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On CY–LG correspondence for (0, 2) toric models

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Abstract

We conjecture a description of the vertex (chiral) algebras of the (0, 2) nonlinear sigma models on smooth quintic threefolds. We provide evidence in favor of the conjecture by connecting our algebras to the cohomology of a twisted chiral de Rham sheaf. We discuss CY/LG correspondence in this setting.

Keywords: Toric varieties; Quintic; Nonlinear sigma model; Vertex algebra; Chiral de Rham complex

1. Introduction

The goal of this paper is to show that the vertex algebra approach to toric mirror symmetry is suitable for working with the $(0,2)$ theories. Compared to their $(2,2)$ cousins, $(0,2)$ nonlinear sigma models are poorly understood. There has been a renewed recent interest in them, see for example [9]. This paper aims to provide a concrete tool for various calculations in the theories. We focus our attention on the quintic case, but most of our techniques are applicable in a much wider context.

Let us review the basics of the vertex algebra approach to mirror symmetry. In the very important paper [12] Malikov, Schechtman and Vaintrob have constructed the so-called chiral de Rham complex, which is a sheaf of vertex (in physics literature *chiral*) algebras over a given smooth

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manifold X. Its cohomology should be viewed as the large Kähler limit of the space of states of the half-twisted theory for the type II string models with target X, see [10].[1]

The chiral de Rham complex $\mathrm{MSV}(X)$ is defined locally. Thus, it does not carry the information about instanton corrections. It is expected that one should be able (in the simply connected case) to construct a deformation of its cohomology that would incorporate these corrections, along the lines of the construction of quantum cohomology. However, this construction is not presently known.

In the case when X is a hypersurface in a Fano toric variety, an ad hoc deformation has been defined in [1], motivated by Batyrev's mirror symmetry. Specifically, let M_1 and N_1 be dual lattices (in this paper this simply means free abelian groups), and let Δ and $\Delta^\vee$ be dual reflexive polytopes in them. Consider extended dual lattices $M = M_1 \oplus \mathbb{Z}$ and $N = N_1 \oplus \mathbb{Z}$ and cones $K = \mathbb{R}_{\geqslant 0}(\Delta, 1) \cap M$ and $K^\vee = \mathbb{R}_{\geqslant 0}(\Delta^\vee, 1) \cap N$ in them. Then the vertex algebras of mirror symmetry are defined in [1] as the cohomology of the lattice vertex algebra $\mathrm{Fock}_{M\oplus N}$ by the differential

$$D_{f,g} = \mathrm{Res}_{z=0}\left(\sum_{m\in\Delta} f_m m^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} + \sum_{n\in\Delta^\vee} g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}\right)$$

where f_m and g_n are complex parameters. This construction may be extended to a more general setting of Gorenstein dual cones. The resulting algebras have numerous nice properties, studied in [2]. In particular, they admit $N = 2$ structures and their chiral rings can be calculated. This approach is somewhat different from the gauged linear sigma model approach of [14] since it is based on the classical description of toric varieties in terms of their fans, as opposed to the homogeneous coordinate ring construction of Cox.

This paper is dealing with a certain generalization of the theory known as $(0,2)$ nonlinear sigma model. One major difference is that the tangent bundle TX is replaced by another vector bundle E with the same first and second Chern classes. The influential paper of Witten [14] describes such theories for the case of the hypersurfaces in the projective space. In this paper we will specifically focus on the quintic threefolds in $\mathbb{P}^4$, although our techniques are valid in any dimension.

As in [14, (6.39)–(6.40)], we consider a homogeneous polynomial G of degree 5 in the homogeneous coordinates x_i on $\mathbb{P}^4$ and five polynomials G^i of degree four in these coordinates with the property $\sum_i x_i G^i = 0$. Equivalently, we consider five polynomials of degree four $R^i = \partial_i G + G^i$. Witten has constructed (physically) a one-dimensional family of $(0,2)$ theories that interpolates between the Calabi–Yau and the Landau–Ginzburg phases. The Calabi–Yau theory in question is defined by the quintic $G = 0$, but with a vector bundle that is a deformation of the tangent bundle, given by G^i. We argue that the half-twisted theories for these data are given by the cohomology of the lattice vertex algebra $\mathrm{Fock}_{M\oplus N}$ by the differential

$$D_{(F^\cdot),g} = \mathrm{Res}_{z=0}\left(\sum_{\substack{m\in\Delta\\ 0\leqslant i\leqslant 4}} F_m^i m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} + \sum_{n\in\Delta^\vee} g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}\right)$$

[1] There is an alternative interpretation of chiral de Rham complex in the works of Heluani and coauthors, see for example [4]. We thank the referee for pointing this out to us.

where $F^i = x_i R^i$ are degree 5 polynomials that generalize the logarithmic derivatives of the equation of the quintic (see Section 3 for details). Equivalently, one can take the cohomology of $\mathrm{Fock}_{M \oplus K^\vee}$ by the above differential $D_{(F^\cdot),g}$. We denote these vertex algebras by $V_{(F^\cdot),g}$. In the case when $F^i = x_i \partial_i f$ are logarithmic derivatives of some degree five polynomial f, we have $V_{(F^\cdot),g} = V_{f,g}$, i.e. these algebras generalize the usual vertex algebras of mirror symmetry.

We consider a natural "limit" of the algebras $V_{(F^\cdot),g}$ for fixed F^i, given by the cohomology of the so-called partial (*deformed* in [1]) lattice vertex algebra $\mathrm{Fock}^{\Sigma}_{M \oplus N}$ by the above differential $D_{(F^\cdot),g}$. Our main result is Theorem 5.1.

Theorem 5.1. *The cohomology of* $\mathrm{Fock}^{\Sigma}_{M \oplus K^\vee}$ *with respect to* $D_{(F^\cdot),g}$ *is isomorphic to the cohomology of a twisted chiral de Rham sheaf on the quintic* $\sum_{i=0}^{4} F^i = 0$ *given by* R^i.

The twisted chiral de Rham sheaf in question is the one studied in [6–8]. It appears that our construction provides, rather unexpectedly, a specific choice among such sheaves, which was pointed to us by Malikov. In another limit we expect to see the Landau–Ginzburg phase of the theory. Thus, the CY/LG correspondence considered in [14] is manifest in our construction.

The paper is organized as follows. In Section 2, we recall the construction of [1] as it applies to the case of quintics in $\mathbb{P}^4$. We recall the Calabi–Yau and Landau–Ginzburg correspondence in this setting. In Section 3, we define the vertex algebras for the $(0, 2)$ sigma model of the quintic, see Definition 3.1. Section 4 is devoted to the proof of the technical result Theorem 4.1 which is necessary to apply the method of [1] to this setting. Theorem 4.1 may be of independent interest, as it gives a novel way of constructing a twisted chiral de Rham sheaf in some cases. In Section 5, we prove the main Theorem 5.1. In Section 6, we discuss further properties of the vertex algebras for $(0, 2)$ models on the quintic that follow from the techniques of [1] and [2]. Specifically, we focus on the description of their chiral rings. Finally, in Section 7 we sketch some future directions of research.

2. Overview of vertex operator algebras of mirror symmetry for the quintic

For a smooth manifold X, the chiral de Rham complex $\mathrm{MSV}(X)$ is a sheaf of vertex algebras on X constructed in [12]. In a given coordinate system near a point on X this sheaf is generated by $4 \dim X$ free fields b^i, ϕ^i, ψ_i, a_i with the operator product expansions (OPEs)

$$a_i(z)b^j(w) \sim \delta_i^j (z-w)^{-1}, \qquad \phi^i(z)\psi_j(w) \sim \delta_j^i (z-w)^{-1}$$

and all the others nonsingular. Here the fields a and b are bosonic and fields ϕ and ψ are fermionic. The b fields transform like coordinates on X. Products of b and ϕ transform under the coordinate changes as differential k-forms (where k is the number of ϕ factors). Products of b and ψ transform as polyvector fields.

The sheaf $\mathrm{MSV}(X)$ carries a natural conformal structure, in fact it contains a natural $N = 1$ algebra in it. If, in addition, X is a Calabi–Yau manifold, then depending on a choice of nowhere vanishing holomorphic volume form (up to constant), the $N = 1$ structure can be extended to $N = 2$ structure, see [12].

For a manifold X, the cohomology $H^*(\mathrm{MSV}(X))$ of the chiral de Rham complex on it provides a fascinating invariant. It inherits the vertex algebra structure from the chiral de Rham complex. Its natural $N = 1$ structure is extended to a natural $N = 2$ structure when X is a Calabi–Yau (in fact if X is in addition compact, then the choice of the volume form is unique up to scaling, so the $N=2$ structure is canonically defined). From the string theory point of view $H^*(\mathrm{MSV}(X))$ can be thought of as a large Kähler limit of the space of half-twisted type II string theory with target X, see [10].

We will now review the (fairly) explicit description of the cohomology of the chiral de Rham complex for a smooth quintic in $\mathbb{P}^4$, which was obtained in [1]. We will also describe the cohomology of the chiral de Rham complex for the canonical bundle W over $\mathbb{P}^4$.

Consider the dual lattices M and N defined as

$$M := \left\{(a_0,\dots,a_4)\in\mathbb{Z}^5,\ \sum a_i = 0 \bmod 5\right\};\qquad N := \mathbb{Z}^5 + \mathbb{Z}\left(\frac{1}{5},\dots,\frac{1}{5}\right)$$

with the usual dot product pairing. We introduce elements $\deg = (1,\dots,1)\in M$ and $\deg^\vee = (\frac{1}{5},\dots,\frac{1}{5})$ in N.

The cone K in M is defined by the inequalities $a_i \geqslant 0$. The intersection of K with the hyperplane $\bullet\cdot\deg^\vee = 1$ is the polytope $\Delta\in M$. This is a four-dimensional simplex which is the convex hull of $(5,0,0,0,0),\dots,(0,0,0,0,5)$. The dual cone $K^\vee$ in N is also defined by non-negativity of the coordinates. The polytope $\Delta^\vee = K^\vee\cap\{\deg\cdot\bullet = 1\}$ is the simplex with vertices $(1,0,0,0,0),\dots,(0,0,0,0,1)$. The only other lattice point of $\Delta^\vee$ is $\deg^\vee$.

Remark 2.1. The lattice points in Δ correspond to monomials of degree 5 in homogeneous coordinates on $\mathbb{P}^4$ while the lattice points in $\Delta^\vee$ correspond to codimension one torus strata on the canonical bundle W over $\mathbb{P}^4$.

We will now describe briefly the construction of the vertex algebras $\mathrm{Fock}_{M\oplus N}$ and $\mathrm{Fock}^\Sigma_{M\oplus N}$, following [1]. We start with the vertex algebra $\mathrm{Fock}_{0\oplus 0}$ generated by 10 free bosonic and 10 free fermionic fields based on the lattice $M\oplus N$ with operator product expansions

$$m^{bos}(z)n^{bos}(w)\sim\frac{m\cdot n}{(z-w)^2},\qquad m^{ferm}(z)n^{ferm}(w)\sim\frac{m\cdot n}{(z-w)}$$

and all other OPEs nonsingular. We then consider the lattice vertex algebra $\mathrm{Fock}_{M\oplus N}$ with additional vertex operators $\mathrm{e}^{\int m^{bos}(z)+n^{bos}(z)}$ (with the appropriate cocycle, see [1]). They satisfy

$$\mathrm{e}^{\int m_1^{bos}(z)+n_1^{bos}(z)}\mathrm{e}^{\int m_2^{bos}(w)+n_2^{bos}(w)} = (z-w)^{m_1\cdot n_2+m_2\cdot n_1}\mathrm{e}^{\int m_1^{bos}(z)+n_1^{bos}(z)+m_2^{bos}(w)+n_2^{bos}(w)}\tag{2.1}$$

with the normal ordering implicitly applied. Here the right-hand side needs to be expanded at $z=w$.

Consider the (generalized) fan Σ in N given as follows. Its maximum-dimensional cones are generated by $\deg^\vee$, $-\deg^\vee$ and four out of the five vertices of $\Delta^\vee$. It is the preimage in N of the fan of $\mathbb{P}^4$ given by the images of the generators of $\Delta^\vee$ in $N/\mathbb{Z}\deg^\vee$. Then define the partial lattice vertex algebra $\mathrm{Fock}^\Sigma_{M\oplus N}$ by setting the product in (2.1) to zero if n_1 and n_2 do not lie in the same cone of Σ. We similarly define the vertex algebras $\mathrm{Fock}_{M\oplus K^\vee}$ and $\mathrm{Fock}^\Sigma_{M\oplus K^\vee}$.

The following results have been proved in [1].

Proposition 2.2. *Let $W\to\mathbb{P}^4$ be the canonical bundle. Then the cohomology of the chiral de Rham complex* $\mathrm{MSV}(W)$ *is isomorphic to the cohomology of* $\mathrm{Fock}^\Sigma_{M\oplus K^\vee}$ *with respect to the* *differential*

$$D_g = \mathrm{Res}_{z=0} \sum_{n\in\Delta^\vee} g_n n^{ferm}(z) \mathrm{e}^{\int n^{bos}(z)}$$

for any collection of nonzero numbers g_n, $n \in \Delta^\vee$.

Proposition 2.3. *The cohomology of the chiral de Rham complex of a smooth quintic $F(x_0,\dots,x_4) = 0$ which is transversal to the torus strata is given by the cohomology of* $\mathrm{Fock}^{\Sigma}_{M\oplus K}$ *by the differential*

$$D_{f,g} = \mathrm{Res}_{z=0}\left(\sum_{m\in\Delta} f_m m^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} + \sum_{n\in\Delta^\vee} g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}\right)$$

where g_n are arbitrary nonzero numbers and f_m is the coefficient of F by the corresponding monomial.

The cohomology of the chiral de Rham complex should be viewed as just an approximation to the true physical vertex algebra of the half-twisted theory. It has been conjectured in [1] that the effect of adding instanton corrections to this algebra must correspond to the removal of the superscript $^\Sigma$ in the calculation of the cohomology. Crucially, while the cohomology of $\mathrm{Fock}^{\Sigma}_{M\oplus K^\vee}$ with respect to $D_{f,g}$ is independent from g (as long as all g_n are nonzero), the cohomology of $\mathrm{Fock}_{M\oplus K^\vee}$ with respect to $D_{f,g}$ depends on it.

Definition 2.4. Fix F and the corresponding f_m. As the g_n vary, consider the family of vertex algebras $V_{f,g}$ which are the cohomology of $\mathrm{Fock}_{M\oplus K^\vee}$ with respect to the differential

$$D_{f,g} = \mathrm{Res}_{z=0}\left(\sum_{m\in\Delta} f_m m^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} + \sum_{n\in\Delta^\vee} g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}\right).$$

We call this a family of vertex algebras of mirror symmetry associated to the quintic $F = 0$.

The vertex algebras of mirror symmetry provide a useful way of thinking about the so-called Calabi–Yau and Landau–Ginzburg (CY–LG) correspondence for the $N = 2$ theories related to the quintic, which we describe below.

There are a priori six parameters in Definition 2.4, that correspond to the values of g_n for $n = \deg^\vee$ or the vertices v_i of the simplex $\Delta^\vee$. However, up to torus symmetry, the algebra depends only on $(\prod_i g_{v_i})/g^5_{\deg^\vee}$, where v_i are the vertices of $\Delta^\vee$. Indeed, for any linear function $r : N \to \mathbb{C}$ one can rescale $\mathrm{e}^{\int n^{bos}(z)}$ to $\mathrm{e}^{r(n)}\mathrm{e}^{\int n^{bos}(z)}$. This will not change the OPEs of any fields in question. This shows that the collection g_n can be replaced by $g_n\mathrm{e}^{r(n)}$ for any r.

Let us now pick a piecewise-linear real-valued function ρ which is strongly convex on Σ. If we rescale $\mathrm{e}^{\int n^{bos}(z)}$ to $\mathrm{e}^{\lambda\rho(n)}\mathrm{e}^{\int n^{bos}(z)}$ for $\lambda \to \infty$, we see that the OPEs of the new vertex operators start to approach those for $\mathrm{Fock}^{\Sigma}_{M\oplus K^\vee}$. This implies that as the ratio $(\prod_i g_{v_i})/g^5_{\deg^\vee}$ approaches 0, the vertex algebras of mirror symmetry approach (in some rather weak sense) the cohomology of the chiral de Rham complex on the quintic. Specifically, while it is not known if the family of algebras stays flat after taking the quotient by $D_{f,g}$, it is still reasonable to think of the cohomology of chiral de Rham complex of $F=0$ as a limit of $V_{f,g}$. Similarly, as this ratio approaches to 0 one gets to the so-called orbifold point on the Kähler moduli space of the theory, which is in the Landau–Ginzburg region of the moduli space. While the $D_{f,g}$ cohomology in fact jumps at the orbifold point (see [5]), we still want to think of the family $V_{f,g}$ as interpolating between the Calabi–Yau and the Landau–Ginzburg phases of the theory.

3. Vertex algebras of (0, 2) nonlinear sigma models for the quintic

In the influential paper [14] Witten has, in particular, considered a CY–LG correspondence for some $(0,2)$ models. The key observation of our paper is that we can very naturally modify the vertex algebras of mirror symmetry for the quintic to accommodate this larger class of theories. The goal of this section is to give a definition of the vertex algebras of the $(0,2)$ sigma models for the quintic, analogous to Definition 2.4.

Specifically, in [14, (6.39)–(6.40)] Witten considered a homogeneous polynomial G of degree 5 in the variables x_i and five polynomials G^i in variables x_i with $\sum_i x_i G^i=0$ and has constructed (physically) a one-dimensional family of theories that interpolates from the Calabi–Yau to the Landau–Ginzburg phases. The Calabi–Yau theory in question is defined by the quintic $G=0$, but with the vector bundle that is a deformation of the tangent bundle, given by G^i.

Clearly, the above data are equivalent to a collection of five polynomials of degree four in x_i which are given by $R^i=\partial_i G+G^i$. Indeed, G can then be uniquely recovered as $\frac{1}{5}\sum_i x_i R^i$. Equivalently, we may consider five polynomials $F^i=x_i R^i$ which are of degree 5 with the property that $F^i|_{x_i=0}=0$. In this language, the quintic is simply $\sum_i F^i=0$.

Definition 3.1. As in Section 2 consider the vertex algebra $\mathrm{Fock}_{M\oplus K^\vee}$. Define by m_i the basis of $M_{\mathbb{Q}}$ which is dual to the basis of $N_{\mathbb{Q}}$ given by the vertices of $\Delta^\vee$. Consider the differential

$$D_{(F^\cdot),g}=\mathrm{Res}_{z=0}\Bigg(\sum_{\substack{m\in\Delta\\0\leqslant i\leqslant 4}} F^i_m m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)}+\sum_{n\in\Delta^\vee} g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}\Bigg)$$

where g_n are six generic complex numbers and F^i_m is the coefficient of the monomial of degree 5 of F^i that corresponds to m. We call the corresponding cohomology spaces $V_{(F^\cdot),g}$ the vertex algebras of the $(0,2)$ sigma model on $\sum_i F^i=0$.

The above definition implicitly assumes that $D_{(F^\cdot),g}$ is a differential, but this requires a verification.

Proposition 3.2. *The above-defined $D_{(F^\cdot),g}$ is a differential and the cohomology inherits the structure of a vertex algebra.*

Proof. We need to show that all modes of the corresponding field of the algebra anticommute with each other. This means verifying that the OPEs of $F^i_m m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)}$ and $g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}$ with each other and themselves are nonsingular. The only interesting cases are the OPEs between the above two operators. There are three possibilities: $n=\deg^\vee$, n is a vertex of $\Delta^\vee$ that corresponds to i and n is some other vertex.

Case 1. $n = \deg^\vee$. Because $m \cdot \deg^\vee = 1$, the OPE of the bosonic terms $\mathrm{e}^{\int m^{bos}(z)}$ and $\mathrm{e}^{\int n^{bos}(z)}$ will start with $(z-w)^1$, which counteracts the $(z-w)^{-1}$ from the fermionic terms.

Case 2. n is a vertex of $\Delta^\vee$ equal to i. Because $F^i|_{x_i=0} = 0$, we may assume that m corresponds to a monomial that is divisible by x_i. Thus, $m \cdot n \geqslant 1$ and we proceed as in the previous case.

Case 3. n is some other vertex of $\Delta^\vee$. Then $m_i \cdot n = 0$ and the fermionic OPE has no pole at $z = w$. The bosonic OPE has no pole either, because $m \cdot n \geqslant 0$. Thus the OPE is nonsingular. □

Remark 3.3. If one uses the same N-part of the differential $D_{f,g}$ but attempts to consider various elements of $M^{ferm}(z)\mathrm{e}^{\int \Delta^{bos}(z)}$ for the M-part, the condition of being a differential is equivalent to it being given by Definition 3.1 for some F^i with $F^i|_{x_i=0} = 0$.

Remark 3.4. In the original setting of the vertex algebras of mirror symmetry, the cohomology with respect to $D_{f,g}$ inherited an $N = 2$ structure from $\mathrm{Fock}_{M\oplus K^\vee}$ which was generated by the fields $M^{ferm} \cdot N^{bos} - \partial_z \deg^{ferm}$ and $M^{bos} \cdot N^{ferm} - \partial_z (\deg^\vee)^{ferm}$. Typically, this structure does not super-commute with the differential $D_{(F^\cdot),g}$ and thus does not descend to the cohomology $V_{(F^\cdot),g}$. However, part of the structure still descends, as is shown below.

Proposition 3.5. *Consider the Virasoro algebra and affine $U(1)$ algebras on* $\mathrm{Fock}_{M\oplus K^\vee}$ *which are given by*

$$L(z) := \sum_i m_i^{bos} n_i^{bos} + \sum_i \left(\partial_z m_i^{ferm}\right) n_i^{ferm} - \partial_z \left(\deg^\vee\right)^{bos},$$

$$J(z) := \sum_i m_i^{ferm} n_i^{ferm} + \deg^{bos} - \left(\deg^\vee\right)^{bos}.$$

Here m_i and n_i are elements of a dual basis. These fields commute with $D_{(F^\cdot),g}$ and thus descend to $V_{(F^\cdot),g}$.

Proof. The parts of the differential that correspond to $n \in \Delta^\vee$ have already been considered in [1]. The OPEs of the remaining terms with J are computed by

$$m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} J(w) \sim \frac{(-m_i^{ferm}\mathrm{e}^{\int m^{bos}(z)} + m_i^{ferm}\mathrm{e}^{\int m^{bos}(z)})}{(z-w)} \sim 0.$$

The OPEs with L are a bit more bothersome. We have

$$\begin{aligned} m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} L(w) &\sim (z-w)^{-1} m_i^{ferm}(z)\left(-m^{bos}(w)\mathrm{e}^{\int m^{bos}(z)}\right) \\ &\quad + (z-w)^{-1}\left(-\partial_z m_i^{ferm}\mathrm{e}^{\int m^{bos}(z)}\right) + \partial_w\left((z-w)^{-1} m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)}\right) \\ &\sim (z-w)^{-2} m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} + (z-w)^{-1}\left(-m_i^{ferm} m^{bos} - \partial_z m_i^{ferm}\right)\mathrm{e}^{\int m^{bos}} \\ &\sim (z-w)^{-2} m_i^{ferm}(w)\mathrm{e}^{\int m^{bos}(w)} \end{aligned}$$

which shows that the differential acts trivially on the corresponding field. □

Remark 3.6. Given the match of the data, the reader should already find it plausible that the algebras $V_{(F^{\cdot}),g}$ are the algebras of the (0, 2) models considered in [14]. In what follows we will strengthen their connection to the (0, 2) models by showing that analogous "limit" algebra which is the cohomology of $\mathrm{Fock}^{\Sigma}_{M \oplus K^{\vee}}$ with respect to $D_{(F^{\cdot}),g}$ is isomorphic to the cohomology of an analog of the chiral de Rham complex defined for deformations of the chiral de Rham complex in [7,8]. We closely follow [1] and overcome the fairly minor technical difficulties that occur along the way.

4. A cohomology construction of a twisted chiral de Rham sheaf in a particular case

Let X be a smooth manifold. Let E be a vector bundle on X such that $c_1(E) = c_1(TX)$ and $c_2(E) = c_2(TX)$. Assume further that $\Lambda^{\dim X} E$ is isomorphic to $\Lambda^{\dim X} TX$, and, moreover, pick a choice of such isomorphism. Then one can construct a collection of sheaves $\mathrm{MSV}(X, E)$ of vertex algebras on X, which differ by regluings given by elements of $H^1(X, (\Lambda^2 TX^{\vee})^{closed})$, see [6–8]. Locally, any such sheaf is again generated by b^i, a_i, ϕ^i, ψ_i, however ϕ^i and ψ_i now transform as sections of $E^{\vee}$ and E respectively. The OPEs between the ϕ and ψ are governed by the pairing between sections of $E^{\vee}$ and E. The sheaves $\mathrm{MSV}(X, E)$ carry a natural structure of graded sheaves of vertex algebras. If, in addition, X is a Calabi–Yau, and one fixes a choice of the nonzero holomorphic volume form, then each of the sheaves $\mathrm{MSV}(X, E)$ acquires a conformal structure, as well as an additional affine $U(1)$ current $J(z)$ on it.

The goal of this section is to construct more explicitly a twisted chiral de Rham sheaf of (X, E) for a particular class of X and E. Specifically, if X is a codimension one subvariety in a smooth variety Y and E is determined by a global holomorphic one-form on a line bundle W over Y, then we will be able to calculate $\mathrm{MSV}(X, E)$ in terms of the usual chiral de Rham complex on W.

Let $\pi : W \to Y$ be a line bundle over an n-dimensional manifold Y with zero section $s : Y \to W$. Let α be a holomorphic one-form on W which is linear with respect to the natural $\mathbb{C}^*$ action on W, i.e. for $\lambda \in \mathbb{C}^*$ the following holds $\lambda^*\alpha = \lambda\alpha$. Consider the locus $X \subset Y$ of points y such that $\alpha(s(y))$ as a function on the tangent space $TW_{s(y)}$ is zero on the vertical subspace.

Locally, we have coordinates $(y_1, \dots, y_n)$ on Y. The bundle W is trivialized so that the coordinates near $s(y)$ are $(y_1, \dots, y_n, y_{n+1})$. The homogeneity property of α implies that it is given by

$$\alpha = \sum_i y_{n+1} P_i(y_1, \dots, y_n)\, dy_i + P(y_1, \dots, y_n)\, dy_{n+1}. \tag{4.1}$$

In these coordinates X is locally given by $P(y_1, \dots, y_n) = 0$. We assume that X is a smooth codimension one submanifold of Y.

Consider the subbundle E of $TY|_X$ which is locally defined as the kernel of $s^*\alpha$. We will assume that it is of corank 1. In the local description above this means that P_i and P are not simultaneously zero. If $P_i = \partial_i P$ then E is simply TX. The goal of the rest of this section is to show how a twisted chiral de Rham sheaf $\mathrm{MSV}(X, E)$ can be defined in terms of the usual chiral de Rham complex of W.

The global one-form α on W gives rise to a fermion field $\alpha(z)$ in the chiral de Rham complex of W. Its residue $\mathrm{Res}_{z=0}\, \alpha(z)$ gives an endomorphism of $\mathrm{MSV}(W)$ and of its pushforward $\pi_* \mathrm{MSV}(W)$ to Y.

Theorem 4.1. *The cohomology sheaf of* $\pi_*\mathrm{MSV}(W)$ *with respect to* $\mathrm{Res}_{z=0}\,\alpha(z)$ *is isomorphic to a twisted chiral de Rham sheaf of* (X, E).

Remark 4.2. We are working in the holomorphic category, using strong topology. The analogous statement in Zariski topology will be addressed in Remark 4.16.

Remark 4.3. We identify vector bundles with their sheaves of holomorphic sections. The weight one component of the pushforward to Y of the sheaf of holomorphic 1-forms on W can be included into a short exact sequence of locally free sheaves on Y

$$0 \to TY^\vee \otimes W^\vee \to \left(\pi_* TW^\vee\right)_1 \to W^\vee \to 0.$$

The global section α as above induces a global section of $W^\vee$. Its zero set is precisely X. When the above sequence restricts to X, the section $\alpha|_X$ can be identified with a section of $TY^\vee|_X \otimes W^\vee|_X$, which gives a map $TY|_X \to W^\vee|_X$. We assume that this map is surjective and the kernel is the bundle E. Thus we have

$$0 \to E \to TY\big|_X \to W^\vee\big|_X = N(X \subseteq Y) \to 0.$$

Consequently, $c(E) = c(TX)$ and the cohomological obstruction of [7] vanishes. However (as was pointed to us by Malikov), it is still rather surprising that one can make a particular choice of the twisted chiral de Rham sheaf, distinguished from its possible regluings by elements of the cohomology group $H^1(X, (\Lambda^2 TY^\vee)^{closed})$. In the case $E = TX$ such a choice exists by [7,8] but there is no clear explanation for this phenomenon in general.

The proof of Theorem 4.1 proceeds in several steps. First, we calculate the cohomology with respect to $\mathrm{Res}_{z=0}\,\alpha(z)$ for small conformal weights. Then we calculate the OPEs of the fields we have found to show that they satisfy the free bosons and free fermions OPEs of the twisted chiral de Rham sheaf. This implies that the corresponding Fock space sits inside the cohomology. Then we calculate the new L and J fields in terms of these free fields. Finally, we use induction on the sum of the conformal weight and the fermion number to show that the cohomology algebra contains no additional fields.

We work in local coordinates as in (4.1). We have the fields ϕ^i, ψ_i, a_i as well as the fields b^i that correspond to the variables y_i. In these coordinates, we have

$$\mathrm{Res}_{z=0}\,\alpha(z) = \mathrm{Res}_{z=0}\left(\sum_{i=1}^{n} b^{n+1}(z) P_i\big(\mathbf{b}(z)\big)\phi^i(z) + P\big(\mathbf{b}(z)\big)\phi^{n+1}(z)\right).$$

Observe that $\mathrm{Res}_{z=0}\,\alpha(z)$ has conformal weight (-1) and fermion number 1. There is also an additional integer grading by the acton of $\mathbb{C}^*$ and this differential has weight 1 with respect to it. Let us calculate the cohomology for small conformal weights.

Lemma 4.4. *The cohomology sheaf of* $\pi_*\mathrm{MSV}(W)$ *with respect to* $\mathrm{Res}_{z=0}\,\alpha(z)$ *is supported on* X. *The conformal weight zero and fermion number zero subsheaf is isomorphic to* $\mathcal{O}_X$.

Proof. For an open set $U_Y \subseteq Y$, the subsheaf of $\pi_* \mathrm{MSV}(W)$ of conformal weight zero and fermion number zero is the sheaf of holomorphic functions on $\pi^{-1}U_Y$. The fields that can map to it under $\mathrm{Res}_{z=0}\,\alpha(z)$ are of conformal weight one and fermion number (-1). These are linear combinations of ψ's with coefficients that are functions in b's, which correspond to the vector fields on $\pi^{-1}U_Y$. The result of applying $\mathrm{Res}_{z=0}\,\alpha(z)$ amounts to pairing of α with that vector field. Thus the image is the ideal generated by $y_{n+1}P_i$ and P. Since P and P_i have no common zeroes, this is the same as the ideal generated by y_{n+1} and P. The quotient is then naturally isomorphic to the sheaf of holomorphic functions on $U_X = U_Y \cap X$. □

Lemma 4.5. *The conformal weight one and the fermion number* (-1) *cohomology sheaf of* $\pi_* \mathrm{MSV}(W)$ *with respect to* $\mathrm{Res}_{z=0}\,\alpha(z)$ *is naturally isomorphic to the sheaf of sections of* E.

Proof. In the notations of the proof of Lemma 4.4, the conformal weight one and fermion number (-1) subspace of $\pi_* \mathrm{MSV}(W)(U_Y)$ is the space of vector fields on $\pi^{-1}U_Y$. The kernel of the map consists of all fields which contract to 0 by α. These fields are given locally by $\sum_{i=1}^n Q_i \partial_i + Q\partial_{n+1}$ with

$$\sum_i y_{n+1} P_i Q_i + QP = 0. \tag{4.2}$$

Here Q_i, Q are functions of $(y_1, \ldots, y_{n+1})$. Observe that Q is necessarily divisible by y_{n+1}, so we have $Q = y_{n+1}\tilde{Q}$ and

$$\sum_i P_i Q_i + \tilde{Q} P = 0.$$

Since the functions P_i and P have no common zeroes, the corresponding Koszul complex is acyclic, and the solutions to the above equation are generated, as a module over the functions on $\pi^{-1}U_Y$, by $(Q_i = P, \tilde{Q} = -P_i)$, which corresponds to $P\partial_i - y_{n+1}P_i\partial_{n+1}$ and $(Q_i = P_j, Q_j = P_i)$ which correspond to $P_j\partial_i - P_i\partial_j$ for $1 \leqslant i, j \leqslant n$.

We need to take a quotient of this space by the image of the space of conformal weight two and fermion number (-2). These are made from second exterior powers of the tangent bundle. The action is the contraction by α. Consequently, the image is the submodule generated by

$$y_{n+1}P_i\partial_{n+1} - P\partial_i, \qquad y_{n+1}P_i\partial_j - y_{n+1}P_j\partial_i. \tag{4.3}$$

Let us consider the quotient module. By using $y_{n+1}P_i\partial_{n+1} - P\partial_i$ we can reduce the quotient to the quotient of the module spanned by $P_j\partial_i - P_i\partial_j$ by the fields spanned by $y_{n+1}P_i\partial_j - y_{n+1}P_j\partial_i$ as well as any linear combinations of the fields $y_{n+1}P_i\partial_{n+1} - P\partial_i$ which have no ∂_{n+1}. These are precisely the terms of the form P times any linear combination of $P_j\partial_i - P_i\partial_j$. This means that we are taking the quotient of the space of sections of the tangent bundle on Y that satisfy $\sum_i Q_i P_i = 0$ by P times these sections. We observe that the result is precisely the sections of the vector bundle E on $U_X = X \cap U_Y$. □

Similarly we can handle the conformal weight zero and fermion number 1 case.

Lemma 4.6. *The conformal weight one and the fermion number* 1 *cohomology sheaf of* $\pi_* \mathrm{MSV}(W)$ *with respect to* $\mathrm{Res}_{z=0}\,\alpha(z)$ *is naturally isomorphic to the sheaf of sections of* $E^\vee$.

Proof. For conformal weight zero and fermion number 1, we are looking at the quotient of the sheaf of differential one forms on W by the image of $\mathrm{Res}_{z=0}\,\alpha(z)$ of the sheaf of fields of the form $f_i^j(b)\phi^i\psi_j + g^i(b)a_i$. The quotient by the image of the first kind of fields is simply the fields of the restriction of TW to X. Indeed, these are simply obtained by multiplying the cokernel of the differential at conformal weight zero and fermion number zero by ϕ^i.

For $1 \leqslant j \leqslant n$ we have

$$\mathrm{Res}_{z=0}\,\alpha(z)a_j = \sum_i b^{n+1}\partial_j P_i\phi^i + \partial_j P\phi^{n+1}.$$

Since b^{n+1} is trivial in the cohomology, we can reduce this to $\partial_j P\phi^{n+1}$. Since the functions $\partial_j P$ have no common zeroes (because X is smooth), we see that ϕ^{n+1} lies in the image and is trivial in cohomology. It remains to take the quotient by the module generated by $\mathrm{Res}_{z=0}\,\alpha(z)a_{n+1}$. We have

$$\mathrm{Res}_{z=0}\,\alpha(z)a_{n+1} = \sum_{i=1}^{n} P_i\phi^i.$$

Thus we see that the cohomology fields of this conformal weight and fermion number are naturally isomorphic to the sections of the dual bundle of E. $\square$

Remark 4.7. The above calculations give us the fields that will correspond to the free fermions of $\mathrm{MSV}(X, E)$. Let us calculate their OPEs. Clearly, OPEs of the fields from Lemma 4.6 with each other are trivial and similarly for the OPEs of the fields from Lemma 4.5. Let us calculate the OPE of a field from Lemma 4.6 with a field from Lemma 4.5. We can take an old ϕ^j to be a representative of a field from Lemma 4.6. Then its OPE with $\sum_i P_i\psi_i$ is

$$\sim \frac{1}{z-w}P_i.$$

Since the pairing between E and $E^\vee$ is induced from pairing between TX and $TX^\vee$, we see that our new fields have the pairings expected for the fields of $\mathrm{MSV}(X, E)$.

Assume for a moment that $P_n \neq 0$ and $P = y_n$. Then the following fields will provide the generators of the cohomology with respect to $\mathrm{Res}_{z=0}\,\alpha(z)$. We will show that they always generate the cohomology a bit later, in Lemma 4.12. For now we will just study their OPEs.

Definition 4.8. For $1 \leqslant j \leqslant n-1$ consider

$$\hat{b}^j := b^j, \qquad \hat{\phi}^j := \phi^j, \qquad \hat{\psi}_j := \psi_j - P_j P_n^{-1}\psi_n,$$
$$\hat{a}_j := a_j - \sum_{i=1}^{n}(\partial_j P_i)P_n^{-1}\phi^i\psi_n - \frac{1}{2}P_n^{-2}\partial_j P_n(P_n)'.$$

Here $(P_n)' = \partial_z P_n$ refers to the differentiation with respect to the variable on the world-sheet. Also in the $i = n$ term for the summation for $\hat{a}_j$ we implicitly assume normal ordering.

Lemma 4.9. *The fields $\hat{b}^j$, $\hat{a}_j$, $\hat{\phi}^j$, $\hat{\psi}_j$ lie in the kernel of* $\mathrm{Res}_{z=0}\,\alpha(z)$ *and thus descend to the cohomology. We have*

$$\hat{a}_j(z)\hat{b}^k(w) \sim \frac{\delta_j^k}{z-w}, \qquad \hat{\phi}^k(z)\hat{\psi}_j(w) \sim \frac{\delta_j^k}{z-w}$$

with other OPEs nonsingular.

Proof. These are routine calculations using Wick's theorem for OPEs of products of free fields. It is important to use $\partial_j P = \partial_j y_n = 0$ for j in the above range.

We will do the more tricky of the calculations and leave the rest to the reader. For example, let us calculate the OPE of $\hat{a}_j(z)$ and $\alpha(w)$. We have

$$\begin{aligned}
\hat{a}_j(z)\alpha(w) &\sim \left(a_j(z) - \sum_{i=1}^{n}(\partial_j P_i)P_n^{-1}\phi^i(z)\psi_n(z)\right) \\
&\quad \times \left(\sum_{k=1}^{n} b^{n+1}(w)P_k(w)\phi^k(w) + b^n(w)\phi^{n+1}(w)\right) \\
&\sim (z-w)^{-1}\left(\sum_{k=1}^{n} b^{n+1}\partial_j P_k\phi^k - \sum_{i,k=1}^{n}(\partial_j P_i)P_n^{-1}b^{n+1}P_k\phi^i\delta_n^k\right) \sim 0.
\end{aligned}$$

The OPE of $\hat{a}$ and $\hat{\psi}$ fields is computed as follows:

$$\begin{aligned}
\hat{a}_j(z)\hat{\psi}_k(w) &\sim \left(a_j(z) - \sum_{i=1}^{n}(\partial_j P_i)(z)P_n^{-1}(z)\phi^i(z)\psi_n(z)\right)\bigl(\psi_k(w) - P_k(w)P_n^{-1}(w)\psi_n(w)\bigr) \\
&\sim (z-w)^{-1}\bigl(-\partial_j\bigl(P_k P_n^{-1}\bigr)\psi_n + (\partial_j P_k)P_n^{-1}\psi_n - P_n^{-1}(\partial_j P_n)P_k P_n^{-1}\psi_n\bigr) \sim 0.
\end{aligned}$$

In the above calculations we ignored the dependence of the terms of the coefficient at $(z-w)^{-1}$ on z versus w, since the difference is nonsingular.

By far the most complicated calculation is the OPE of $\hat{a}_j(z)\hat{a}_k(w)$. This OPE has poles of order two at $z=w$. We need to be careful with the second order terms to include the dependence on the variables. The coefficient at $(z-w)^{-2}$ is coming from the double pairings of the $\phi^n\psi_n$ terms and the pairing between the a's and the $(P_n)'$ terms. It is given by

$$\begin{aligned}
&(\partial_j P_n)(z)P_n^{-1}(z)(\partial_k P_n)(w)P_n^{-1}(w) \\
&\quad \sim -\frac{1}{2}P_n^{-2}(w)(\partial_k P_n)(w)\bigl(\partial_j P_n(w)\bigr) - \frac{1}{2}P_n^{-2}(z)\bigl(\partial_j P_n(z)\bigr)\bigl(\partial_k P_n(z)\bigr).
\end{aligned} \tag{4.4}$$

The above expression is zero at $z=w$. However, these pairings contribute to the coefficient by $(z-w)^{-1}$. Specifically, (4.4) contributes

$$\partial_w\big((\partial_j P_n)P_n^{-1}\big)(\partial_k P_n)P_n^{-1} - \frac{1}{2}\partial_w\big(P_n^{-2}(\partial_j P_n)(\partial_k P_n)\big)$$
$$= \frac{1}{2}P_n^{-2}(\partial_k P_n)(\partial_j P_n)' - \frac{1}{2}P_n^{-2}(\partial_j P_n)(\partial_k P_n)'. \tag{4.5}$$

Note that the pairing between $a_j(z)$ and $-\frac{1}{2}P_n^{-2}(w)\partial_k P_n(w)(P_n)'(w)$ additionally contributes to $(z-w)^{-1}$ term as follows. We have

$$-\frac{1}{2}P_n^{-2}(w)(\partial_k P_n)(w)\partial_w\big((z-w)^{-1}\partial_j P_n(w)\big)$$
$$\sim -\frac{1}{2}P_n^{-2}(w)(\partial_k P_n)(w)\partial_j P_n(w)(z-w)^{-2} - \frac{1}{2}P_n^{-2}(\partial_k P_n)(\partial_j P_n)'(z-w)^{-1} \tag{4.6}$$

of which only the first term was accounted for in (4.4). Similarly, the OPE of $-\frac{1}{2}P_n^{-2}(z)\partial_j P_n(z)(P_n)'(z)$ and $a_k(w)$ will yield

$$\frac{1}{2}P_n^{-2}(z)(\partial_j P_n)(z)\partial_z\big((z-w)^{-1}(\partial_k P_n)(z)\big)$$
$$\sim -\frac{1}{2}P_n^{-2}(z)(\partial_j P_n)(z)(\partial_k P_n)(z)(z-w)^{-2} + \frac{1}{2}P_n^{-2}(\partial_j P_n)(\partial_k P_n)'(z-w)^{-1} \tag{4.7}$$

of which the second term is not accounted for in (4.4). Note that the second terms of (4.6) and (4.7) cancel the contribution of (4.5).

There are additional contributions to the $(z-w)^{-1}$ term of the OPE that come from other pairings in Wick's theorem. We need to consider the pairings of a_i and a_k with the functions of b's. We also need to consider the results of pairings of $\phi^n\psi_n$ terms with $\phi^i\psi_n$ terms. The coefficient at $(z-w)^{-1}$ is then calculated to be

$$\sum_{i=1}^n \partial_k\big((\partial_j P_i)P_n^{-1}\big)\phi^i\psi_n - \sum_{i=1}^n \partial_j\big((\partial_k P_i)P_n^{-1}\big)\phi^i\psi_n$$
$$-\sum_{i=1}^n(\partial_j P_n)P_n^{-2}\partial_k P_i\phi^i\psi_n + \sum_{i=1}^n(\partial_k P_n)P_n^{-2}\partial_j P_i\phi^i\psi_n$$
$$+\frac{1}{2}\partial_k\big(P_n^{-2}\partial_j P_n\big)(P_n)' - \frac{1}{2}\partial_j\big(P_n^{-2}\partial_k P_n\big)(P_n)' = 0. \qquad \square \tag{4.8}$$

In the next lemma we will calculate a Virasoro and the $U(1)$ current fields for the fields of Definition 4.8.

Definition 4.10. Define

$$\hat{J} := \sum_{j=1}^{n-1}\hat{\phi}^j\hat{\psi}_j, \qquad \hat{L} := \sum_{j=1}^{n-1}(\hat{b}^j)'\hat{a}_j + \sum_{j=1}^{n-1}(\hat{\phi}^j)'\hat{\psi}_j,$$

where we are using the normal ordering from the modes of the free $\hat{}$ fields.

Lemma 4.11. *We have the following equalities in the cohomology of* $\mathrm{MSV}(U_W)$ *by* $\mathrm{Res}_{z=0}\,\alpha(z)$

$$\hat{J}=\sum_{j=1}^{n+1}\phi^j\psi_j-\left(b^{n+1}a_{n+1}+\phi^{n+1}\psi_{n+1}\right)-(\ln P_n)',$$

$$\hat{L}=\sum_{j=1}^{n+1}\left(b^j\right)'a_j+\sum_{j=1}^{n+1}\left(\phi^j\right)'\psi_j+\frac{1}{2}(\ln P_n)''-\left(b^{n+1}a_{n+1}+\phi^{n+1}\psi_{n+1}\right)'$$

where on the right-hand side we are using the normal ordering with respect to the free fields on $\pi_*\mathrm{MSV}(W)$.

Proof. Let us first try to calculate $\hat{J}$. To calculate the normal ordered products, we subtract the singular terms of the OPEs to get

$$\hat{J}=\sum_{j=1}^{n-1}\hat{\phi}^j\hat{\psi}_j=\sum_{j=1}^{n-1}\phi^j\psi_j-\sum_{j=1}^{n-1}P_jP_n^{-1}\phi^j\psi_n=\sum_{j=1}^{n+1}\phi^j\psi_j-\phi^{n+1}\psi_{n+1}-\sum_{j=1}^{n}P_jP_n^{-1}\phi^j\psi_n.$$

Consider

$$\begin{aligned}
&\alpha(z)P_n^{-1}(w)a_{n+1}(w)\psi_n(w)\\
&\quad=\left(\sum_{i=1}^{n}b^{n+1}(z)P_i(z)\phi^i(z)+b^n(z)\phi^{n+1}(z)\right)P_n^{-1}(w)a_{n+1}(w)\psi_n(w)\\
&\quad\sim-(z-w)^{-2}P_n(z)P_n^{-1}(w)+(z-w)^{-1}\left(b^{n+1}a_{n+1}-\sum_{i=1}^{n}P_i\phi^iP_n^{-1}\psi_n\right)\\
&\quad\sim-(z-w)^{-2}+(z-w)^{-1}\left(b^{n+1}a_{n+1}-\sum_{i=1}^{n}P_i\phi^iP_n^{-1}\psi_n-P_n^{-1}P_n'\right).
\end{aligned}$$

Thus,

$$\mathrm{Res}_{z=0}\,\alpha(z)\left(P_n^{-1}a_{n+1}\psi_n\right)=b^{n+1}a_{n+1}-\sum_{j=1}^{n}P_jP_n^{-1}\phi^j\psi_n-P_n^{-1}P_n',$$

so the field $\hat{J}$ is equivalent to $\sum_{j=1}^{n+1}\phi^j\psi_j-\phi^{n+1}\psi_{n+1}-b^{n+1}a_{n+1}-(\ln P_n)'$.

The calculation for $\hat{L}$ is similar though more complicated. The difference between it and the right-hand side of Lemma 4.11 turns out to equal the image under $\mathrm{Res}_{z=0}\,\alpha(z)$ of the field

$$P_n^{-1}\psi_na_{n+1}'+\sum_{j=1}^{n}(\partial_nP_j)P_n^{-1}\phi^j\psi_n\psi_{n+1}'-\psi_{n+1}'a_n.$$

Details are left to the reader. □

In the following lemma, we will show that the free fields $\hat{b}^j$, $\hat{\phi}^j$, $\hat{\psi}_j$, $\hat{a}_j$ locally generate the cohomology of $\mathrm{MSV}(W)$ by $\mathrm{Res}_{z=0}\,\alpha(z)$.

Lemma 4.12. *Let $x \in X$ be a point. Pick a small open subset $U_X \subset X$ containing x. We can pick coordinates on Y such that $y_n = P$. By changing y_i to $y_i + y_n$ and possibly shrinking U_X we can also assume that $P_n \neq 0$ on U_X. Pick U_Y to be an open subset on Y with $U_Y \cap X = U_X$ and denote by U_W the preimage of U_Y in W. Then the cohomology of* $\mathrm{MSV}(U_W)$ *with respect to* $\mathrm{Res}_{z=0}\,\alpha(z)$ *is generated by the* $4(n-1)$ *free fields $\hat{b}^j$, $\hat{\phi}^j$, $\hat{\psi}_j$, $\hat{a}_j$, for $1 \leqslant j \leqslant n-1$.*

Proof. From Lemma 4.9 we see that the above fields generate a subalgebra of the cohomology. Since we have a description of the cohomology of the conformal weight zero piece as the functions on U_X, we see that their OPEs imply that this subalgebra is the usual Fock space representation, namely polynomials in negative modes of $\hat{b}$, $\hat{a}$, $\hat{\psi}$ and nonpositive modes of $\hat{\phi}$, tensored with functions on U_X for the zero modes of $\hat{b}$.

Let us show that there are no additional cohomology elements. We will first handle the part of the cohomology where the fermion number plus the conformal weight of $\pi_*\mathrm{MSV}(W)$ is zero. This is the cohomology of the algebra of polyvector fields on U_W with respect to the contraction by α. We have already seen this at fermion number (-1) in Lemma 4.5. This is a Koszul complex for the ring $\mathcal{O}(U_W)$ and functions $y_{n+1}P_i$ and y_n. We can think of it as an exterior algebra over the ring of $\mathcal{O}(U_W)$ of the vector space with the basis $\hat{\psi}_j$, $1 \leqslant j \leqslant n-1$, ψ_n, ψ_{n+1}. Then we have the Koszul complex for $y_{n+1}P$ and y_n for the ring $\mathcal{O}(U_W)$ tensored with the exterior algebra in $\hat{\psi}_j$. It remains to observe that this Koszul complex has cohomology only at the degree zero term which is equal to $\mathcal{O}(U_X)$.

We will proceed by induction on the conformal weight plus fermion number. Conformal weight plus fermion number is simply the eigenvalue of the operator H which is the coefficient of z^{-2} of $L(z) - J(z)'$. By Lemma 4.11, this operator H is equal to the z^{-2} coefficient of $\hat{L}(z) - \hat{J}(z)'$. We can write

$$\hat{b}^j(z) = \sum_{n\in\mathbb{Z}} \hat{b}^j[n] z^{-n}, \qquad \hat{a}_j(z) = \sum_{n\in\mathbb{Z}} \hat{a}_j[n] z^{-n-1},$$
$$\hat{\phi}^j(z) = \sum_{n\in\mathbb{Z}} \hat{\phi}^j[n] z^{-n-1}, \qquad \hat{\psi}_j(z) = \sum_{n\in\mathbb{Z}} \hat{\psi}_j[n] z^{-n},$$

where the endomorphisms with index $[n]$ change the H-degree of homogeneous elements by $(-n)$. We have

$$\begin{aligned} H = {} & \sum_{n\in\mathbb{Z}_{>0}} \sum_j (-n)\hat{a}_j[-n]\hat{b}^j[n] + \sum_{n\in\mathbb{Z}_{<0}} \sum_j (-n)\hat{b}^j[n]\hat{a}_j[-n] \\ & + \sum_{n\in\mathbb{Z}_{>0}} \sum_j (-n)\hat{\phi}^j[-n]\hat{\psi}_j[n] - \sum_{n\in\mathbb{Z}_{<0}} \sum_j (-n)\hat{\psi}_j[n]\hat{\phi}^j[n]. \end{aligned} \tag{4.9}$$

Suppose we have proved the statement of the lemma for all eigenvalues of H that are less than some positive integer r. If an element v of the cohomology of $\mathrm{MSV}(U_W)$ with respect to $\mathrm{Res}_{z=0}\,\alpha(z)$ has positive H-eigenvalue r, then we have $v = \frac{1}{r}Hv$. Because of the normal ordering, when calculating Hv as in (4.9) one is applying first the modes that decrease the eigenvalue of H and thus by induction send v into the subalgebra generated by the $\hat{}$ fields. Thus Hv lies in this algebra, which furnishes the induction step. □

Remark 4.13. While the cohomology of $\pi_*\mathrm{MSV}(W)$ with respect to $\mathrm{Res}_{z=0}\,\alpha(z)$ is a well-defined sheaf of vertex algebras, the conformal structure is a priori not clear. In general, to define a conformal structure for the twisted chiral de Rham sheaf for a vector bundle E one needs to choose an isomorphism between $\Lambda^{n-1}E$ and $\Lambda^{n-1}TX$ (up to constant multiple). Specifically, one needs to be sure that in the local coordinates the exterior product of $\hat{\phi}^j$ corresponds to the exterior product of $d\hat{b}^j$ under the dual of the above isomorphism. There is a natural choice of isomorphism here that works globally for X as follows. We can think of the restriction $\alpha|_X$ as a section of $W^\vee \otimes TY|_X^\vee$, or a map $TY|_X \to W|X$. Then it defines a short exact sequence of bundles on X

$$0 \to E \to TY|_X \to W|X \to 0.$$

This provides a natural identification of $\Lambda^{n-1}E$ and $\Lambda^{n-1}TX$. Locally this amounts to the multiplication by P_n. In the notations above $\hat{\phi}$ and $d\hat{b}$ are not compatible, which accounts for the presence of the extra term $\frac{1}{2}(\ln P_n)''$ in $\hat{L}$. Consequently, for the globally defined conformal structure on the cohomology, we need to use $\hat{L}$ and $\hat{J}$ that are defined for P_n that's constant on X, in which case the extra terms in Lemma 4.11 do not appear.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.12 the cohomology is locally isomorphic to a free field vertex algebra. The conformal weight zero and fermion one and weight one and fermion number (-1) parts are naturally isomorphic to the sheaves of sections of $E^\vee$ and E by Lemmas 4.6 and 4.5 respectively. The statement now follows from [7]. □

Remark 4.14. Let us examine in more detail the field

$$\beta = b^{n+1}a_{n+1} + \phi^{n+1}\psi_{n+1}$$

featured prominently in Lemma 4.11. The action of $\mathbb{C}^*$ on W canonically defines a vector field which in local coordinates looks like $\psi_{\mathbb{C}^*} = b^{n+1}\psi_{n+1}$. Consider the OPE of the field

$$Q(z) = \sum_{i=1}^{n+1} a_i\phi^i$$

with $\psi_{\mathbb{C}^*}$. We get

$$Q(z)\beta(w) \sim (z-w)^{-2} + (z-w)^{-1}\beta(w).$$

Consequently, β is the image of $\psi_{\mathbb{C}^*}$ under the map $\mathrm{Res}_{z=0}\,Q(z)$. While Q itself depends on the choice of coordinates, see [12, Eq. (4.1c)], its residue does not. Thus, β is independent of the choice of the coordinate system.

Remark 4.15. If in addition the bundle W is the canonical line bundle on Y, then the total space of W is a Calabi–Yau. It has a natural nondegenerate volume form which is the derivative of the image in $\Lambda^n TW^\vee$ of the tautological section of $\pi^*\Lambda^n TY^\vee$. Thus, the J field on W is well-defined as is the field $\hat{J}$ on X. In fact, Lemma 4.11 shows that $\hat{J}$ is the image in the cohomology of the field

$$J-\beta$$

where β is defined in the above remark. The field $J-\beta$ descends naturally to X, which in this case is also a Calabi–Yau. The particular case when α was a gradient of a global function, linear on fibers, was considered in [1]. In this case, we get the usual (not twisted) chiral de Rham complex on X, with $N=2$ structure.

Remark 4.16. We observe that Theorem 4.1 holds in the algebraic setting. Namely, if Y, W, X and α are algebraic, then the statement holds for sheaves of vertex algebras in Zariski topology. Indeed, the calculations of Lemmas 4.4–4.6 are unchanged. We can pick rational functions y_i and Zariski open subsets U_X, U_Y and U_W as before, so that they generate the m/m^2 at all points in U_W. Then the partial derivatives of rational functions make sense as rational functions and the calculations of Lemmas 4.11 and 4.12 and Theorem 4.1 go through as well.

Proposition 4.17. *For* any affine *Zariski open subset U_Y the cohomology of* $\pi_*\mathrm{MSV}(W)$ *on U_Y by* $\mathrm{Res}_{z=0}\,\alpha(z)$ *is isomorphic to the sections of* $\mathrm{MSV}(X,E)$ *on $U_X=U_Y\cap X$.*

Proof. We can cover U_Y by smaller subsets on which the statement holds. Then the statement holds on their intersections by localization. The Čech complexes for $\pi_*\mathrm{MSV}(W)$ and $\mathrm{MSV}(X,E)$ for this cover of U_Y have no higher cohomology, because these sheaves are filtered with quasi-coherent quotients. Then the snake lemma finishes the proof. □

Remark 4.18. It appears plausible that one can replace the line bundle W by a vector bundle and apply the calculations of this section to subvarieties $X\subseteq Y$ which are defined by sections of a vector bundle. In particular, the approach should work for complete intersections of hypersurfaces.

Remark 4.19. It would be interesting to study to what extent one can use this approach to *define* the (twisted) chiral de Rham sheaf for hypersurfaces with some mild singularities.

5. Deformations of the cohomology of twisted chiral de Rham sheaf and CY/LG correspondence

In this section we want to show that the vertex algebras $V_{(F^\cdot),g}$ of Definition 3.1 are in some sense deformations of the cohomology of a twisted chiral de Rham sheaf constructed in [6–8] and further studied in [13]. Specifically, we will show that the cohomology of the chiral de Rham sheaf for the vector bundle on the quintic considered in [14] is equal to the cohomology of $\mathrm{Fock}^{\Sigma}_{M\oplus K^\vee}$ by the operator $D_{(F^\cdot),g}$ defined in Section 3. Our method also shows how one can produce more examples of calculations of cohomology of twisted chiral de Rham sheaf on hypersurfaces and complete intersections.

Let x_i, $0 \leqslant i \leqslant 4$ be homogeneous coordinates in $\mathbb{P}^4$. Let $F^i = x_i R^i$, $0 \leqslant i \leqslant 4$ be homogeneous polynomials of degree 5 as in Section 3. Consider the lattice vertex algebra $\mathrm{Fock}^{\Sigma}_{M \oplus K^\vee}$ and the operator

$$D_{(F^\cdot),g} = \mathrm{Res}_{z=0}\left(\sum_{\substack{m\in\Delta \\ 0\leqslant i\leqslant 4}} F^i_m m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)} + \sum_{n\in\Delta^\vee} g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}\right)$$

from Section 3.

Theorem 5.1. *The cohomology of* $\mathrm{Fock}^{\Sigma}_{M \oplus K^\vee}$ *with respect to* $D_{(F^\cdot),g}$ *is isomorphic to the cohomology of a twisted chiral de Rham sheaf on the quintic* $\sum_{i=0}^4 F^i = 0$ *given by* R^i.

Proof. Consider the canonical bundle $\pi : W \to \mathbb{P}^4$. Over the chart $x_j \neq 0$ on $\mathbb{P}^n$ the coordinates on W are $\frac{x_i}{x_j}$, $i \neq j$ and s_j. The coordinate changes are $s_k = s_j(\frac{x_k}{x_j})^5$. The data $(F^i = x_i R^i)$ give rise to a 1-form in an affine chart $x_k \neq 0$ defined as

$$\alpha_k = x_k^{-4} s_k \sum_{i=0}^4 R^i(x)\, d\left(\frac{x_i}{x_k}\right) + \frac{1}{5} x_k^{-5} \sum_{i=0}^4 x_i R^i\, ds_k.$$

It is easily checked that these forms glue together to a global 1-form α on W which is of weight one with respect to the $\mathbb{C}^*$ action on the fibers. The vector bundle E on $X = \{\sum_i F^i = 0\}$ constructed from this form in Section 4 is isomorphic to the bundle considered in [14].

All further arguments are essentially identical to those of [1]. One considers the cover of $\mathbb{P}^4$ and its canonical bundle W by toric affine charts. The cone $K^\vee$ is subdivided by a fan Σ. The cones of this fan correspond to toric charts on W. It was already seen in [1] that for a chart that corresponds to a face σ of $K^\vee$, the sections of the chiral de Rham complex on W correspond to the cohomology of $\mathrm{Fock}_{M \oplus \sigma}$ with respect to

$$D_g = \mathrm{Res}_{z=0} \sum_{n\in\Delta^\vee\cap\sigma} g_n n^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}.$$

By Proposition 4.17 the cohomology of a twisted chiral de Rham sheaf $\mathrm{MSV}(X, E)$ of $X = \{\sum_i x_i R^i = 0\}$ is isomorphic to the cohomology of $\mathrm{Fock}_{M \oplus \sigma}/D_g$ by the $\mathrm{Res}_{z=0}\,\alpha(z)$. It is a routine calculation to check that this corresponds precisely to the cohomology via

$$\mathrm{Res}_{z=0} \sum_{i=0}^4 \sum_{m\in\Delta} F^i_m m_i^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)}.$$

The spectral sequence for the cohomology of the sum degenerates, as in [1, Proposition 7.11]. This shows that the sections of the twisted chiral de Rham sheaf over the open chart are isomorphic to the cohomology of $\mathrm{Fock}_{M \oplus \sigma}$ by $D_{(F^\cdot),g}$.

Toric Čech cohomology as in [1, Theorem 7.14] finishes the proof. □

Remark 5.2. We observe that the fields J and L defined in Proposition 3.5 correspond precisely to the fields J and L of the twisted chiral de Rham complex. This is simply a matter of going through the calculations. The field β of Remark 4.14 turns out to be $(\deg^\vee)^{bos}$. The $\deg^{bos}$ part in Proposition 3.5 comes from the description of chiral de Rham complex in logarithmic coordinates, see [1, Proposition 6.4].

We are now ready to remark on Calabi–Yau/Landau–Ginzburg correspondence for $(0,2)$ theories. Consider the vertex algebras which are the cohomology of $\mathrm{Fock}_{M\oplus K^\vee}$ by $D_{(F^\cdot),g}$ as $(F^\cdot)$ is fixed and g varies. If we fix g_n for $n\neq \deg^\vee$ and let $g_{\deg^\vee}$ go to ∞, then in the limit the action of $D_{(F^\cdot),g}$ starts to resemble its action on $\mathrm{Fock}^\Sigma_{M\oplus K^\vee}$, after an appropriate reparametrization. This is the Calabi–Yau limit of the theory. The Landau–Ginzburg limit occurs for $g_{\deg^\vee}=0$, as in the $N=2$ case.

Remark 5.3. We do not know whether passing from the cohomology of $\mathrm{Fock}^\Sigma_{M\oplus K^\vee}$ by $D_{(F^\cdot),g}$ to the cohomology of $\mathrm{Fock}_{M\oplus K^\vee}$ by $D_{(F^\cdot),g}$ does not change the dimension of the graded pieces of the cohomology. From the physical point of view it is conceivable that instanton corrections result in some reduction of the dimension of the state space of the half-twisted $(0,2)$ theory.

6. Chiral rings

In this section we discuss the consequences of the machinery of [1,2] as it applies to the algebras $V_{(F^\cdot),g}$.

First, we observe that we can replace the cone $K^\vee$ by the whole lattice N.

Proposition 6.1. *The algebra $V_{(F^\cdot),g}$ can be alternatively described as the cohomology of $\mathrm{Fock}_{M\oplus N}$ or $\mathrm{Fock}_{K\oplus N}$ by $D_{(F^\cdot),g}$.*

Proof. This statement for the usual algebras $V_{f,g}$ is called the Key Lemma in [1] because of its importance to mirror symmetry. The argument is unchanged after one replaces the Koszul complex for $\mathbb{C}[K]$ and logarithmic derivatives of F by the Koszul complex for $\mathbb{C}[K]$ and F_i. □

As in the $N=2$ case we define operators H_A and H_B by

$$H_A=\mathrm{Res}_{z=0}\, zL(z),\qquad H_B=\mathrm{Res}_{z=0}\big(zL(z)+J(z)\big).$$

We then define the chiral rings of the theory as the parts of the vertex algebra where $H_A=0$ or $H_B=0$. The calculations of the paper [2] apply directly to this more general setting. Consider the commutative ring $\mathbb{C}[K\oplus K^\vee]$. Consider the quotient $\mathbb{C}[(K\oplus K^\vee)_0]$ by the ideal spanned with monomials with positive pairing. Consider the endomorphism $d_{(F^\cdot),g}$ on $\mathbb{C}[(K\oplus K^\vee)_0]\otimes \Lambda^* M_{\mathbb{C}}$ defined by

$$\sum_{i=0}^{4}\sum_{m\in\Delta} F^i_m[m]\otimes(m_i\wedge)+\sum_{n\in\Delta^\vee} g_n[n]\otimes(\mathrm{contr}.n). \tag{6.1}$$

It is a differential by a calculation similar to Proposition 3.2.

Theorem 6.2. *For generic $F^{\cdot}$ and g the eigenvalues of H_A and H_B on $V_{(F^{\cdot}),g}$ are nonnegative integers. The $H_A = 0$ part is given as the cohomology of the corresponding eigenspace of* $\mathrm{Fock}_{K \oplus K^{\vee} - \deg^{\vee}}$. *As a vector space, this is isomorphic to the cohomology of* $\mathbb{C}[(K \oplus K^{\vee})_0] \otimes \Lambda^* M_{\mathbb{C}}$ *by $d_{(F^{\cdot}),g}$ from* (6.1). *The $H_B = 0$ part comes from the corresponding eigenspace of* $\mathrm{Fock}_{K - \deg \oplus K^{\vee}}$. *As a vector space it is isomorphic to the cohomology of* $\mathbb{C}[(K \oplus K^{\vee})_0] \otimes \Lambda^* N_{\mathbb{C}}$ *by an operator similar to* (6.1) *where one replaces all wedge products by contractions and vice versa.*

Proof. One follows the argument of [2]. □

Remark 6.3. It would be interesting to compare this description of chiral rings to other known statements about the $(0,2)$ theories, see for example [9]. It also appears that the work of [11] is closely related to this paper.

Remark 6.4. It is possible, in the quintic case, to completely calculate the products in the chiral rings. However, this will be in the set of coordinates that is somewhat different from the usual Kähler parameters. We plan to return to this topic in future research.

7. Concluding comments

The main philosophical outcome of this paper is a simple observation that $(0,2)$ string theory in toric setting (at the level of half-twisted theory) is quite amenable to explicit calculations. The quintic case is however somewhat special, because one is dealing with a smooth ambient variety.

The most general possible toric framework to which one can hope to extend this setup should also combine the almost dual Gorenstein cones explored in [3]. From this perspective the most generic ansatz that we wish to make is the following.

Consider dual lattices M and N with elements $\deg \in M$ and $\deg^{\vee} \in N$. Consider *subsets* Δ and $\Delta^{\vee}$ in M and N respectively with the properties

$$\Delta \cdot \deg^{\vee} = \deg \cdot \Delta^{\vee} = 1, \qquad \Delta \cdot \Delta^{\vee} \geqslant 0.$$

In addition, the cones generated by Δ and $\Delta^{\vee}$ should be almost dual to each other, in some sense. It is possible that the technical definition of [3] would still be appropriate, but since it might not be, we feel that it may not be wise to present it here.

Consider the lattice vertex algebra $\mathrm{Fock}_{M \oplus N}$. Pick a basis m_i of M and n_i of N. Then one needs to consider collections of complex numbers F^i_m and G^i_n for all $i, m \in \Delta, n \in \Delta^{\vee}$ such that the operator

$$D_{(F^{\cdot}),(G^{\cdot})} = \mathrm{Res}_{z=0}\left(\sum_i \sum_{m \in \Delta} F^i_m m_i^{ferm}(z) \mathrm{e}^{\int m^{bos}(z)} + \sum_i \sum_{n \in \Delta^{\vee}} G^i_n n_i^{ferm}(z) \mathrm{e}^{\int n^{bos}(z)}\right)$$

is a differential on $\mathrm{Fock}_{M \oplus N}$ (and in fact we want the OPE of the above field with itself to be nonsingular).

Then we would like to consider the cohomology of $\mathrm{Fock}_{M \oplus N}$ by the above differential. The hope is that under some almost duality condition the Key Lemma of [1] still works and we can then show that this cohomology satisfies $H_A, H_B \geqslant 0$.

It is not clear what, if any, geometric meaning one would be able to ascribe to a generic family of algebras obtained in this fashion, but they appear to be very natural constructs to study. In this context the $(0, 2)$ mirror symmetry would simply correspond to a switch between M and N.

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References

[1] L. Borisov, Vertex algebras and mirror symmetry, Comm. Math. Phys. 215 (3) (2001) 517–557.
[2] L. Borisov, Chiral rings of vertex algebras of mirror symmetry, Math. Z. 248 (3) (2004) 567–591.
[3] L. Borisov, Berglund–Hübsch mirror symmetry via vertex algebras, preprint, arXiv:1007.2633.
[4] J. Ekstrand, R. Heluani, J. Källén, M. Zabzine, Non-linear sigma models via the chiral de Rham complex, Adv. Theor. Math. Phys. 13 (4) (2009) 1221–1254.
[5] V. Gorbounov, F. Malikov, Vertex algebras and the Landau–Ginzburg/Calabi–Yau correspondence, Mosc. Math. J. 4 (3) (2004) 729–779, 784.
[6] V. Gorbounov, F. Malikov, V. Schechtman, Gerbes of chiral differential operators, Math. Res. Lett. 7 (1) (2000) 55–66.
[7] V. Gorbounov, F. Malikov, V. Schechtman, Gerbes of chiral differential operators. II. Vertex algebroids, Invent. Math. 155 (3) (2004) 605–680.
[8] V. Gorbounov, F. Malikov, V. Schechtman, Gerbes of chiral differential operators. III, in: The Orbit Method in Geometry and Physics, Marseille, 2000, in: Progr. Math., vol. 213, Birkhäuser Boston, Boston, MA, 2003, pp. 73–100.
[9] J. Guffin, Quantum sheaf cohomology, a précis, preprint, arXiv:1101.1305.
[10] A. Kapustin, E. Witten, Electric–magnetic duality and the geometric Langlands program, Commun. Number Theory Phys. 1 (1) (2007) 1–236.
[11] I.V. Melnikov, M.R. Plesser, A (0, 2) mirror map, preprint, arXiv:1003.1303.
[12] F. Malikov, V. Schechtman, A. Vaintrob, Chiral de Rham complex, Comm. Math. Phys. 204 (2) (1999) 439–473.
[13] M.-C. Tan, Two-dimensional twisted sigma models, the mirror chiral de Rham complex, and twisted generalized mirror symmetry, J. High Energy Phys. 7 (2007) 013, 80 pp.
[14] E. Witten, Phases of $N = 2$ theories in two dimensions, Nuclear Phys. B 403 (1–2) (1993) 159–222.