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The jump number and the lattice of maximal antichains

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Abstract

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We expose a relationship of the jump number $s(P)$ and the length of the lattic of maximal antichains $l(\mathrm{MA}(P))$ of an ordered set P:

$$|P| - l(\mathrm{MA}(P)) - 1 \leq s(P) \leq |P| - \tfrac{1}{2}l(\mathrm{MA}(P)) - 1.$$

If P is of height one or if P is N-free, then $s(P)$ even equals $|P| - l(\mathrm{MA}(P)) - 1$. Motivated by this connection, we deduce properties of $\mathrm{MA}(P)$ for P not containing subdiagrams shaped as N or W or as the 6-elementary fence.

1. Introduction

The jump number, a quite natural parameter for ordered sets, has gained a lot of attention in the last ten years (cf. [2]). Computing the jump number can be regarded as a scheduling problem, albeit a special one from the scheduling point of view.

For a linear extension L of P we denote by $B(L)$ the set of bump-(nonsetup)-pairs $\{(a, b) \in P \times P \mid a <_P b,\ a <_L b\}$ and by $b(P)$ the maximal number of bumps realizable by some linear extension L, i.e., $\max\{|B(L)|: L$ a linear extension of $P\}$. The jump number $s(P)$ is then just $|P| - b(P) - 1$, i.e., the minimal number of jumps realizable by some L. For our purpose it is often more convenient to speak of $b(P)$ instead of $s(P)$. So let us bump directly into the subject with this chain of inequalities:

$$\tfrac{1}{2}l(\mathrm{MA}(P)) \leq b(P)) \leq l(\mathrm{MA}(P)) \leq \operatorname{rank} M(P) \leq |P| - w(P).$$

The jump number $s(P)$ is obviously bounded below by $w(P) - 1$, where $w(P)$ denotes the width of P. This was already noticed in the first papers on this subject. Gierz and Poguntke [5] improved this bound by using the rank of the following incidence matrix indexed by elements of P, namely

$$M(P)=\begin{cases}1 & \text{if } x<y,\\ 0 & \text{else.}\end{cases}$$

They proved that $b(P)\leq \operatorname{rank} M(P)\leq |P|-w(P)$ and discussed various cases in which $b(P)$ equals rank $M(P)$. Faigle, Schrader and Gierz continued the research in this direction in several papers [6–8]. We shall expose in this paper the bounds involving $l(\mathrm{MA}(P))$, this is the length of the lattice of maximal antichains (the definitions will be given below). We shall also show that in the case of a height-one order $b(P)$ even equals $l(\mathrm{MA}(P))$. The bound of $b(P)$ by $l(\mathrm{MA}(P))$ is sharper than the older bounds. For this we have to pay the price that computing $l(\mathrm{MA}(P))$ might be difficult. For e.g., in the height-one case computing $b(P)$ and so $l(\mathrm{MA}(P))$ is known to be a NP-complete problem [10]. But nevertheless, for certain classes of P the computation of $l(\mathrm{MA}(P))$ can be easy. This is especially the case if $\mathrm{MA}(P)$ is graded. This led us to investigate $\mathrm{MA}(P)$ more detailed in Section 3. One of the results is that $\mathrm{MA}(P)$ is modular if P does not contain F, the 6-elementary fence, as a sub-diagram, implying that the jump number problem is polynomial for height-one orders not containing F. This improves a result of Faigle and Schrader on M-free height-one orders [8]. In Section 4 we investigate decompositions of P induced by maximal chains of $\mathrm{MA}(P)$ in case that P is W- or N-free. This then will be related to some known results about the jump number of these special classes of orders. In Section 5 greedy kind of extensions will be proposed for finding good linear extensions.

A comment on the development of this paper: Originally we have discovered a connection of the concept lattice of $(P, P, \ngtr)$ to the jump number of P. Later on we learned from Wille that this lattice is isomorphic to the lattice of maximal antichains of P. Thus with speaking about $\mathrm{MA}(P)$ it is now quite hidden that we have gained most of our results by knowledge of formal concept analysis.

For $A\subseteq P$, $\downarrow A$ denotes the order ideal generated by A, i.e. $\{x\in P \mid \text{there exists } y\in A \text{ with } x\leq y\}$, and $\uparrow A$ denotes the order filter generated by A which is defined dually. $\mathrm{MA}(P)$ denotes the set of maximal (not necessary maximal seized!) antichains of P. The lattice of maximal antichains of P, $(\mathrm{MA}(P), \leq)$ is defined by $A_1\leq A_2 :\Leftrightarrow \downarrow A_1\subseteq \downarrow A_2$ for $A_1, A_2\in \mathrm{MA}(P)$. A bit inaccurate we often denote this lattice as $\mathrm{MA}(P)$ instead of $(\mathrm{MA}(P), \leq)$. Fig. 1 represents two examples.

The height $h(P)$ or the length $l(P)$ of an order P is the number of elements of a maximal seized chain of P minus one (we usually use the height for orders and the length for lattices). E.g., $l(\mathrm{MA}(F))=3$ in Fig. 1.

Further terminology: $\downarrow x$ means $\downarrow\{x\}$ and $\downarrow_L A$ means the order ideal of A with respect to the order L. For $X, Y\subseteq P$ we briefly write $X\leq Y$ instead of $x\leq y$ for all $x\in X$ and $y\in Y$. E.g., $X\ngtr Y$ reads $x\ngtr y$ for all $x\in X$ and $y\in Y$; $\min(P)$ and $\max(P)$ denote the sets of minimal and maximal elements, respectively. Everything is assumed to be finite.

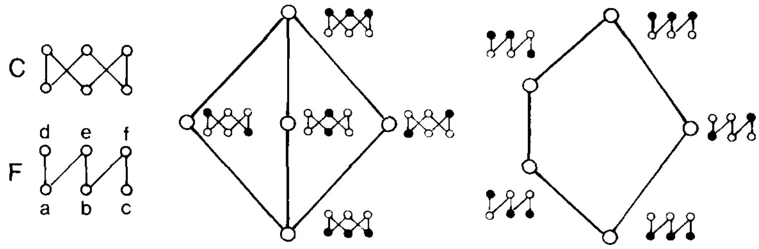

Fig. 1.

2. The main result

The two fundamental lattices related to orders, the distributive lattice of order ideals and the lattice of the MacNeille completion arise naturally in the setting of concept analysis. Due to an observation of Wille the lattice of maximal antichains fits also in this scheme. So let us record:

$\mathfrak{B}(P, P, \leqslant)$ represents the MacNeille completion (completion by cuts) of P.

$\mathfrak{B}(P, P, \ngeqslant)$ represents the lattice of order ideals of P.

$\mathfrak{B}(P, P, \ngtr)$ represents the lattice of maximal antichains of P.

We recall some notions of concept analysis. The triple (G, M, I) is called a *context* where G and M are sets and I is a binary relation of G and M. For $X \subseteq G$ and $Y \subseteq M$, XIY means xIy for all $x \in X$ and $y \in Y$. For $X \subseteq G$ (resp. $Y \subseteq M$) X' (resp. Y') denotes the largest subset of M (resp. G) with XIX' (resp. $Y'IY$). The pair (X, Y) is called a *concept* if $X' = Y$ and $X = Y'$. The concepts of (G, M, I) are ordered by $(X_1, Y_1) \leqslant (X_2, Y_2) :\Leftrightarrow X_1 \subseteq X_2$, and they build the so-called *concept lattice* $\mathfrak{B}(G, M, I)$. For further information we refer to [13].

Proposition 2.1. *$(MA(P), \leqslant)$ is isomorphic to $\mathfrak{B}(P, P, \ngtr)$.*

Proof. An antichain $A \in \mathrm{MA}(P)$ corresponds uniquely to an order ideal C and an order filter D of P, for which $C \cup D = P$ and $C \cap D = \max(C) = \min(D)$. It is established by setting $C := {\downarrow}A$ and $D := {\uparrow}A$ given $A \in \mathrm{MA}(P)$, and by setting $A := C \cap D$ given C and D with the described properties. We shall show that $(C, D) \in \mathfrak{B}(P, P, \ngtr)$ if and only if $C = {\downarrow}C$, $D = {\uparrow}D$ and $C \cap D = \max(C) = \min(D)$ and $C \cup D = P$. Let $(C, D) \in \mathfrak{B}(P, P, \ngtr)$. Let $c \in C$ and $p \in P$ with $p \leqslant c$. Then $c \ngtr d$ for all $d \in D$ and therefore $p \ngtr d$ for all $d \in D$, i.e., $p \in D' = C$. Thus C is an order ideal and dually D is an order filter. Now we show that $C \cap D = \max(C)$ and conclude $C \cap D = \min(D)$ by duality. $\max(C) \subseteq D$ holds, because $x \ngtr c$ for all $x \in \max(C)$ and $c \in C$, which implies $x \in C' = D$. In order to show that $C \cap D \subseteq \max(C)$, we conclude that $x \in C \cap D = C \cap C'$ implies $x \ngtr c$ for all $c \in C$ and hence $x \in \max(c)$. Similarly $C \cup D = P$ follows, because $x \in P \setminus C$ implies $c \ngtr x$ for all $c \in C$ and hence $x \in C' = D$.

On the other hand it is easy to see that $C = {\downarrow}C$, $D = {\uparrow}D$, $C \cap D = \max(c) = \min(D)$ and $C \cup D = P$ implies $(C, D) \in \mathfrak{B}(P, P, \ngtr)$. □

We continue with two technical lemmas. Given an antichain A of P, the completion of A to a maximal antichain is not unique but there exists a unique lowest completion. We can equally speak of the generated order ideals. Then the completion of the ideal I is given by the double prime closure I'' in the context $(G=P, M=P, \not\succ)$ where I is considered as a subset of G. Thus $I''=\{x\in P \mid x \not\succ y$ for all $y \in I'\}$ and $I'=\{y \in P \mid x \not\succ y$ for all $x \in I\}$. E.g., for $P=F$ of Fig. 1 it holds: $\{a, b, c, e\}'=\{c, d, e, f\}$ and $\{a, b, c, e\}''=\{c, d, e, f\}'=\{a, b, c, d, e\}$. The lowest completion of the antichain $\{c, e\}$ is therefore the maximal antichain $\{c, d, e\}$. In the following lemmas we give conditions under which this closure operator generates new elements.

Lemma 2.2. *Let $I_1, I_2 \subseteq P$ with $I_1=\downarrow I_1$, $I_2=\downarrow I_2$ and $I_1 \subseteq I_2$. Then the existence of $x \in \max(I_1)$, $y \in I_2 \backslash I_1$ with $x<y$ implies $I_1'' \subsetneq I_2''$. If $|I_2 \backslash I_1|=1$, then $I_1'' \subsetneq I_2''$ implies the existence of $x \in \max(I_1)$, $y \in I_2 \backslash I_1$ with $x<y$.*

Proof. $I_1'' \subsetneq I_2''$ is equivalent to $I_2' \subsetneq I_1'$. Also for an order ideal I of P it holds $I'=(P \backslash I) \cup \max(I)$. Now we first assume that $|I_2 \backslash I_1|=1$ and $I_2' \subsetneq I_1'$. Then there exists $x \in I_1' \backslash I_2'$. This implies that $x \in (P \backslash I_1) \cup \max(I_1)$ and that there exists $y_0 \in I_2$ with $x<y_0$. The assumption $|I_2 \backslash I_1|=1$ implies $x \in I_1$ which leads to $x \in \max(I_1)$. If we choose an element $y \in P$ with $x<y \leqslant y_0$, then x and y are the wanted elements. Now we assume that there exist $x \in \max(I_1)$, $y \in I_2 \backslash I_1$ with $x<y$. It follows that $x \in I_1'$ because of $x \in \max(I_1)$, but that $x \notin I_2'$ because of $x \in I_2 \backslash \max(I_2)$. Therefore $I_2' \subsetneq I_1'$. □

Lemma 2.3. *Let $I \subseteq P$, $I=\downarrow I$, $x \in \max(I)$, $y \in P$ and $y>x$. Then $y \notin I''$.*

Proof. Assume to the contrary that $y \in I''$. Then $y \not\succ z$ for all $z \in I'$ which means $y \not\succ z$ for all $z \in ((P \backslash I) \cup \max(I))$. But this contradicts $y>x$ with $x \in \max(I)$. □

The following lemma clarifies when two maximal antichains are in cover relation in $\mathrm{MA}(P)$. A height-one order is called *complete* if all minimal elements are below all maximal elements.

Lemma 2.4. *Let $A_1, A_2 \in \mathrm{MA}(P)$. Then $A_1 \prec A_2$ if and only if $(\uparrow A_1 \cap \downarrow A_2) \backslash (A_1 \cap A_2)$ induces a complete height-one suborder, say Q, of P with $\max(Q)=A_2 \backslash A_1$ and $\min(Q)=A_1 \backslash A_2$.*

Proof. If Q is as described and there is $A \in \mathrm{MA}(P)$ with $A_1<A<A_2$, then $A \cap (A_1 \backslash A_2) \neq \emptyset$ and $A \cap (A_2 \backslash A_1) \neq \emptyset$ which is a contradiction to the fact that A is an antichain.

Now we assume that $A_1 \prec A_2$. To begin with we show that $h(\uparrow A_1 \cap \downarrow A_2)=1$. If $h(\uparrow A_1 \cap \downarrow A_2)=0$ then $A_1 \cup A_2$ is an antichain and hence $A_1=A_2$, a contradiction. We suppose that $z_0, z_1, z_2 \in (\uparrow A_1 \cap \downarrow A_2)$ with $z_0<z_1<z_2$. Set

$$I_0:=\downarrow(A_1 \cup \{z_0\}), \quad I_1:=\downarrow(A_1 \cup \{z_0, z_1\}), \quad I_2:=\downarrow(A_1 \cup \{z_0, z_1, z_2\}).$$

By Lemma 2.2 $\downarrow A_1 \subseteq I_0'' \subsetneq I_1'' \subsetneq I_2'' \subseteq \downarrow A_2$, which contradicts $A_1 \prec A_2$.

Now, after it is shown that $h(\uparrow A_1 \cap \downarrow A_2) = 1$, it is clear that

$$(\uparrow A_1 \cap \downarrow A_2) \setminus (A_1 \cap A_2) = (A_1 \setminus A_2) \,\dot\cup\, (A_2 \setminus A_1) = \min(Q) \,\dot\cup\, \max(Q).$$

It remains to show that $x < y$ for all $x \in A_1 \setminus A_2$ and $y \in A_2 \setminus A_1$. We assume that $x \not< y$ for some $x \in A_1 \setminus A_2$ and $y \in A_2 \setminus A_1$. There must exist some $z \in A_2 \setminus A_1$ with $x < z$ since otherwise A_2 would not be a maximal antichain. Now setting $I := \downarrow(A_1 \cup \{y\})$ obviously $A_1 \subsetneq I''$ and also $I'' \subsetneq A_2$ since $x \in \max(I)$ and therefore $z \notin I''$ by Lemma 2.3. But this contradicts $A_1 \prec A_2$. □

Lemma 2.5. *Let $A_0 \prec A_1 \prec \cdots \prec A_n$ be a maximal chain of* $\mathrm{MA}(P)$ *and let Q_1 be the complete height-one suborder of P induced by A_{i-1} and A_i for $i = 1, \ldots, n$. Then*

$$P = \min(P) \,\dot\cup\, \dot\bigcup \max(Q_i) = \max(P) \,\dot\cup\, \dot\bigcup \min(Q_i);$$

and for $i, j \in \{1, \ldots, n\}$, $i > j$ and $x \in \min(Q_i)$ and $y \in \max(Q_j)$ implies $x \not< y$.

Proof. By Lemma 2.4 it follows that $\max(Q_i) = A_i \setminus A_{i-1} = \downarrow A_i \setminus A_{i-1}$ for $i = 1, \ldots, n$. Now

$$P = A_0 \,\dot\cup\, \dot\bigcup (\downarrow A_i \setminus \downarrow A_{i-1}) = \min(P) \,\dot\cup\, \dot\bigcup \max(Q_i).$$

The second equality of the lemma follows dually.

Now let $i, j \in \{1, \ldots, n\}$, $i > j$, $x \in \min(Q_i)$ and $y \in \max(Q_j)$. It is clear that $x \in A_{i-1}$ and $y \in A_j$. Because of $i - 1 \geq j$ it is $\downarrow A_j \subseteq \downarrow A_{i-1}$. Now $x \in A_{i-1} = \max(\downarrow A_{i-1})$ and $y \in A_j \subseteq \downarrow A_{i-1}$ implies $x \not< y$. □

Lemma 2.5 will be of importance throughout this paper. Whenever we speak of certain Q's we refer to this lemma. Now our main result follows.

Theorem 2.6.

$$s(P) \geq |P| - l(\mathrm{MA}(P)) - 1. \tag{1}$$

$$|P| - l(\mathrm{MA}(P)) - 1 \geq |P| - \operatorname{rank} M(P) - 1. \tag{2}$$

If P is a height-one order then $s(P) = |P| - l(\mathrm{MA}(P)) - 1.$ (3)

Proof. (1) We have to show that $b(P) \leq l(\mathrm{MA}(P))$. Let L be an optimal linear extension of P with the bumps $(a_1, b_1), \ldots, (a_n, b_n)$. Setting $I_i := \downarrow_L a_i$ and $J_i := \downarrow_L b_i$ it follows that $I_1'' \subsetneq J_1'' \subseteq I_2'' \subsetneq J_2'' \subseteq \cdots \subseteq I_n'' \subsetneq J_n''$ by Lemma 2.2. This yields a chain of length n in $\mathrm{MA}(P)$.

(2) We have to show that $l(\mathrm{MA}(P)) \leq \operatorname{rank} M(P)$. Consider a maximal seized chain of $\mathrm{MA}(P)$, say of length n. Let the induced Q_i be as in Lemma 2.5. According to this Lemma 2.5 the matrix $M(P)$ can be stated as shown in Fig. 2. Now it is obvious that $M(P)$ has at least rank n.

	min(P)	max(Q₁)	max(Q₂)			max(Qₙ)
min(Q₁)	0	1				
min(Q₂)	0	0	1			
⋮	0	0	0	1		
			⋱			
min(Qₙ)				⋱		1
max(P)	0	0	0			0

Fig. 2.

(3) Let P be of height one and let $A_0 < A_1 < \cdots < A_n$ be a maximal seized chain of MA(P). Let the induced Q_i be as in Lemma 2.5. Because of the last statement of this Lemma 2.5 there must exist a linear extension L of P such that

$$\min(Q_1) <_L \max(Q_1) <_L \cdots <_L \min(Q_n) <_L \max(Q_n) <_L (\min(P) \cap \max(P)).$$

This linear extension has n bumps. Thus

$$b(P) \geq n \quad \text{and} \quad s(P) \leq |P| - l(\mathrm{MA}(P)) - 1. \qquad \square$$

In Fig. 3 two sequencies of orders are represented, P_n and $S_n (n \in \mathbb{N})$ which show how extreme on the one side $b(p)$ and $l(\mathrm{MA}(P))$ and on the other side $l(\mathrm{MA}(P))$ and rank $M(P)$ can differ: $l(\mathrm{MA}(P_n)) = 2n$ and $b(P_n) = n + 1$; $l(\mathrm{MA}(S_n)) = 2$ and rank $M(S_n)$ is n or $n - 1$ depending on the characteristic of the based field.

Habib suggested to us to consider the following proposition.

Proposition 2.7. *If $s(P) = |P| - l(\mathrm{MA}(P)) - 1$ then every optimal linear extension of P is greedy.*

Proof. Let L be an optimal linear extension of P with the bumps (a_i, b_i) for $i = 1, \ldots, n$ and assume that L is not greedy which means that there exists $x, y \in P$ with $x < y$, $x \notin \{a_i \mid i \in \{1, \ldots, n\}\}$ and $\downarrow y \subseteq \{y\} \cup \downarrow_L x$. Moreover $y \notin \{b_i \mid i \in \{1, \ldots, n\}\}$ since $y = b_i$ and $x <_L y$ would imply $x <_L a_i$ and therefore $a_i \in \downarrow y \setminus (\{y\} \cup \downarrow_L x)$, a contradiction. Now let L' denote the linear extension

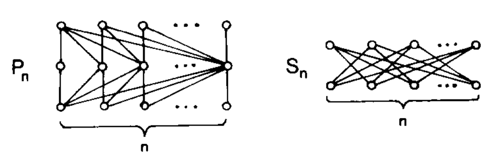

Fig. 3.

build from L by putting y directly above x, i.e., $x <_{L'} y$. Then the chain of ideals

$$\downarrow_{L'} a_1 \subseteq \downarrow_{L'} b_1 \subseteq \cdots \subseteq \downarrow_{L'} a_i \subseteq \downarrow_{L'} B_i \subseteq \downarrow_{L'} x \subseteq \downarrow_{L'} y \subseteq \cdots \subseteq \downarrow_{L'} a_n \subseteq \downarrow_{L'} b_n$$

yields by taking the closure a chain in MA(L) of length $n+1$, which is a contradiction to the assumption that $n = l(\mathrm{MA}(P))$. □

It is known that the jump number is related to maximal cycle free matchings [2, 9]. Here we denote by matching a set of covering pairs $(a_1, b_1), \ldots, (a_n, b_n)$ with $a_i < b_i$. We build a directed graph with the covering pairs as vertices and with $(a_i, b_i) \to (a_j, b_j)$ iff $a_i < a_j$. We call the matching cycle free if the corresponding graph is cycle free. By $m(P)$ we denote the cardinality of a maximal cycle free matching of P.

Proposition 2.8. $l(\mathrm{MA}(P)) = m(P)$.

Proof. Consider a maximal seized chain of MA(P), say of length n, and let the Q_i's be as in Lemma 2.5. From each Q_i choose a covering pair (a_i, b_i). By Lemma 2.5 $a_i \nless b_j$ for $i > j$ and therefore the matching consisting of the pairs (a_i, b_i) $(i = 1, \ldots, n)$ is cycle free. This shows that $l(\mathrm{MA}(P)) \leq m(P)$. Now assume that n covering pairs (a_x, b_x) $(x \in X)$ are cycle free in P. Then the graph restricted to this pairs is cycle free and hence a preorder which can be extended to a linear order, say (a_1, b_1), $(a_2, b_2), \ldots, (a_n, b_n)$, such that there is no arrow from (a_i, b_i) to (a_j, b_j) for $i > j$. This means $a_i \nless b_j$ for $i > j$. But under this condition there exists a linear extension L of P such that

$$a_1 <_L b_1 \leq_L a_2 <_L b_2 \leq_L \cdots \leq_L a_n <_L b_n.$$

This proves $m(P) \leq b(P) \leq l(\mathrm{MA}(P))$. □

We finished this section deducing the lower bound of $b(P)$ in terms of $l(\mathrm{MA}(P))$. PP is defined to be the height one order build from P with $\min(PP) = P \times \{0\}$ and $\max(PP) = P \times \{1\}$ and $(x, 0) < (y, l)$ in PP iff $x < y$ in P.

Lemma 2.9. $MA(P) \cong \mathrm{MA}(PP)$.

Proof. The context $(P, P, \ngtr)$ is a reduced form of the context $(PP, PP, \ngtr)$. The concept lattice does not change when a context is reduced. □

Similar as we have described a cycle free matching we shall now describe what we call cycle free chains. Let $C_1, \ldots, C_r$ be chains of P and let $d(C_i)$ denote the smallest and $u(C_i)$ denote the highest element of C_i. We define a directed graph with the chains as vertices and with $C_i \to C_j$ iff $d(C_i) < u(C_j)$. The chains are called cycle free if this graph is cycle free. A subset S of P is called convex if $x < y < z$ and $x, z \in S$ always implies $y \in S$.

Proposition 2.10. *If $C_1, \ldots, C_r$ are disjoint, convex, cycle free chains of P such that $\sum l(C_i)$ is maximal, then $b(P) = \sum l(C_i)$.*

Proof. An optimal linear extension of P naturally induces a chain partition, say $C_1 \cup \cdots \cup C_r = P$, for which the chains are disjoint, convex and cycle free, and for which $\sum l(C_i) = |B(L)|$. Hence $b(P) \leqslant \sum l(C_i)$.

Now let $C_1, \ldots, C_r$ be disjoint, convex, cycle free chains of P. We assume that the chains are already ordered such that there is no arrow from C_j to C_i for $j > i$. Setting $T_i := (\downarrow C_i) \setminus C_i$, $T_0 := \emptyset$, and $S_i := \bigcup_{j=1}^{i-1} \downarrow T_j$ for $i = 1, \ldots, r$ one has the following extension of P: $S_1 < C_1 < S_2 < C_2 < \cdots < S_r < C_r <$ rest of P. For a linear extension L of this order $|B(L)| \geqslant \sum l(C_i)$. $\square$

Theorem 2.11.

$$\tfrac{1}{2} b(PP) \leqslant b(P) \leqslant b(PP)$$

and

$$\tfrac{1}{2} l(\mathrm{MA}(P) \leqslant b(P) \leqslant l(\mathrm{MA}(P)).$$

Proof. By Lemma 2.9 and Theorem 2.6, $b(PP) = l(\mathrm{MA}(P))$. Since $b(P) \leqslant l(\mathrm{MA}(P))$ was already shown in Theorem 2.6, it remains to prove that $\frac{1}{2} b(PP) \leqslant b(P)$. Let $(a_1, b_1), \ldots, (a_n, b_n)$ be the bumps in PP of some optimal linear extension. Then it is easy to see that the corresponding pairs in P must be covering pairs. These covering pairs induce disjoint chains in P. Let $c_0 < c_1 < \cdots < c_s$ be such a chain. Then we pick the first, the third, the fifth, and so on pair of this chain, i.e. (c_0, c_1), (c_2, c_3), $(c_4, c_5), \ldots$. Doing this for all chains one gets at least $n/2$ pairs together. All these pairs can be considered as chains in P, which are disjoint, convex and cycle free. So $\frac{1}{2}\, b(PP) \leqslant n/2 \leqslant b(P)$. $\square$

With some more effort one can even show that $\frac{1}{2} l(\mathrm{MA}(P)) < b(P)$. One has to take a greedy maximal chain of $\mathrm{MA}(P)$ as described at the end of Section 5 and has to choose special of the Q_i's. This slightly stronger bound is best possible as can be seen from the example of P_n of Fig. 3.

3. More about the lattice of maximal antichains

The first question to be asked is whether $\mathrm{MA}(P)$ has some lattice properties in general. The answer is no. Every lattice can be a lattice of maximal antichains. The following proposition can be found in [1] and is implicitely contained in [14]. Let L be a lattice and $J(L)$ and $M(L)$ be the set of join and meet irreducible elements. We define a height one order P by taking $J(L)$ as the maximal and $M(L)$ as the minimal elements and by setting $m <_P u$ iff $m \not>_L u$ for $u \in J(L)$ and

$m \in M(L)$. Now the context $(P, P, \not>_P)$ can be reduced (see [13]) to $(\max(P), \min(P), \not>_P)$ which is isomorphic to $(J(L), M(L), \leq_L)$. Therefore

$$\mathrm{MA}(P) \cong \mathcal{B}(P, P, \not>_P) \cong \mathcal{B}(J(L), M(L), \leq_L) \cong L.$$

This proves the following proposition.

Proposition 3.1. *For every lattice L there exists a height-one order P such that $L \cong \mathrm{MA}(P)$.*

To analyse $\mathrm{MA}(P)$ it is useful to know some lattice theoretical notions as gluing, the skeleton of a lattice and tolerance relations [9, 15]. We do not assume this in the following but want to state at least this fact (cf. [14]):

Proposition 3.2. $\mathrm{MA}(P)$ *is isomorphic to the skeleton of the lattice of order ideals of P.*

With respect to the jump number our main interest is to determine $l(\mathrm{MA}(P))$. Therefore it is natural to ask under which conditions on P $\mathrm{MA}(P)$ is graded (i.e., has a rank function)? The next proposition says that for lattices in general the property to be graded can be localized up to a certain degree.

Proposition 3.3. *A lattice L is graded if and only if for all $x, y, z \in L$ with $x \prec y, z$ the interval $[x, y \vee z]$ is graded.*

Proof. We are going to show that under the stated conditions $r(x) := l(L) - l([x, 1_L])$ is a rank function. We assume by induction that this holds for all proper filter of L. It remains to show that $l([0_L, 1_L]) = 1 + l([x, 1_L])$ for all $0_L \prec x$. This means that $l([x, 1_L])$ has to equal $l([y, 1_L])$ for all $0_L \prec x, y$. This follows from the identities

$$l([y, 1_L]) = l([y, y \vee z] + l([y \vee z, 1_L],$$
$$l([z, 1_L] = l([z, y \vee z]) + l([y \vee z])$$

and

$$l([y, y \vee z]) = l([z, y \vee z]). \qquad \square$$

The example in Fig. 4 illustrates this proposition. The small 6-elementary lattices are graded and so is the whole lattice.

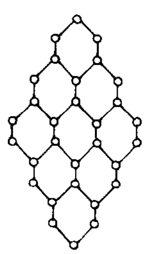

Fig. 4.

The next lemma shows how an interval of MA(P) naturally corresponds to the lattice of maximal antichains of some induced suborder of P.

Lemma 3.4. *Let* $A_1, A_2 \in \mathrm{MA}(P)$, $A_1 \leq A_2$ *and* $Q := \downarrow A_2 \cap \uparrow A_1$. *Then* $\mathrm{MA}(Q) \cong [A_1, A_2]$.

Proof. One has to check that an antichain A which is maximal with respect to the subposet Q of P is also maximal with respect to P. W.l.o.g. let $x \in (\downarrow A_1)\backslash A_1$. Then there exists $a_1 \in A_1$ with $x < a_1$. Now there has to exist some $a \in A$ with $a_1 \leq a$, since otherwise A would not be maximal in Q. Thus it follows $x < a$. $\square$

Now we show that what can be localized in MA(P) in a spanned interval can be localized in P in a height-one suborder.

Lemma 3.5. *Let* $A_1, A_2, C_1, \ldots, C_n \in \mathrm{MA}(P)$ *with* $A_1 < C_i$ *and* $A_2 = \bigvee C_i (i = 1, \ldots, n)$. *Setting* $Q := \downarrow A_2 \cap \uparrow A_1$ *it holds* $h(Q) \leq 1$.

Proof. By Lemma 2.4 it is $h(\downarrow C_i \cap \uparrow A_1) = 1$ for $i = 1, \ldots, n$. This implies $h(\uparrow A_1 \cap \bigcup \downarrow C_i) = 1$ and by Lemma 2.3 it follows that $h(\uparrow A_1 \cap (\bigcup \downarrow C_i)'') = 1$. This together with $(\bigcup \downarrow C_i)'' = \downarrow A_2$ proves the assertion. $\square$

From Proposition 3.3 and Lemma 3.5 we know now that MA(P) is graded iff MA(Q) is graded for all height-one suborders Q of P. Still this does not help so much. In a lot of cases MA(P) is not graded. Even in the case of a boolean lattice B_n with n atoms its lattice of maximal antichains is not graded in general. In MA(B_6) there exist maximal chains of length 9 and 10 between the antichains of all 2- and 3-elementary subsets of B_6. They are described in Fig. 5 by its Q_i's. This answers a question asked by Wille at the Oberwolfach meeting on Orders and Combinatories in 88 to the negative, whether the skeleton of a boolean lattice is always graded.

Q_1	12; 123, 124, 125, 126	Q_1	16; 156, 146, 136, 126
Q_2	13; 134, 135, 136	Q_2	26; 256, 246, 236
Q_3	14; 145, 146	Q_3	34; 346, 345, 234, 134
Q_4	15, 16; 156	Q_4	15; 145, 135, 125
Q_5	23; 234, 235, 236	Q_5	25; 235, 245
Q_6	24; 245, 246	Q_6	35, 36; 356
Q_7	25, 26; 256	Q_7	14, 24; 124
Q_8	34; 345, 346	Q_8	45, 46, 56; 456
Q_9	35, 36; 356	Q_9	12, 13, 23; 123
Q_{10}	45, 46, 56; 456		

Fig. 5.

In the following we show that MA(P) is modular or distributive if P does not contain certain orders as subdiagrams. F denotes the 6-elementary fence and C the 6-elementary crown (see Fig. 1).

Theorem 3.6. MA(P) *is modular if P is F-free.*

Proof. Assume that MA(P) is not modular. Then it is not upper semimodular or not lower semimodular. W.l.o.g. we assume the first case. Then there exists an N_5-sublattice in MA(P) as represented in Fig. 6(a) with the additional property that $A_0 \prec A_1$ and $A_0 \prec A_3$. By Lemma 3.5 the suborder $Q := \uparrow A_0 \cap \downarrow A_4$ is of height one. Setting $Q' := Q \setminus (A_0 \cap A_4)$ and $A_i' = A_i \cap Q' (i = 1, \ldots, 5)$ it is easy to see that the A_i' form a corresponding N_5 in MA(Q') (deleting isolated points from an order does not change its lattice of maximal antichains). Now let $A_i' = C_i \cup D_i$ with $C_i \subseteq \max(Q')$ and $D_i \subseteq \min(Q')$. Observe that $C_0 = \emptyset$ and $D_4 = \emptyset$. Now $A_2 \wedge A_3 = A_0$ implies $C_3 \cap C_2 = C_0 = \emptyset$, and $A_1 \vee A_3 = A_4$ implies $D_1 \cap D_3 = D_4 = \emptyset$. Also, all the sets C_1, $C_2 \setminus C_1$, C_3, D_2, $D_1 \setminus D_2$, D_3 must be non-empty (see Fig. 6(b)). For $c_2 \in C_2 \setminus C_1$ there exists $d_1 \in D_1 \setminus D_2$ with $d_1 < c_2$ since otherwise A_1 would not be a maximal antichain. Similar there exists $c_3 \in C_3$ with $c_3 > d_1$, since otherwise A_3 would not be maximal. Going on with this kind of argument one finds $d_2 \in D_2$ with $d_2 < c_3$, $d_3 \in D_3$ with $d_3 < c_2$ and $c_1 \in C_1$ with $c_1 > d_3$. Since the elements of each A_i are incomparable it follows that $d_2 \nless c_2$, $d_2 \nless c_1$, $d_1 \nless c_1$, $d_3 \nless c_3$. This gives us the wanted fence in Q' and so in P. □

As a consequence of this Theorem the jump number problem of F-free, height-one orders is polynomial. Even more: One can take an arbitrary linear extension L and can easily construct an optimal L' such that $B(L) \subseteq B(L')$. This might be of interest from an applied point of view. Because of additional constraints one might want to have certain bumps to be included and then knows that one can complete them to get an optimal extension. The construction of L' is as follows: We build from L a chain C in MA(P) as in the proof of Theorem 2.6(1). Now C is contained in a maximal chain C' of MA(P) and one can build L' from C' as in the proof of Theorem 2.6(3). It is of course not necessary to know the whole MA(P) for doing this.

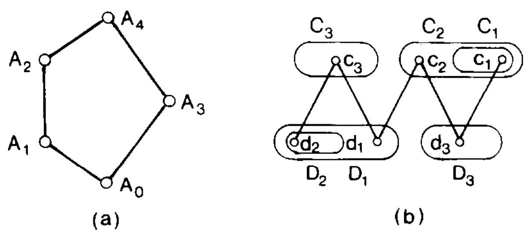

Fig. 6.

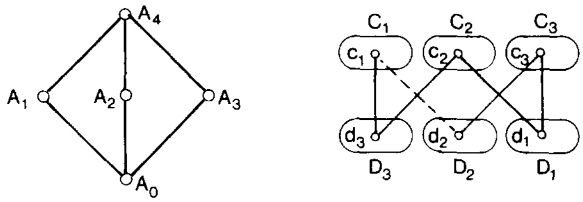

Fig. 7.

Theorem 3.7. MA(P) *is distributive if* P *is* F- *and* C-*free.*

Proof. We preceed as in the proof of Theorem 3.6. We want to show that P contains an F or C if MA(P) is not distributive. Because of Theorem 3.6 we can assume that MA(P) is not distributive but modular. Then it is known that MA(P) must contain a covering dense M_3. Now the same arguments can be applied as in the foregoing proof (see Fig. 7). In this case C_1, C_2, C_3 and D_1, D_2, D_3 must be pairwise disjoint, respectively. As shown in Fig. 7 one finds $c_1 > d_3 < c_2 > d_1 < c_3 > d_2$. If $d_2 \nless c_1$ then P contains F, and if $d_2 < c_1$ then P contains C. □

Theorem 3.8. MA(P) *is a chain if and only if* P *is an interval order.*

Proof. P is an interval order iff the relation '$>$' is a Ferrers relation, i.e., $b_1 > a_1$ and $b_2 > a_2$ implies $b_1 > a_2$ or $b_2 > a_1$. Now the relation '$>$' is a Ferrers relation, iff its complement '$\ngtr$' is a Ferrers relation. The concept lattice of $(P, P, \ngtr)$ is a chain iff the relation '$\ngtr$' is a Ferrers relation. □

4. Further discussion

To begin with we further investigate the context $(P, P, \ngtr)$. We define $L(x) := (\downarrow x)\setminus\{x\}$ and $U(x) := (\uparrow x)\setminus\{x\}$.

Lemma 4.1. *For* $(G, M, I) = (P, P, \ngtr)$ *and* $x \in G$ *it holds* $\{x\}' = P\setminus L(x)$.

Proof. By definition $\{x\}' = \{m \in P \mid x \ngtr m\}$, which equals $P\setminus L(x)$. □

The join irreducible concepts are characterized in the next lemma.

Lemma 4.2. $(I, I') \in J(\mathfrak{B}(P, P, \ngtr))$ *iff* $I = \{x\}''$ *for some* $x \in P$ *and for all* $S \subseteq P$, $L(x) \neq \bigcup \{L(y) \mid L(y) \subsetneq L(x), y \in S\}$.

Proof. More general for a context (G, M, I) it holds $(A, B) \in J(\mathfrak{B}(G, M, I))$ iff $A = \{g\}''$ for some $g \in G$ and for all $S \subseteq G$, $\{g\}' \neq \bigcap \{\{h\}' \mid \{g\}' \subsetneq \{h\}', h \in S\}$. Now Lemma 4.2 is just a special case where Lemma 4.1 and the De Morgan Law is used. □

Lemma 4.3. *If P is W-free then $(P, P, \not>)$ is row reduced or, which is the same, $(\{x\}'', \{x\}') \in J(\mathfrak{B}(P, P, \not>))$ for all $x \in P$.*

Proof. We shall show that for W-free orders the second condition of Lemma 4.2 is always fulfilled. By $\mathrm{LC}(x)$ we denote the lower covers of x. We assume that there exist $x \in P$ and $S \subseteq P$ such that $L(x) = \bigcup \{L(y) \mid L(y) \subsetneq L(x), y \in S\}$. Then $\mathrm{LC}(x) \subseteq \bigcup \{\mathrm{LC}(y) \mid y \in S\}$ and $\mathrm{LC}(y) \subsetneq \mathrm{LC}(x)$ for all $y \in S$. Now there must exist $y, z \in S$ such that neither $\mathrm{LC}(y) \subseteq \mathrm{LC}(z)$ nor $\mathrm{LC}(z) \subseteq \mathrm{LC}(y)$. But this implies that there also exist $u, v \in P$ with $u \prec y$, $u \prec x$, $v \prec x$, $v \prec z$, $z \not\geq u$, $y \not\geq v$, which yields a W. □

Lemma 4.4. *If a context (G, M, I) is row reduced and $\mathfrak{B}(G, M, I)$ is distributive then the partitions of G induced by a maximal chain is the same for all maximal chains.*

Proof. For a maximal chain in a lattice we consider the naturally induced partition of the join irreducibles. For a distributive lattice the partition is the trivial one, independent of a specific maximal chain. For a row reduced context the intent of a concept is the union of the intents of all concepts which are join irreducible and below this concept. Both facts together yield Lemma 4.4. □

The next proposition and lemma can be considered as analytic structure theorems for W- and N-free orders. They say that a $W(N)$-free order has a characteristical partition of P (of the covering pairs of P into the Q's). It should be interesting to connect this to structure theorems like: Every N-free lattice is glued together over points by some Q_i's. But we are not going to study this here.

In order to indicate that the Q_i's of Lemma 2.5 depend on a specific chain $C \in \mathrm{MA}(P)$ we write $Q_i(C)$ in the following.

Proposition 4.5. *if P Is W-free, then*

$$\{\max(Q_i(C)) \mid 1 \leq i \leq n\} = \{\max(Q_i(C')) \mid 1 \leq i \leq n\}$$

for all maximal chains C and C' of $\mathrm{MA}(P)$.

Proof. Just by combining Theorem 3.7, Lemma 4.3 and Lemma 4.4. □

Corollary 4.6. *If P is N-free, then $\{Q_i(C) \mid 1 \leq i \leq n\} = \{Q_i(C') \mid 1 \leq i \leq n\}$ for all maximal chains C and C' of* $\mathrm{MA}(P)$.

Proof. This follows by Proposition 4.5 and its dual version. □

Lemma 4.7. *Let P be N-free. If $x \prec y$, $m = x$ and $g = y$, then $(\{m\}', \{m\}'') <$ $(\{g\}'', \{g\}')$ in $\mathfrak{B}(P, P, \not>)$. Each covering pair $x \prec y$ is contained in some Q_i.*

Proof. We set $I := \{m\}'$ and $F := \{g\}'$. Then $I = P \setminus U(x)$ and $F = P \setminus L(y)$ by Lemma 4.1 and its dual version. We define $\mathrm{UC}(x) := \{u \in P \mid x \prec u\}$ and $\mathrm{LC}(y) := \{l \in P \mid l \prec y\}$. Now

$$\max(I) \setminus F = \max(P \setminus U(x)) \cap L(y) = \mathrm{LC}(y)$$

and

$$\min(F) \setminus I = \min(P \setminus L(y)) \cap U(x) = UC(x).$$

Since P is N-free it holds $\mathrm{LC}(y) < UC(x)$. This and the foregoing show that $(\{m\}', \{m\}'') < (\{g\}'', \{g\}')$ by applying Lemma 2.4. The second statement is a direct consequence. □

Rival [11] discovered that the jump number result of [4] for series parallel orders are more general valid for N-free orders. The following theorem gives information about the possible optimal extensions for a N-free order P by using the decomposition of P induced by the lattice of maximal antichains. We want to mention that several other structural investigations of N-free orders are known which have led to similar results as the following one.

Theorem 4.8. *Let P be N-free and let $b_i \in \max(Q_i(P))$ for $i = 1, \ldots, n$. Then there exists a linear extension L of P and there exist $a_i \in \min(Q_i(P))$ such that $\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\} = B(L)$.*

Proof. We shall precede by induction, but before doing this we do the following consideration. Let L be a linear extension of an N-free order P. Let (a, b) be a bump of L and let Q be the height one order which contains (a, b). Then we can construct a linear extension L' which is as L but with the difference that all elements of $\max(Q) \cap \max(P)$ (this set might be empty) follow immediately the element a in L' and that $s(L) = s(L')$. It is possible to let all elements of $\max(Q)$ follow a since $x < y \in \max(Q)$ implies $x \leq z$ for some $z \in \min(Q)$ and we know that $z <_L a$ (remember that an edge of the diagram of P which goes down from some $y \in \max(Q)$ must lead to some $x \in \min(Q)$ by Lemma 4.7). The jump number does not change because after an element of $\max(Q)$ one has to jump anyway.

Now we consider some complete height one order Q of P with $\max(Q) \subseteq \max(P)$ and define $P_1 := P \setminus \max(Q)$. Obviously, the Q_i's of P are the Q_i's of P_1 added Q. By induction we can assume that the statement of this Theorem holds for P_1. Because of the N-freeness $\min(Q) \subseteq \min(P_1)$. For all Q_i of P_1 with $\max(Q_i) \cap \min(Q) \neq \emptyset$ we change the optimal linear extension as described at the beginning to end up with a linear extention, say L. Now let b (this is some b_i, since Q corresponds to some Q_i) be the given element of $\max(Q)$. Let a be the greatest element of $\min(Q)$ with respect to L. Then one can build an optimal linear extension of P by adding all elements of $\max(Q)$ starting with b right after a in L. This yields the wanted optimal extension. □

Corollary 4.9. *If P is N-free, then* $s(P) = |P| - l(\mathrm{MA}(P)) - 1$.

This could also be deduced from the fact that N-free orders are defect optimal [6].

In [8] (cf. [12]) a matroid structure $\mathrm{Md}(P)$ is deduced from P in connection with the jump number problem. We describe $\mathrm{Md}(P)$ by its set of basis. $\{b_1, \ldots, b_n\}$ is a basis of $\mathrm{Md}(P)$ iff there exists an optimal linear extension L of P and $a_1, \ldots, a_n \in P$ with $\{(a_1, b_1), \ldots, (a_n, b_n)\} = B(L, P)$.

Proposition 4.10. *Let P be N-free or let P be of height one and W-free. Then $\{b_1, \ldots, b_n\}$ is a basis of* $\mathrm{Md}(P)$ *iff* $(b_1, \ldots, b_n) \in \times_{i=1,\ldots,n} \max(Q_i(P))$.

Proof. Let P be N-free. Let L be an optimal linear extension and $a_1, \ldots, a_n \in P$ with $\{(a_1, b_1), \ldots, (a_n, b_n)\} = B(L)$. Then L induces a maximal chain C in $\mathrm{MA}(P)$ with $(a_i, b_i) \in Q_i(C)$ for $i = 1, \ldots, n$. Thus $b_i \in \max(Q_i(C))$ and by Corollary 4.6 we can assume that $\max(Q_i(P)) = \max(Q_i(C))$.

Now let $(b_1, \ldots, b_n) \in \times_{i=1,\ldots,n} \max(Q_i(P))$. By Theorem 4.8 there exists a linear extension L of P such that there exist $a_1, \ldots, a_n \in P$ with $\{(a_1, b_1), \ldots, (a_n, b_n)\} = B(L)$. This L is optimal because $n = l(\mathrm{MA}(P)) = b(P)$.

For W-free height-one orders the proof is the same because of Proposition 4.5 and because the statement of Theorem 4.8 holds for height-one W-free orders, too. □

Faigle and Schrader [8] have investigated M-free orders, whereas we discuss here W-free orders. Since the deduced matroid is not defined in a selfdual way, this really makes a difference. The M-free case can be analyzed by our methods, too, but does not lead to such a result like Proposition 4.10, and we therefore renounce to show this.

5. Aspects of greediness

For an optimal linear extension it is reasonable to climb up as often as possible. This vague idea has led to the notion of greedy linear extensions. It is well known that there are always optimal extensions which are greedy. The greedy concept simply says that one can climb up whenever it is possible. We are going to strengthen this idea by selecting and concentrating on some special bumps. As a simple example assume that we start an extension of a N with its left minimal element. Then the greedy continuation does not lead to an optimal extension. If one would have planned the bump on the right sight of the N, one could have avoid the unlucky start. Let L be a linear extension of P and $y <_L x <_L z$ then we say that x is *bump free* in L if $(y, x) \notin B(L)$ and $(x, z) \notin B(L)$. If (a, b) is the first bump in L, i.e., a and b are less than or equal than all other bump element, then we call b the *first bump element* in L.

Lemma 5.1. *Let L be a linear extension of P, let $x \in P$ be bump free and let L' be the linear extension which is as L except that x is pushed up as far as possible, then $b(L') \geqslant b(L)$.*

Proof. Let z be the unique successor of x in L'. Since x was pushed up as far as possible, it is $x <_P z$ and therefore $(x, z) \in B(L')$. It follows that $\{b \in P \mid (a, b) \in B(L)\} \subsetneq \{b \in P \mid (a, b) \in B(L')\}$ and so $|B(L)| \leqslant |B(L')|$. □

Lemma 5.2. *Let $b \in P$ be the first bump element of a linear extension L. Then there exists a linear extension L' with first bump element b and $\downarrow_{L'} b = \downarrow_P b$ and $b(L') \geqslant b(L)$.*

Proof. The elements of $\downarrow_{L'} b \setminus \downarrow_P b$ can be pushed over b and Lemma 5.1 applies. □

Lemma 5.3. *Let b be the first bump element of some linear extension L of P and let $b' \in P$ be such that $L(b') \subsetneq L(b)$. Then there exists a linear extension L' of P such that b' is the first bump element in L' and $b(L') \geqslant b(L)$.*

Proof. One can change the order of the elements below b in L such that L' starts with the elements of $L(b')$, is followed by b' and is continued as in L. One might have lost a bump with b' as the lower element but one wins a bump with b' as upper element. Thus $b(L) \leqslant b(L')$. □

Proposition 5.4. *There exists an optimal linear extension L of P such that the first bump element b is contained in some antichain which is an atom in* MA(P), *and such that $\downarrow_L b = \downarrow_P b$.*

Proof. $(\min(P) \setminus L(b)) \cup \{b\}$ is a maximal antichain of P if and only if there exists no b' with $L(b') \subsetneq L(b)$. Now the foregoing lemmas apply. □

With this on hand we describe the following heuristic algorithm for finding 'good' linear extensions:

Choose as the first bump-element b an element which fulfills the property described in Proposition 5.4, i.e., b has to be in some $A \in \mathrm{MA}(P)$ with $0 < A$ in $\mathrm{MA}(P)$. Let L start with the element of $L(b)$ followed by b. Climb up from b as far as possible up to the element $\hat{b}$ (thus $\hat{b}$ might equal b if no bump is possible). Then restart the algorithm with $P := P \setminus \downarrow \hat{b}$.

It can easily be seen that there always exist an optimal linear extension arising from this algorithm. This algorithm improves the usual greedy one in the sense that one has less freedom in choices to find an optimal extension. Fig. 8(a) represents an example in which the linear extension resulting from the algorithm is even unique.

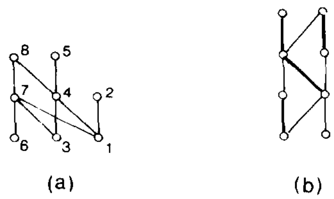

Fig. 8.

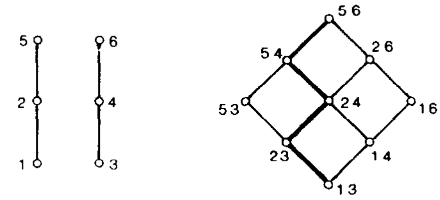

Fig. 9.

The fact that the first bump pair is contained in some Q induced from $0 < A$ in $\mathrm{MA}(P)$ leads to concecture that there is always an optimal linear extension such that all bump pairs are contained in Q's. The example in Fig. 8(b) however shows that this does not hold in general.

Now we shall discuss another aspect of greediness. Assume that we have determined a maximal chain C of $\mathrm{MA}(P)$. We know that $\frac{1}{2}l(C) \leqslant b(P) \leqslant l(C)$, but of course we want to construct from C a linear extension L with as many bumps as possible. If the linear extension is build from C we have the freedom to choose in a heuristically reasonable way the pairs from each Q. We plan to discuss this aspect in another paper. In the following we want to discuss the possibility of reorganizing the order of the Q's in C. The idea is as follows. We consider all chains in $\mathrm{MA}(P)$ which lead to the same partition in Q's as C does. We shall show how they can be described and then we can easily pick a suitable one out of these. Consider e.g. Fig. 9. The indicated maximal chain C leads to the linear extension of P given by the order of the numbers and this is a quite bad one. If we would have taken a chain going along an outside of $\mathrm{MA}(P)$ the result would have been optimal.

Let $Q = \{Q_1, \ldots, Q_n\}$. We define a preorder on Q by $P_Q = (Q, \to)$ with $Q_i \to Q_j$ iff there exists $x \in \min(Q_i)$ and $y \in \max(Q_j)$ with $x < y$.

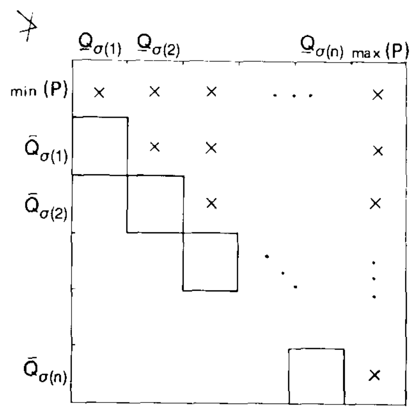

Fig. 10. $\underline{Q}_i := \min(Q_i)$ and $\bar{Q}_i := \max(Q_i)$.

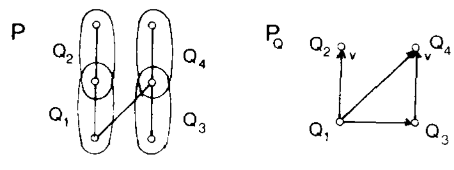

Fig. 11.

Lemma 5.5. *The maximal chains of* MA(P) *with the same induced* Q *correspond uniquely to the linear extension of* P_Q.

Proof. We consider $\mathscr{B}(P, P, \not>)$. Now a maximal chain in $\mathscr{B}(P, P, \not>)$ which induces Q corresponds uniquely to a permutation $\sigma : Q \to Q$ such that $\max(Q_{\sigma(i)}) \not> \min(Q_{\sigma(j)})$ for $i > j$. But such a permutation σ corresponds uniquely to the linear extension $Q_{\sigma(1)}, Q_{\sigma(2)}, \ldots, Q_{\sigma(n)}$ of P_Q. $\square$

The arrows have different qualities. We specify this. We say that $Q_i \to_v Q_j$ (v like vertical) iff $Q_i \cap Q_j \neq \emptyset$. For example see Fig. 11.

Now we can define some greedy maximal chains in MA(P). We say that a maximal chain C in MA(P) is greedy if the corresponding linear extension of P_Q follows always vertical arrows if possible. From an heuristic point of view it is reasonable to choose a greedy chain of MA(P) and to deduce of it a 'good' linear extension.

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