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The pagenumber of k-trees is $O(k)$

Joseph L. Ganley[a,*], Lenwood S. Heath[b]

[a] Simplex Solutions Inc., 521 Almanor Avenue, Sunnyvale, CA 94085, USA

[b] Department of Computer Science, Virginia Tech., Blacksburg, VIA 24061, USA

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Abstract

A k-tree is a graph defined inductively in the following way: the complete graph K_k is a k-tree, and if G is a k-tree, then the graph resulting from adding a new vertex adjacent to k vertices inducing a K_k in G is also a k-tree. This paper examines the book-embedding problem for k-trees. A book embedding of a graph maps the vertices onto a line along the spine of the book and assigns the edges to pages of the book such that no two edges on the same page cross. The pagenumber of a graph is the minimum number of pages in a valid book embedding. In this paper, it is proven that the pagenumber of a k-tree is at most $k+1$. Furthermore, it is shown that there exist k-trees that require k pages. The upper bound leads to bounds on the pagenumber of a variety of classes of graphs for which no bounds were previously known.

Keywords: Book embedding; Pagenumber; k-trees; Treewidth; Graph embedding

1. Introduction

Given a graph $G=(V,E)$, a *p-page book embedding* of G consists of two functions:

- A function $\sigma : V \to \{1,\dots,|V|\}$ that orders the vertices along the spine of the book and
- A function $\rho : E \to \{1,\dots,p\}$ that assigns the edges to pages in the book.

These functions must be such that no two edges assigned to the same page cross.

The *pagenumber* $\mathscr{P}(G)$ of G is the minimum p such that there exists a p-page book embedding of G. Table 1 gives the pagenumber of a few classes of graphs.

In this paper we prove that every k-tree can be embedded in $k+1$ pages and that there are k-trees that require k pages. The upper bound leads to bounds on the

* Corresponding author.
E-mail address: joseph@ganley.org (J.L. Ganley)

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Table 1
Pagenumber $\mathscr{P}$ of some classes of graphs

Graph class	$\mathscr{P}$
Outerplanar	1 [1]
Trees	1 [4]
Planar	4 [14]
K_n	$\lfloor n/2 \rfloor$ [4]

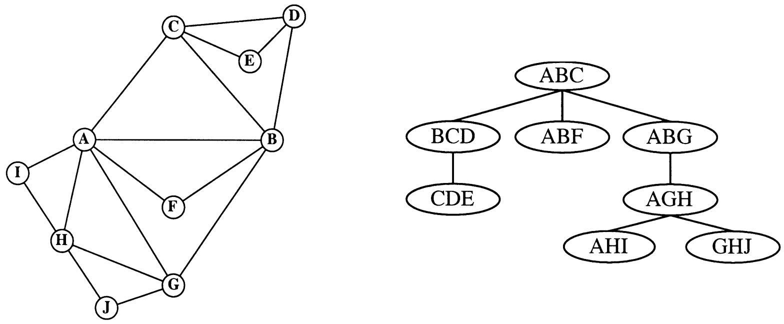

Fig. 1. A 2-tree and a tree-decomposition thereof.

pagenumber of several other classes of graphs for which no bounds on pagenumber were previously known.

2. *k*-trees

A *k-tree* is defined inductively in the following way:

- The k-vertex complete graph K_k is a k-tree.
- If G is a k-tree, then adding a new vertex to G that is adjacent to k vertices that induce a K_k in G results in a k-tree.

A *partial k-tree* is a subgraph of a k-tree. The notion of a partial k-tree is intimately related to the notion of *treewidth* [2]. Specifically, a graph has treewidth k precisely if it is a partial k-tree. A k-tree is a maximal graph with treewidth k. Our use of these terms should be taken to imply minimality, i.e., if a graph G is a partial k-tree, or equivalently has treewidth k, then it is implied that G is not a partial $(k-1)$-tree.

A *tree-decomposition* of a graph $G=(V,E)$ consists of a tree $T=(I,F)$ and a family $\{X_i\colon i \in I\}$ of subsets of V such that [2]:

(1) $\bigcup_{i\in I} X_i = V$,

(2) For all $(u,v) \in E$, there exists an $i \in I$ with $u,v \in X_i$, and

(3) For all $i,j,k \in I$, if j is on the path from i to k in T, then $X_i \cap X_k \subset X_j$.

If G has treewidth k, then there exists a tree-decomposition of G with $\max_{i\in I}|X_i|=k+1$.

Fig. 1 illustrates a 2-tree and a tree-decomposition thereof.

3. The pagenumber of *k*-trees

We now prove that k trees have pagenumber at most $k+1$.

Theorem 1. *If G is a k-tree, then* $\mathscr{P}(G) \leqslant k+1$.

Proof. Consider a width-k tree-decomposition of G. Perform a depth-first search of T, and let the ordering $\sigma(v)$ of each vertex v be determined by the first time an $i \in I$ with $v \in X_i$ is encountered in the depth-first search.

Now, consider a family of subtrees T_v of T, where T_v is the subtree induced by the vertices $i \in I$ such that $v \in X_i$. By property (3) of tree-decompositions, each T_v is connected, i.e. a single tree. Now, consider coloring the subtrees such that no two overlapping subtrees have the same color. The intersection graph of a set of subtrees of a tree is a chordal graph [5]. That is, if one builds a graph in which the vertices correspond to subtrees of a tree and in which there is an edge between two vertices if and only if their corresponding subtrees intersect, then that graph is chordal. Chordal graphs are perfect, implying (among other things) that their chromatic number is equal to their maximum clique size. Since $|X_i| = k+1$ for all $i \in I$, the family of subtrees can be colored using $k+1$ colors (a clique of size greater than $k+1$ in the subtree intersection graph would imply that some $|X_i| > k+1$).

We now use the coloring of the subtrees of T to color the edges of G. Let $c(T_v)$ be the color assigned to subtree T_v. Color each edge $(u,v) \in E$ with the color $c(u,v)$ as follows:

$$c(u,v) = \begin{cases} c(T_u) & \text{if } \sigma(u) < \sigma(v), \text{ and} \\ c(T_v) & \text{if } \sigma(v) < \sigma(u). \end{cases}$$

We claim that relative to the ordering σ defined above, no two edges of the same color cross. Suppose to the contrary that edges (a,b) and (c,d) have the same color and that they cross, i.e. (without loss of generality), $\sigma(a) < \sigma(c) < \sigma(b) < \sigma(d)$. This would imply that the corresponding paths in the subtrees T_a and T_c intersect, contradicting the assumption that they have the same color. (Note, however, that edges can nest, as in Fig. 2.)

Thus, the ordering in the book embedding is σ as defined above, and each edge is assigned to a page corresponding to its color. Since there are $k+1$ colors, this results in a $(k+1)$-page book embedding, proving the theorem. $\square$

Fig. 2 illustrates a 3-page embedding of the 2-tree shown in Fig. 1, computed according to Theorem 1.

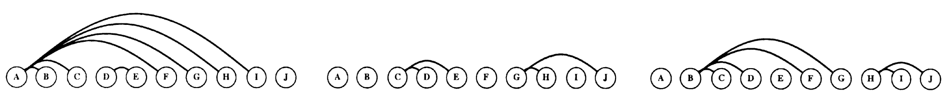

Fig. 2. A 3-page embedding of the 2-tree shown in Fig. 1.

We now show that there are k-trees for which the embedding described by Theorem 1 is within 1 of optimal.

Theorem 2. *There are k-trees that require k pages.*

Proof. Muder et al. [7] prove that

$$\mathscr{P}(K_{mn,n}) \geqslant \left(\frac{m}{m+1}\right) n.$$

The graph $K_{mk,k}$ is a partial k-tree (simply add mk vertices all adjacent to the root K_k), so taking the limit as m approaches infinity implies the theorem. □

It is unlikely that there exists a nontrivial worst-case lower bound on the pagenumber of a partial k-tree. For example, the $n \times n$ grid graph has pagenumber 2 [4] but has treewidth exactly n [10].

4. Pagenumber of other graph classes

Theorem 1 implies upper bounds on pagenumber for a number of classes of graphs that have known treewidth. No upper bounds on pagenumber have been previously shown for these classes of graphs. We now describe these upper bounds, which are summarized in Table 2.

An *almost-tree* with parameter k is a graph G such that removing at most k edges from each biconnected component of G results in a tree. Bodlaender [3] proves that almost-trees with parameter k have treewidth at most $k+1$. Thus, by Theorem 1 they have pagenumber at most $k+2$.

A graph G has *bandwidth* k if there exists a linear ordering σ of the vertices in G such that for every edge (u,v) in G, it is the case that $|\sigma(u) - \sigma(v)| \leqslant k$. Bodlaender

Table 2
Upper bounds on pagenumber $\mathscr{P}$ of some classes of graphs (ω is maximum clique size). Note that stronger bounds are known for bandwidth-k and Halin graphs

Graph class	$\mathscr{P} \leqslant$
Almost-trees (k)	$k+2$
Bandwidth k	$k+1$
Cutwidth k	$k+1$
Halin	4
k-Outerplanar	$3k$
Chordal	ω
Split	ω
Undirected path	ω
Directed path	ω
Interval	ω
Circular arc	2ω
Proper circular arc	$2\omega - 1$

[3] proves that bandwidth-k graphs have treewidth at most k, and thus by Theorem 1 they have pagenumber at most $k+1$. Note, however, that Swaminathan et al. [12] have recently proved that, in fact, the pagenumber of bandwidth-k graphs is exactly $k-1$.

Similarly, a graph G has *cutwidth* k if there exists a linear ordering σ of the vertices in G such that

$$\max_{v \in V} |\{(u,w)\colon \sigma(u) \leqslant \sigma(v) \text{ and } \sigma(v) < \sigma(w)\}| \leqslant k.$$

Bodlaender [3] proves that cutwidth-k graphs have treewidth at most k, and thus by Theorem 1 they have pagenumber at most $k+1$.

A graph G is a *Halin graph* if G can be decomposed into a tree with no degree-2 vertices and a cycle around the leaves of the tree. Halin graphs are partial 3-trees [13], so by Theorem 1 they have pagenumber at most 4. However, Halin graphs are planar and Hamiltonian, which Bernhart and Kainen [1] prove implies that they in fact have pagenumber 2.

A graph G is k-outerplanar if it has a planar embedding such that removing all vertices on the infinite face results in a $(k-1)$-outerplanar graph (a standard outerplanar graph is 1-outerplanar). Bodlaender [3] proves that k-outerplanar graphs have treewidth at most $3k-1$, and thus by Theorem 1 they have pagenumber at most $3k$.

A graph is *chordal* if every cycle on four or more edges has a *chord*, i.e. an edge connecting two nonadjacent vertices on the cycle. Robertson and Seymour [11] prove that if ω is the size of a maximum clique in a chordal graph G, then G has treewidth at most $\omega-1$, so Theorem 1 implies a bound of ω on its pagenumber. *Split graphs* (chordal graphs whose complement is also chordal), *undirected and directed path graphs* (intersection graphs of paths in an undirected or directed tree), and *interval graphs* (intersection graphs of intervals on the real line) are all chordal, so the same bounds apply for these classes as well. Furthermore, the complete bipartite graph described in the proof of Theorem 2 is a subgraph of an interval graph, so the lower bound of Theorem 2 applies for all the classes of chordal graphs described above (all of which contain the interval graphs as a subclass).

A *circular arc graph* is the intersection graph of a set of arcs on a circle. Bodlaender [3] proves that the treewidth of a circular arc graph is at most $2\omega-1$, where again ω is the size of a maximum clique, and thus Theorem 1 implies that its pagenumber is at most 2ω. A *proper circular arc graph* is one with a circular arc representation in which no arc properly contains another. In the same paper Bodlaender proves that proper circular arc graphs have treewidth at most $2\omega-2$, and thus Theorem 1 implies a bound of $2\omega-1$ on their pagenumber.

5. Conclusions

Perhaps the most pressing issue remaining is the gap between our upper bound of $k+1$ and our lower bound of k. We conjecture that, in fact, every k-tree has a k-page book embedding. This conjecture has been shown to be true for $k = 1$ (trees) and $k = 2$ (series–parallel graphs) by Chung et al. [4].

An important issue raised by these results is computation of book embeddings of partial k-trees. Our results provide a near-optimal algorithm for book embedding of a k-tree, but the pagenumber of a *partial* k-tree can fall anywhere in the range 1 to $k+1$. It seems likely that an algorithm exists that computes an optimal book embedding of a partial k-tree in polynomial time for fixed k (though the time complexity may well be exponential in k).

The book embedding problem can be reformulated in terms of *stack layouts*. An s-stack layout of a graph is an ordering of the vertices and an assignment of each edge to one of s stacks, such that if the vertices are traversed in order, then each edge can be pushed onto its stack when its left endpoint is encountered and popped from its stack when its right endpoint is encountered. A p-page book embedding and a p-stack layout are precisely equivalent, with the same ordering and with pages corresponding to stacks.

A related problem is *queue layout*, where the stacks are replaced by queues [6]. This can be cast in terms of book embeddings by requiring that no two edges in the same page nest. One might wonder whether bounds on the number of queues required for a k-tree can be found, similar to those presented here for stacks. Fairly strong evidence exists suggesting that the answer is "no". Pemmaraju [8] examines a planar subclass of the 3-trees called *stellated triangles* and provides evidence that their queuenumber is unbounded. Note, however, that Rengarajan and Madhavan [9] have proven that the queuenumber of 2-trees is at most 3.

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