth conditions on $(\partial h)/(\partial t)$, $(\partial h)/(\partial x_i)$, $(\partial^2 h)/(\partial x_i\, \partial x_j)$, Pardoux [8] showed that (4.4) has a unique solution ψ and further that if p is defined by (4.3), then p is the unnormalized conditional density. For this, Pardoux also assumes that the drift coefficient b of (X_t) has at most linear growth and the diffusion coefficient a of (X_t) is bounded.

Baras, Blankenship and Mitter [3] and Baras, Blankenship and Hopkins [2] have also considered this problem for the unbounded case. However, their results do not seem to be satisfactory from the point of view of filtering theory for the following reasons:

(i) Under certain conditions, in both these papers, they show that the robust form of Zakai's equation (4.4) has a unique solution. However, they do not address themselves to the problem of identifying this solution as the unnormalized conditional density. This part of the problem is by no means trivial and does need a lot of work even under more restrictive conditions.

(ii) Baras, Blankenship and Mitter [3] do need the existence of derivatives of h and growth conditions on them even for proving existence and uniqueness for (4.4). Our conditions are weaker than theirs.

(iii) In [2], Baras, Blankenship and Hopkins have considered the one dimensional problem. Even in this case, their conditions are complicated and involve relative growth of a, b, h and their derivatives. Moreover, the examples discussed in [2] implicitly imply that their conditions allow them to consider the case of 'polynomial' drift and diffusion coefficients of arbitrarily large degree. But, if a, b grow too fast, it is well known that the martingale problem for (a, b) is not well posed and thus the condition A5 in [2] is violated. (The authors seem to be aware of this difficulty as their comments on p. 205 of [2] show.)

It may not be out of place here to point out that the generality of our result (as far as h is concerned) is a consequence of the finitely additive white noise model adopted in this paper. Specifically, since we do not need to use the transformation (4.3) to get a PDE, it is not necessary for us to assume the existence of derivatives of h. Furthermore, the potential term in the white noise version of the Zakai equation which is also the exponent appearing in the Bayes formula is bounded above in our set up, regardless of any growth condition on h.

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