Coordinatization of B-matroids

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Abstract


We prove that the class of C-matroids whose circuits intersect cocircuits on finite sets is closed under the taking of minors, and we show that through the concept of matroids with coefficients it is possible to coordinatize B-matroids.

1. Introduction

C-matroids and B-matroids are two classes of nonfinitary infinite matroids introduced by Higgs [7]. The class of B-matroids, a subclass of C-matroids, retains many of the properties of finite matroids. In particular, this class is closed under orthogonality, restriction and contraction. As is obvious from the definition, C-matroids are closed under orthogonality. Still as far as I know, it is an open problem to decide whether they are closed under restriction and contraction or not. This notwithstanding, in the case where circuits intersect cocircuits on finite sets the answer is positive.

The concept of matroids with coefficients developed by Dress [3, 4] applies to those C-matroids whose circuits intersect cocircuits on finite sets, in which case it is possible to use an inner product on $K^E$ by means of which duality is defined.

But this is not the case, in general, for, on the one hand, a circuit and a cocircuit of a C- or B-matroid do not necessarily intersect on a finite set and, on the other hand, it is difficult to deal with the sum of an infinite number of coefficients when the coefficient domain is a field.

To remove these difficulties, the algebraic structures which are to serve as coefficient domains are assumed to be positive, which means that no nonzero element has an additive inverse and that there are no zero divisors. As will be shown, it is possible to coordinatize C-matroids over such coefficient domains. Furthermore, the aforementioned conditions do not lead to any loss of generality.
2. Notations and definitions

For the convenience of the reader we recall some basic definitions and properties of Klee matroids.

**Definition 2.1.** A preclosure $f$ on a set $E$ is a map from $2^E$ to $2^E$ satisfying:

(i) for any $X \subseteq E$: $X \subseteq f(X)$;

(ii) for any $X, Y \subseteq E$: $(X \subseteq Y)$ implies $(f(X) \subseteq f(Y))$.

**Definition 2.2.** Let $f$ be a preclosure on a set $E$ and let $X$ be a subset of $E$.

(i) We say that $X$ is an $f$-independent if for any $e \in X$ we have $e \notin f(X \setminus \{e\})$, otherwise it is an $f$-dependent.

(ii) An $f$-dependent inclusion-minimal with this property is called an $f$-circuit.

(iii) We say that $X$ is spanning if $f(X) = E$.

(iv) A nonspanning inclusion-maximal with this property is called an $f$-hyperplane.

In what follows we give six conditions on preclosures which generalize properties of finite matroid closures.

**Definition 2.3.** A preclosure $f$ on a set $E$ is:

(i) weakly idempotent if it satisfies: (WI) for any $x \in f(Y)$: $f(\{x\} \cup Y) = f(Y)$;

(ii) idempotent if it satisfies: (I) for any $X \subseteq f(Y)$: $f(X \cup Y) = f(Y)$, or, equivalently, $f \circ f(Y) = f(Y)$;

(iii) weakly exchanging if it satisfies: (WE) for any $p \in f(Y)$, if $p \notin f(Y \setminus \{x\})$ then $x \in f(\{p\} \cup (Y \setminus \{x\}))$;

(iv) exchanging if it satisfies: (E) for any $p \in f(Y)$, if $p \notin f(Y \setminus X)$ then there exists $x \in X$ such that $x \in f(\{p\} \cup (Y \setminus \{x\}))$;

(v) a C-preclosure if it satisfies: (C) for any $p \in f(Y)$, there exists $U \subseteq Y$ such that $p \in f(U)$, and $U$ is an $f$-independent inclusion-minimal with these properties;

(vi) an H-preclosure if it satisfies: (H) for any $p \notin f(Y)$, there exists $V \supseteq Y$ such that $p \notin f(V)$ and $\{p\} \cup V$ is spanning, $V$ being inclusion-maximal with these properties.

In the remainder of this paper an (IECH)-preclosure will be referred to as a C-matroid.

Now let us introduce preclosure duality.

**Proposition 2.4.** Let $f$ be a preclosure on $E$. We denote by $f^*$ the map from $2^E$ to $2^E$ defined by

$$f^*(X) = X \cup \{x \in E | x \notin f(E \setminus (X \cup \{x\}))\}.$$  

It is a preclosure, called the dual of $f$, which satisfies $f^{**} = f$. 
It is easy to check that the \( f \)-circuits are exactly the complements of the \( f^* \)-hyperplanes. Note that conditions (\( wE \)) and (\( wI \)) are dual to each other and that the same is true for (\( E \)) and (\( I \)) and for (\( C \)) and (\( H \)).

The next result shows that, under conditions (\( wI \)), (\( E \)) and (\( C \)), a preclosure is characterized by its circuits in the same way as the closure of a finite matroid is.

**Theorem 2.5** (Klee [9]). The circuits of an \((wI, E, C)\)-preclosure form a clutter satisfying the strong elimination axiom and, conversely, if \( \mathcal{C} \) is a clutter satisfying the strong elimination axiom, then the map \( f \) from \( 2^E \) to \( 2^E \) defined by: for any \( X \in 2^E \):
\[
f(X) = X \cup \{ x \in E \setminus X \mid \exists C \in \mathcal{C}; e \in C \subseteq X \cup \{ e \} \}
\]
is the only \((wI, E, C)\)-preclosure whose circuits are the elements of \( \mathcal{C} \).

Now we turn to the minors.

**Definition 2.6.** Let \( f \) be a preclosure on a set \( E \), and let \( E' \) be a subset of \( E \).

The restriction \( f_{E'} \) of \( f \) to \( E' \) is defined by
\[
\text{for any } X \subseteq E': f_{E'}(X) = f(X) \cap E'.
\]

The contraction \( f^{E'} \) of \( f \) to \( E' \) is defined by
\[
\text{for any } X \subseteq E': f^{E'}(X) = f(X \cup (E \setminus E')) \cap E'.
\]

We leave it to the reader to check that both \( f_{E'} \) and \( f^{E'} \) are preclosures. In Section 3 we will make use of the fact that conditions (\( wI \)), (\( wE \)), (\( I \)) and (\( E \)) are preserved under restriction and contraction ([11, p. 264]).

This section ends with the definition of B-matroids and their relation to closures.

**Definition 2.7.** Let \( E \) be a set and let \( \mathcal{I} \) be a subset of \( 2^E \). \( \mathcal{I} \) is the set of independents of a B-matroid if it satisfies:

(i) \( \emptyset \in \mathcal{I} \);

(ii) for any \( X \in \mathcal{I} \) and any \( Y \subseteq X \): \( Y \in \mathcal{I} \);

(iii) if \( Y \subseteq X \subseteq E \) and \( Y \in \mathcal{I} \) then there is a maximal \( \mathcal{I} \)-subset \( B \) of \( X \) such that \( Y \subseteq B \subseteq X \);

(iv) for all \( X \subseteq E \), if \( B_1 \) and \( B_2 \) are maximal \( \mathcal{I} \)-subsets of \( X \) and \( e \in B_1 \setminus B_2 \), then there exists \( f_e \in B_2 \setminus B_1 \) such that \( B_1 \setminus e + f \) is a maximal \( \mathcal{I} \)-subset of \( X \).

**Proposition 2.8** (Oxley [12]). Let \( E \) be a set and let \( \mathcal{I} \) be the set of independents of a B-matroid. The map from \( 2^E \) to \( 2^E \) defined by
\[
\text{for all } X \subseteq E: f(X) = X \cup \{ e \in E \mid I \cup \{ e \} \notin \mathcal{I} \text{ for some } I \subseteq X \text{ such that } I \in \mathcal{I} \}
\]
is a C-matroid. Conversely, if \( f \) is an \((IE)\)-preclosure on \( E \) such that the set \( \mathcal{I} \) of \( f \)-independents satisfies condition 2.7(iii), then \( \mathcal{I} \) is the set of independents of a B-matroid.
3. A class of C-matroids closed under restriction and contraction

Theorem 3.1. Let \( f \) be a C-matroid such that any circuit intersects any cocircuit on a finite set. The minors of \( f \) are C-matroids.

Proof. In this proof we make use of some properties of minors of \((wI, E, C)- preclosures which are stated and proved in the following three lemmas.

Lemma 3.2. The restriction of a \((wI, E, C)- preclosure is a \((wI, E, C)- preclosure.

Proof. Let \( f \) be a \((wI, E, C)- preclosure and let \( E' \) be a subset of \( E \). Denote by \( \mathcal{C} \) the set of circuits of \( f \), and by \( \mathcal{C}' \) the subset of \( \mathcal{C} \) whose elements are those of \( \mathcal{C} \), which are contained in \( E' \). By Theorem 2.5, \( \mathcal{C} \) is a clutter satisfying the strong elimination axiom which implies that the same holds for \( \mathcal{C}' \). It is clear from the definition of the restriction and from Theorem 2.5 that for any \( X \subseteq E' \): \( f_c(X) = X \cup \{ e \in E' \mid \exists C \in \mathcal{C}': e \in C \subseteq X \cup \{ e \} \} \). Thus, the conclusion follows from Theorem 2.5. \( \Box \)

A preclosure \( f \) is said to satisfy condition \((C_f)\) if for any \( e \in f(X) \) there exists a finite subset \( U \) of \( X \) such that \( e \in f(U) \).

Lemma 3.3. Let \( f \) be a \((wI, E, C)- preclosure and let \( E' \) be a subset of \( E \) such that, for any \( f \)-circuit \( C \), we have \( |C \cap E'| < \aleph_0 \). The contraction of \( f \) to \( E' \) satisfies \((C_f)\).

Proof. Let \( X \) be a subset of \( E' \) and \( e \in f^{E'}(X) \). If \( e \in X \), we have \( e \in f^{E'}(\{ e \}) \) and \( \{ e \} \subseteq X \). Otherwise, according to Theorem 2.5, there exists an \( f \)-circuit \( C \) such that \( e \in C \subseteq X \cup (E \setminus E') \cup \{ e \} \). Now put \( U = (C \cap E') \setminus \{ e \} \); our hypothesis implies \( |U| < \aleph_0 \) and, clearly, we have \( e \in f^{E'}(U) \) and \( U \subseteq X \). \( \Box \)

Lemma 3.4. Let \( f \) be a \((wI, E, C)- preclosure on a set \( E \) and let \( E' \) be a subset of \( E \). The circuits of \( f^{E'} \) are the nonempty subsets of \( E' \) of the form \( C \cap E' \), where \( C \) is an \( f \)-circuit inclusion-minimal with this property.

Proof. Assume \( C' \) to be a nonempty subset of \( E' \) of the form \( C \cap E' \), where \( C \) is an \( f \)-circuit inclusion-minimal with this property. It is easy to check that \( C' \) is an \( f^{E'} \)-dependent. Now assume that there exists an \( f^{E'} \)-dependent \( C'' \) such that \( C'' \subseteq C' \). Necessarily, there exists \( e \in C'' \) such that \( e \in f^{E'}(C'' \setminus \{ e \}) \); hence, \( e \in f((C'' \setminus \{ e \}) \cup (E \setminus E')) \). In view of Theorem 2.5, there exists an \( f \)-circuit \( \gamma \) such that \( e \in \gamma \subseteq C'' \cup (E \setminus E') \); hence, \( e \in \gamma \cap E' \subseteq C'' \subseteq C' \), in contradiction to our first assumption. Therefore, \( C' \) is an \( f^{E'} \)-circuit. Conversely, let \( C' \) be an \( f^{E'} \)-circuit. Necessarily, there exists \( e \in C' \) such that \( e \in f^{E'}(C' \setminus \{ e \}) \); hence, \( e \in f((C' \setminus \{ e \}) \cup (E \setminus E')) \). Consequently, by Theorem 2.5, there exists an \( f \)-circuit \( C \) such that \( e \in C \subseteq C' \cup (E \setminus E') \), which implies \( e \in C \cap E' \subseteq C' \). Since \( C \cap E' \) is an \( f^{E'} \)-dependent, we have \( C' = C \cap E' \). Now the conclusion follows from the fact that all nonempty subsets of \( E' \) of the form \( \Gamma \cap E' \), where \( \Gamma \) is an \( f \)-circuit, are \( f^{E'} \)-dependents.
Since the dual of a C-matroid is a C-matroid and \((f'_E)^* = f^E\), it suffices to prove Theorem 3.1 for a restriction. Let \(f\) be a C-matroid on \(E\) such that any circuit intersects any cocircuit on a finite set and let \(E'\) be a subset of \(E\). According to Lemma 3.2 and the last remark in Section 2, it remains to show that \(f_{E'}\), satisfies condition (H). Take \(Y \subseteq E'\) and \(e \in E' \setminus f_{E'}(Y)\). We have \(e \notin f(Y)\); hence, there exists an \(f\)-hyperplane \(H\) such that \(f(Y) \subseteq H\) and \(e \notin H\). Clearly, \(f_{E'}(Y) \subseteq H \cap E'\) and \(e \notin H \cap E'\).

Now put \(X = E' \setminus H\), since \(E \setminus H\) is an \(f\)-cocircuit, under our assumption, any \(f_{E'}\)-circuit intersects \(X\) on a finite set. According to Lemmas 3.2 and 3.3, this implies that \((f_{E'})^X\) is an \((W, E, C_t)\)-preclosure which in turn implies that it is a C-matroid (see [9, p. 140]). We claim, in view of Lemma 3.4, that \(e\) is not a loop of \((f_{E'})^X\). Consequently, there exists an \((f_{E'})^X\)-hyperplane of \(H'\) such that \(e \notin H'\). Obviously, the fact that \(H'\) is an \((f_{E'})^X\)-flat implies that \(H'' = H' \cap (H \cap E')\) is an \(f_{E'}\)-flat and, moreover, \(f_{E'}(Y) \subseteq H''\) and \(e \notin H''\). It is easy to check that \(H''\) is an \(f_{E'}\)-hyperplane, which completes the proof. 

4. Coordinatization of B-matroids

We shall show that the Dress concept of matroids with coefficients is fitted for the coordinatization of B-matroids in two steps. As an introduction to a more general concept, we begin with a coordinatization over semirings in which we will not make use of infinite sums.

The following algebraic structure is a particular case of fuzzy rings, which are the coefficient domains introduced by Dress in [4]. In a fuzzy ring, distributivity does not necessarily hold; however, semirings are sufficient for our purposes [14, 1.2.3.12.2].

**Definition 4.1.** An \(\Omega\)-semiring \(R = (R, +, ., e, \Omega)\) consists of a set \(R\) together with two compositions \(+\) and \(.\), a specified element \(e\) and a specified subset \(\Omega\) of \(R\) with the following properties:

- (i) \((R, +)\) and \((R, .)\) are abelian semigroups, with neutral \(0\) and \(1\) respectively;
- (ii) \(0.r = 0\) for all \(r \in R\);
- (iii) \(e^2 = 1\);
- (iv) for all \(r, s, s' \in R\): \(r.(s+s') = r.s + r.s'\);
- (v) \(\Omega\) is a proper ‘ideal’ of \(R\), i.e. \(\Omega + \Omega \subseteq \Omega\) and \(R.\Omega \subseteq \Omega\) and \(0 \in \Omega\), \(1 \notin \Omega\) hold;
- (vi) for any unit \(\alpha\) of \(R\), \(1 + \alpha \in \Omega\) iff \(\alpha = e\);
- (vii) for any \(r, r', s, s' \in R\) if \(r + s \in \Omega\) and \(r' + s' \in \Omega\) then \(r.r' + e.s.s' \in \Omega\).

Any commutative ring \(R\) gives a natural \(\Omega\)-semiring \((R, +, ., -1, \{0\})\); moreover, when \(R\) is a field, finite matroids with coefficients in \(R\) are nothing but coordinatizations of ordinary matroids over \(R\). Conversely, if \(R = (R, +, ., e, \Omega)\) is an \(\Omega\)-semiring such that \(\Omega - \{0\}\) then \(e = -1\) and \(R\) is a commutative ring.

To extend the results of Dress [3, 4] we will, for technical reasons clear when one looks closely at the proofs, get rid of additive inverses and zero divisors.
Definition 4.2. An $\Omega$-semiring $R = (R, +, \cdot, e, \Omega)$ is said to be positive if it satisfies:

(i) for any $u, v \in R$: $u + v = 0$ iff $u = v = 0$;

(ii) for any $u, v \in R$: $u \cdot v = 0$ iff $u = 0$ or $v = 0$.

Examples 4.3. (1) Let $G$ be an abelian group. The set $\mathbb{N}(G)$ defined by

$$\left\{ \bigoplus_{g \in G} n_g \cdot g \mid \forall g \in G: n_g \in \mathbb{N} \text{ and } \sum_{g \in G} n_g < \infty \right\}$$

is provided with the following two compositions $+$ and $\cdot$:

$$\left( \bigoplus_{g \in G} n_g \cdot g \right) + \left( \bigoplus_{g \in G} m_g \cdot g \right) = \bigoplus_{g \in G} (n_g + m_g) \cdot g,$$

$$\left( \bigoplus_{g \in G} n_g \cdot g \right) \cdot \left( \bigoplus_{g \in G} m_g \cdot g \right) = \bigoplus_{g \in G} \left( \sum_{g \cdot k = g} n_h \cdot m_k \right) \cdot g.$$

If $R = (R, +, \cdot, e, \Omega)$ is an $\Omega$-semiring then $\left( \mathbb{N}(U(R)), +, e, \Omega' \right)$, where $U(R)$ is the unit group of $R$ and

$$\Omega' = \left\{ \bigoplus_{x \in U(R)} n_x \cdot x \mid \sum_{x \in U(R)} n_x \cdot x \in \Omega \right\}$$

is a positive $\Omega$-semiring. Furthermore, $R$ and $\mathbb{N}(U(R))$ are equivalent as far as the theory of matroids with coefficients is concerned [14, 1.2.3.12.2]. In order to coordinatize $B$-matroids, we need an algebraic structure in which 'sums' of an infinite number of coefficients are well defined. With these 'infinite sums' in store, orthogonality between a circuit with coefficients and a cocircuit with coefficients of infinite intersection can be defined via an inner product. The structure of a positive $\Omega$-semiring is a feasible starting point to build on a map $\Sigma$ from the set of families of coefficients indexed by subsets of a given infinite set into the coefficient domain, and this, given the present example, without any loss of generality.

(2) The following tables define an $\Omega$-semiring $R_{\Omega}$ on the set $\{0, 1, \omega\}$, with $e = 1$ and $\Omega = \{0, \omega\}$. Ordinary matroids are in one-to-one correspondence with matroids with coefficients in $R_{\Omega}$.

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(3) The following tables define an $\Omega$-semiring $R_\epsilon$ on the set $\{0, 1, \epsilon, \omega\}$, with $\Omega=\{0, \omega\}$. Oriented matroids are in one-to-one correspondence with matroids with coefficients in $R_\epsilon$.

\[\begin{array}{c|cccc}
+ & 0 & 1 & \epsilon & \omega \\
\hline
0 & 0 & 1 & \epsilon & \omega \\
1 & 1 & 1 & \omega & \omega \\
\epsilon & \epsilon & \omega & \epsilon & \omega \\
\omega & \omega & \omega & \omega & \omega \\
\end{array}\]

(4) The following tables define an $\Omega$-semiring $R_{\omega, \epsilon}$ on the set $\{0, 1, \epsilon, r, \omega\}$, with $\Omega=\{0, \omega\}$. Weakly oriented matroids are in one-to-one correspondence with matroids with coefficients in $R_{\omega, \epsilon}$.

\[\begin{array}{c|cccc}
+ & 0 & 1 & \epsilon & r & \omega \\
\hline
0 & 0 & 1 & \epsilon & r & \omega \\
1 & 1 & r & \omega & \omega & \omega \\
\epsilon & \epsilon & \omega & r & \omega & \omega \\
r & r & \omega & \omega & \omega & \omega \\
\omega & \omega & \omega & \omega & \omega & \omega \\
\end{array}\]

**Notation 4.4.** In the remainder of this section, $E$ is a set and $(R, +, \cdot, \epsilon, \Omega)$ is a positive $\Omega$-semiring.

For any $v \in R^E$, we put $\text{supp}(v)=\{e \in E \mid v(e) \neq 0\}$ and $\text{supp}^\text{inv}(v) = \{e \in E \mid v(e) \in U(R)\}$.

Let $v, v' \in R^E$ and $f \in E$, $v \bigwedge_f v'$ is the element of $R^E$ defined by

\[
\forall e \in E: \quad v \bigwedge_f v'(e) = \begin{cases} 
0 & \text{if } e = f, \\
v(f) \cdot v(e) + \epsilon \cdot v(f) \cdot v'(e) & \text{if } e \neq f.
\end{cases}
\]

For any subset $V$ of $R^E$, we put

\[\hat{V} = \left\{ \left( \ldots \left( v_0 \wedge_{e_1} v_1 \right) \wedge_{e_2} \ldots \right) \wedge_{e_n} \right\}_{n \geq 0}, \]

\[\{v_0, \ldots, v_n\} \subseteq V, \{e_1, \ldots, e_n\} \subseteq E \}

The inner product $\langle \cdot \rangle$ is defined for $v, v' \in R^E$ such that $(v(e) \cdot v'(e))_{e \in E}$ has finite support as follows:

\[\langle v \mid v' \rangle = \sum_{e \in E} v(e) \cdot v'(e).\]
We say that $v$ is orthogonal to $v'$ when either $(v(e).v'(e))_{e \in E}$ has infinite support or $(v(e).v'(e))_{e \in E}$ has finite support and $\langle v|v' \rangle \in \Omega$.

For any subset $V$ of $R^E$, we put

$$V(E) = \{ v \in V | \text{supp}(v) = \text{suppinv}(v) \}.$$ 

$V(E)_{\text{min}}$ is the subset of $V(E)$ whose elements are those with minimal nonempty support.

**Definition 4.5.** Let $V$ be a subset of $R^E$, we say that $(E, V)$ presents a matroid with coefficients in $R$ if we have:

(i) for every $\alpha \in U(R)$ and every $v \in V$, $\alpha \cdot v \in V$;
(ii) for every $v \in V$ and every $e \in E$ such that $v(e) \neq 0$, there exists $v' \in V(E)_{\text{min}}$ satisfying $e \in \text{supp}(v') \subseteq \text{supp}(v)$;
(iii) $\{ \text{supp}(v)| v \in V(E)_{\text{min}} \}$ is the circuit set of a C-matroid.

When $R$ is a field, the concept of matroids with coefficients coincides with the Tutte representation of matroids over this field.

Let $M$ be a matroid with coefficients in $R$ presented by $(E, V)$. Denote by $f$ the C-matroid whose circuit set is $\{ \text{supp}(v)| v \in V(E)_{\text{min}} \}$ and by $\mathcal{H}$ its set of hyperplanes.

**Lemma 4.6** (Dress [4, 3.2]). Let $v, v_1, v_2 \in R^E$ be such that $v \perp v_i$ for $i = 1, 2$. If $\text{supp}(v) \cap (\text{supp}(v_1) \cup \text{supp}(v_2)) \subseteq \text{suppinv}(v)$ and $f \in E$ then $v \perp (v_1 \wedge_f v_2)$.

**Proof.** Since $R$ is a positive $\Omega$-semiring, we have

$$f \in E \setminus (\text{supp}(v_1) \cup \text{supp}(v_2))$$

$$\text{supp}(v_1 \wedge_f v_2) = 0,$$

$$f \in \text{supp}(v_1) \Delta \text{supp}(v_2)$$

$$\text{supp}(v_1 \wedge_f v_2) = \text{supp}(v_1),$$

where $f \in \text{supp}(v_j) \setminus \text{supp}(v_i)$ and $\{i, j\} = \{1, 2\}$;

$$f \in \text{supp}(v_1) \cap \text{supp}(v_2)$$

$$\text{supp}(v_1 \wedge_f v_2) = (\text{supp}(v_1) \cup \text{supp}(v_2)) \setminus f.$$

In the first case, obviously $v \perp (v_1 \wedge_f v_2)$. In the following case, orthogonality between $v$ and $v_1 \wedge_f v_2$ arises from $v \perp v_i$. In the last case, if $|\text{supp}(v_i) \cap \text{supp}(v)| \geq \aleph_0$ for $i = 1$ or 2, then $|\text{supp}(v) \cap \text{supp}(v_1 \wedge_f v_2)| \geq \aleph_0$. Hence, by definition, $v \perp (v_1 \wedge_f v_2)$; otherwise, $|\text{supp}(v_i) \cap \text{supp}(v)| < \aleph_0$, for $i = 1$ and 2; thus, the conclusion follows from [4, 3.2].

In the following lemma we build the dual of $M$. 

Lemma 4.7 (Dress [4, 5.1]). For any hyperplane \( H \in \mathcal{H} \), any \( e \in E \setminus H \), and any \( z \in U(R) \) there exists one and only one \( s = s^{\mathcal{H}, e, z}_H \in V^\perp \), such that \( s(e) = z \), \( \text{supp}(s) = \text{suppinv}(s) = E \setminus H \), and for this \( s \) we have \( s \in (V)^\perp \). Moreover, \( s \) depends only on \( V(E)_{\text{min}} \).

**Proof.** The uniqueness of \( s \) which leads to its definition is established as in [4, 5.1]. In the same way as in his proof, we show that, for any \( v \in V \), such that \( |\text{supp}(v) \setminus H| < N_0 \), \( v \perp s \). Otherwise, orthogonality between \( v \) and \( s \) follows from our definition of orthogonality. □

Theorem 4.8 (Dress [4, 5.3]). If \((E, V)\) presents a matroid \( M \) with coefficients in \( R \), then \((E, V^\perp)\) presents a matroid with coefficients in \( R \) and \( V^\perp(E)_{\text{min}} = \{ s^{\mathcal{H}, e, z}_H \in \mathcal{H}^*, e \in E \setminus H, z \in U(R) \} \).

**Proof.** Assume \( v^* \in (V^\perp)^\perp \) and \( e \in f((v^*)^{-1}(0)) \setminus (v^*)^{-1}(0) \). In view of Theorem 2.5, there exists \( v \in V(E)_{\text{min}} \) such that \( e \in \text{supp}(v) \subseteq (v^*)^{-1}(0) \cup \{ e \} \); hence, \( \text{supp}(v) \cap \text{supp}(v^*) = \{ e \} \). But, by Lemma 4.6, \( v \perp v^* \); therefore, \( v(e), v^*(e) \in \Omega \). Since \( v(e) \in U(R) \), we have necessarily \( v^*(e) \in \Omega \). Consequently, if \( v^*(e) \notin \Omega \) then \( e \notin f((v^*)^{-1}(0)) \). By Lemma 4.7, there exists \( s \in V^\perp \) such that \( \text{supp}(s) = \text{suppinv}(s) = E \setminus H \), so that \( e \in \text{supp}(s) \subseteq \text{supp}(v^*) \). Since \( f^* \) is a \( C \)-matroid whose set of circuits is \( \{ E \setminus H | H \in \mathcal{H} \} \), it is easy to check that \( V^\perp(E)_{\text{min}} = \{ s^{\mathcal{H}, e, z}_H \in \mathcal{H}^*, e \in E \setminus H, z \in U(R) \} \). □

It is clear from Lemma 4.7 that the matroid with coefficients presented by \((E, V^\perp)\) depends only on \( V(E)_{\text{min}} \). This theorem suggests the next definition.

**Definition 4.9.** Let \( M \) be a matroid with coefficients in \( R \) presented by \((E, V)\). The matroid with coefficients in \( R \) presented by \((E, V^\perp)\) is called the dual of \( M \), and is denoted by \( M^* \).

We know that \( f^{**} = f \), and it is easy to check that \( V(E)_{\text{min}} = \{ s^{V^\perp}_{H, e, z} / H \in \mathcal{H}^*, e \in E \setminus H, z \in U(R) \} \); hence, \( M^{**} = M \).

We put \( V_M = V(E)_{\text{min}} \) and \( V^M = (V_M)^\perp \). We leave it to the reader to check that if \((E, V^\perp)\) presents a matroid with coefficients and \( V^\perp(E)_{\text{min}} = V(E)_{\text{min}} \) then \( V_M \subseteq V^\perp \subseteq V^M \). Therefore, \( V_M \) is called the minimal presentation and \( V^M \) is called the maximal presentation of \( M \).

As for matroids, weakly oriented matroids and oriented matroids, it is possible to define matroids with coefficients in terms of dual pairs.

**Theorem 4.10** (Dress [3]). Let \( E \) be a set, \( R = (R, +, \cdot, \varepsilon, \Omega) \) a positive \( \Omega \)-semiring and \( V \) and \( V^* \) two subsets of \( R^E \). There exists a matroid \( M \) with coefficients in \( R \), on the set
E, whose minimal presentation is V, such that V* is the minimal presentation of its dual iff:

(i) \( V = V(\mathcal{E}) \) and \( V^* = V^*(\mathcal{E}) \);

(ii) for every \( v \in V \) (\( v^* \in V^* \)) and every \( \alpha \in U(R) \), we have \( \alpha v \in V \) (\( \alpha v^* \in V^* \));

(iii) for every \( v \in V \) and every \( v^* \in V^* \), we have \( v \perp v^* \);

(iv) \( \{\text{supp}(v) \mid v \in V\} \) is the set of circuits of a C-matroid whose set of cocircuits is \( \{\text{supp}(v) \mid v \in V^*\} \).

**Proof.** Obviously, conditions (i)–(iv) are necessary. Their sufficiency is proved along the same line of argument as in Theorem 4.8. □

The concept of minors plays an important part in matroid theory. As already mentioned we do not know whether C-matroids are closed under the taking of minors; consequently, we will define minors of matroids with coefficients only for those whose closure is associated with a B-matroid or is a C-matroid for which any circuit intersects any cocircuit on a finite set.

In the remainder of this section, \( V \subseteq R^E \) is the minimal presentation of a matroid with coefficients \( M \), whose closure is either associated with a B-matroid or is a C-matroid whose circuits intersect cocircuits on finite sets. We denote by \( F \) a subset of \( E \) and by \( V^* \) the minimal presentation of \( M^* \). We put

\[
V \setminus F = \{ v \in V \mid v \notin F \mbox{ and } \text{supp}(v) \cap F = \emptyset \}.
\]

**Proposition 4.11.** \( (V \setminus F, (V^* \setminus F)_{\text{min}}) \) presents a dual pair of matroids with coefficients in \( R \).

**Proof.** Conditions (i)–(iii) of Theorem 4.10 are obviously satisfied. It is clear from the proof of Lemma 3.2 that \( \{\text{supp}(v) \mid v \in V \setminus F\} \) is the set of circuits of \( f_{E \setminus F} \) and, by Lemma 3.4, \( \{\text{supp}(v) \mid v \in (V^* \setminus F)_{\text{min}}\} \) is the set of circuits of \( (f^*)_{E \setminus F} \). We know that \( f_{E \setminus F} \) and \( (f^*)_{E \setminus F} \) are dual to each other. If \( f \) is the closure of a B-matroid then, according to [12, 3.2.8–3.2.10], (iv) is satisfied. Now, if \( f \) is the closure of a C-matroid whose circuits intersect cocircuits on finite sets, the same conclusion follows from Theorem 3.1. □

**Definition 4.12.** The matroid with coefficients in \( R \) presented by \( (E \setminus F, V \setminus F) \) is called the restriction of \( M \) to \( E \setminus F \), and is denoted by \( M \setminus F \). The one presented by \( (E \setminus F, (V_{E \setminus F})_{\text{min}}) \) is called the contraction of \( M \) to \( E \setminus F \), and is denoted by \( M / F \).

**Corollary 4.13.**

\[
(M \setminus F)^* = M^* / F, \quad (M / F)^* = M^* \setminus F.
\]

In the case where any circuit intersects any cocircuit on a finite set it is possible to coordinatize a C-matroid over nonpositive structures such as fields.
5. *E*-summable semirings

In this section a notion of ‘infinite sums’ comes into play. Together with $\Omega$-semirings they form the building blocks of the structure of an $E$-summable semiring.

**Definition 5.1.** Let $E$ be an infinite set. An $E$-summable semiring $R$ consists of an $\Omega$-semiring $(R, +, \cdot, \circ, \Omega)$ together with a map $\Sigma$ from the set of families of elements of $R$, indexed by subsets of $E$, into $R$, satisfying the following conditions:

(i) if $I \subseteq E$ admits $(I_j)_{j \in J}$, where $J \subseteq E$, as a partition and $(r_i)_{i \in I}$ is a family of coefficients in $R$ then

$$\sum_{i \in I} r_i = \sum_{j \in J} \left( \sum_{i \in I_j} r_i \right);$$

(ii) if $I, J \subseteq E$ and $(r_i)_{i \in I} \in R^I$, $(\rho_j)_{j \in J} \in R^J$ are such that there exists a bijection $\Phi : I \rightarrow J$ satisfying: $\rho_{\Phi(i)} = r_i$ for any $i \in I$, then

$$\sum_{i \in I} r_i = \sum_{j \in J} \rho_j;$$

(iii) if $I \subseteq E$ is such that $|I| \geq 2$ and $(r_i)_{i \in I} \in R^I$ has finite support then $\sum_{i \in I} r_i$ is the sum with respect to + of the elements $r_i$ whose indices belong to the support of this family; if $I = \{ j \} \subseteq E$ and $r_j \in R$ then $\sum_{i \in I} r_i = r_j$; as a convention, we put $\sum_{i \in \emptyset} = 0$;

(iv) if $I \subseteq E$ and $(r_i)_{i \in I} \in \Omega^I$ then $\sum_{i \in I} r_i \in \Omega$;

(v) if $I \subseteq E$ and $(r_i)_{i \in I} \in R^I$ and $r \in R$ then $r \cdot (\sum_{i \in I} r_i) = \sum_{i \in I} r \cdot r_i$.

In what follows, the set $\mathbb{N}$ of nonnegative integers is identified with a subset of the infinite set $E$, and $R$ is an $E$-summable semiring.

**Proposition 5.2.** If $I \subseteq E$ and, for any $i \in I$, one has $r_i = 0$ then $\sum_{i \in I} r_i = 0$.

**Proof.** Put $J_i = \emptyset$ for any $i \in I$. Obviously, $(J_i)_{i \in I}$ is a partition of $\emptyset$; hence, by Definition 5.1(i),

$$\sum_{i \in I} r_i = \sum_{i \in I} \left( \sum_{j \in J_i} r_i \right).$$

So, the conclusion follows from $\sum_{\emptyset} = 0$ (Definition 5.1(iii)). $\Box$

**Proposition 5.3.** For any $u, v \in R$: $u + v = 0$ iff $u = v = 0$.

**Proof.** Assume $u, v \in R$ to satisfy $u + v = 0$. Denote by $(x_n)_{n \in \mathbb{N}}$ the sequence defined by $x_n = u$ for $n \equiv 0 (2)$ and $x_n = v$ for $n \equiv 1 (2)$. Since, for any $i \in \mathbb{N}$, $x_{2i} + x_{2i+1} = u + v = 0$, we have, according to Definition 5.1(i) and (iii) and Proposition 5.2, $\sum_{n \in \mathbb{N}} x_n = 0$. In the
same manner, we prove \( \sum_{n \in \mathbb{N}} x_n = 0 \), where \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \). Now, by Definition 5.1(i) and (iii), we have

\[
0 - \sum_{n \in \mathbb{N}} x_n - u + \sum_{n \in \mathbb{N}^*} x_n - u + 0 - u.
\]

By symmetry, we have also \( v = 0 \). The converse is obvious. \( \Box \)

The following definition is concerned with a class of \( E \)-summable semirings which allows one to define a matroid with coefficients independently of its dual, while for arbitrary \( E \)-summable semirings we have, so far, to define pairs of dual matroids with coefficients.

**Definition 5.4.** An \( E \)-summable semiring \( R \) is said to be finitary if \( \Sigma \) maps any family \((r_i)_{i \in I} \in R^I \) (\( I \subseteq E \)), with infinite support, in \( \Omega \).

**Example 5.5.** Let us give an example of a nonfinitary \( E \)-summable semiring built on the positive \( \Omega \)-semiring \( R_0 \) (Example 4.3(3)). We define a map \( \Sigma \) from the set of families of elements of \( R_0 \) indexed by subsets of \( E \) into \( R_0 \) as follows: let \( I \subseteq E \) and \((r_i)_{i \in I} \in (R_0)^I \); if \( I = \emptyset \) or, for any \( i \in I \), one has \( r_i = 0 \) then \( \sum_{i \in I} r_i = 0 \); if there exists \( i \in I \) such that \( r_i = 1 \) and \[ \{ r_i / i \in I \} \subseteq \{ 0, 1 \} \{ \{ 0, \varepsilon \} \} \) then \( \sum_{i \in I} r_i = 1 \) (\( \varepsilon \)); if there exists \( i \in I \) such that \( r_i = \omega \) or there exist \( i, j \in I \) such that \( r_i = 1 \) and \( r_j = \varepsilon \) then \( \sum_{i \in I} r_i = \omega \). It is straightforward to check that \((R_0, +, \Sigma, \varepsilon, \omega \{ 0, \omega \})\) is an \( E \)-summable semiring. So, the theory of oriented matroids extends to the class of B-matroids; indeed, a B-matroid with coefficients in \((R_0, +, \Sigma, \varepsilon, \omega \{ 0, \omega \})\) can be seen as an oriented B-matroid.

**Example 5.6.** This example shows that the concept of matroids with coefficients developed in Section 4 is a particular case of matroids with coefficients in an \( E \)-summable semiring. Let \( R \) be a positive \( \Omega \)-semiring and let \( \theta \) be an extra element. We put: for any \( r \in R \): \( \theta + r = \theta = r + \theta \); for any \( r \in R \setminus \{ 0 \} \): \( \theta . r = r = r . \theta \); \( 0 . \theta = 0 = \theta . 0 \). We denote by \( \Sigma \) the map from the set of families of elements of \( R \cup \{ \theta \} \), indexed by a subset of the infinite set \( E \), into \( R \cup \{ \theta \} \) defined by: if \( I = \emptyset \) or \( I \subseteq E \) and, for any \( i \in I \), one has \( r_i = 0 \) then \( \sum_{i \in I} r_i = 0 \); if \( I \subseteq E \) and \((r_i)_{i \in I} \in (R \cup \{ \theta \})^I \) has finite support then \( \sum_{i \in I} r_i \) is the sum of the elements \( r_i \) whose indices belong to the support of this family; otherwise, \( \sum_{i \in I} r_i = \theta \). We leave it to the reader to check that \( R(E) = (R \cup \{ \theta \}, +, \Sigma, \varepsilon, \Omega \cup \{ \theta \}) \) is an \( E \)-summable semiring.

The next proposition is the converse of this construction.

**Proposition 5.7.** If \( R \) is an \( E \)-summable semiring and if there exists \( \omega \in \Omega \) satisfying for any \( r \in R \setminus \{ 0 \} \): \( \theta \cdot r = \omega = r \cdot \omega \), such that \( \Sigma \) maps every family \((r_i)_{i \in I} \in R^I \) (\( I \subseteq E \)), with infinite support, on \( \omega \), then \( R \) is a positive \( \Omega \)-semiring.
Proof. According to Proposition 5.3 it remains to show that $R$ has no zero divisor. Let us show that for every $r \in R$ one has $\omega + r = \omega$. Take $r \in R$ and let $(x_n)_{n \in \mathbb{N}}$ denote the sequence defined by $x_n = r$ for $n \in \mathbb{N}$. If $r = 0$, there is nothing to prove; if $r \neq 0$ then, under our assumption, we have $\sum_{n \in \mathbb{N}} x_n = \omega$. By Definition 5.1(i) and (iii), we have $r + \sum_{n \in \mathbb{N}} x_n = \omega$. But, by hypothesis, $\sum_{n \in \mathbb{N}} x_n = \omega$; hence, we have $\omega + r = \omega$. Now assume $u, v \in R \setminus \{0\}$ to be such that $u \cdot v = 0$ and denote by $(y_n)_{n \in \mathbb{N}}$ the sequence defined by $y_n = v$ for $n \in \mathbb{N}$. By hypothesis, we have $u \cdot \sum_{n \in \mathbb{N}} y_n = u \cdot \omega = \omega$. But $u \cdot \sum_{n \in \mathbb{N}} y_n = \sum_{n \in \mathbb{N}} u \cdot y_n = 0$ ($u \cdot y_n = u \cdot v = 0$), so that $\omega = \omega$. Therefore, we have $1 = 1 + 0 = 1 + \omega = \omega = 0$, a contradiction from which the conclusion follows (1 $\notin \Omega$ and $0 \in \Omega$).

In what follows, we generalize the coordinatization of B-matroids, defined on an infinite set $E$, over positive $\Omega$-semirings to the broader class of $E$-summable semirings. In the remainder of this paper, $E$ is an infinite set and $(R, +, \cdot, C, E, \Sigma, s, \Omega)$ is an $E$-summable semiring.

Apart from the definition of orthogonality, we refer the reader to Notation 4.4 for the notational conventions.

The inner product $\langle | \rangle$ is defined for $v, v' \in R^E$ by

$$\langle v | v' \rangle = \sum_{e \in E} v(e) \cdot v'(e).$$

We say that $v$ is orthogonal to $v'$, when $\langle v | v' \rangle \in \Omega$.

Definition 5.8. We say that a pair $(V, V^*)$ of subsets of $R^E$ presents a dual pair of matroids with coefficients in $R$ if the following are true:

(i) $V = V(E)$ and $V^* = V^*(E)$;
(ii) for every $v \in V (v^* \in V^*)$ and every $x \in U(R)$, we have $x \cdot v \in V (x \cdot v^* \in V^*)$;
(iii) for every $v \in V$ and every $v^* \in V^*$, we have $v \perp v^*$;
(iv) $\{\text{supp}(v) | v \in V\}$ is the set of circuits of a C-matroid whose set of cocircuits is $\{\text{supp}(v^*) | v \in V^*\}$.

Lemma 4.6 is true in the context of $E$-summable semirings.

Now we give the elimination axiom for matroids with coefficients.

Definition 5.9. The subset $V$ of $R^E$ satisfies ($\forall$) if for every $v \in V$ with $v(e) \notin \Omega$, there exists $v' \in V(E)_{\text{min}}$ such that $e \in \text{supp}(v') \subseteq \text{supp}(v)$.

Proposition 5.10. If $(V, V^*)$ presents a dual pair of matroids with coefficients in $R$ then, for any $v \in (V^*)^\perp$ such that $v(e) \notin \Omega$, there exists $v' \in V$ such that $e \in \text{supp}(v') \subseteq \text{supp}(v)$.

Proof. See the proof of Theorem 4.8. $\Box$
Corollary 5.11. If \((V, V^*)\) presents a dual pair of matroids with coefficients in \(R\) then \(V\) and \(V^*\) satisfy (\(\mathcal{D}\)).

Proof. By Lemma 4.6, \(\hat{V} \subseteq (V^*)^\perp\) and \(\hat{V}^* \subseteq V^\perp\); hence, the conclusion follows from Proposition 5.11. \(\square\)

Proposition 5.12. If \(V, V'\) and \(V''\) are three subsets of \(R^E\) such that \((V, V')\) and \((V, V'')\) present dual pairs of matroids with coefficients in \(R\) then \(V' = V''\).

Proof. Take \(v' \in V'\). Since we have, according to Definition 5.8(iv), \(\{\text{supp}(u') \mid u' \in V'\} = \{\text{supp}(u'') \mid u'' \in V''\}\), there exists \(v'' \in V''\) such that \(\text{supp}(v') = \text{supp}(v'')\). By hypothesis, there exists a hyperplane \(H\) of the C-matroid associated with \(V\) such that \(\text{supp}(v') = E \setminus H = \text{supp}(v'')\). In view of Definition 5.8(i) and (ii), we may assume without loss of generality that there exists \(e \in E \setminus H\) such that \(v'(e) = v''(e)\). Take \(f \in E \setminus (H \cup \{e\})\). Since \(H\) is a hyperplane, there exists, according to Definition 5.8(iv), \(v \in V\) such that \(f \in \text{supp}(v) \subseteq H \cup \{e, f\}\); hence, \(\text{supp}(v) \setminus H = \{e, f\}\). Since \(v \perp v'\) and \(v \perp v''\), we have \(v(e).v'(e) + v(f).v'(f) \in \Omega\) and \(v(e).v''(e) + v(f).v''(f) \in \Omega\). Now, by Definition 4.1(vi), we have \(v(f).v'(f) = v(e).v'(e) - v(e).v''(e) = v(f).v''(f)\); hence, \(v'(f) = v''(f)\). So, we have \(v' = v''\), which proves that \(V' \subseteq V''\). By symmetry, we have also \(V'' \subseteq V'\); hence, \(V' = V''\). \(\square\)

Definition 5.13. If \((V, V^*)\) presents a dual pair of matroids with coefficients in \(R\) then \((E, V^*)\) is said to present the dual \(M^*\) of the matroid \(M\) presented by \((E, V)\).

As a consequence of Proposition 5.12, we have \(M^{**} = M\).

We have already mentioned, before Definition 5.4, that a matroid with coefficients in a finitary \(E\)-summable semiring can be defined independently of its dual. This is proved in the next theorem by means of axiom (\(\mathcal{D}\)).

Theorem 5.14. A subset \(V\) of \(R^E\) presents a matroid with coefficients in a finitary \(E\)-summable semiring \(R\) iff:

(i) \(V = V(E)\);
(ii) \(V = U(R).V'\);
(iii) Axiom (\(\mathcal{D}\)) holds;
(iv) \(\{\text{supp}(v) \mid v \in V\}\) is the set of circuits of a C-matroid.

Proof. The necessity arises from Definition 5.8 and Corollary 5.11. The proof of the sufficiency is outlined in the following remarks. As in Lemma 4.7, we prove the existence, for every hyperplane \(H\), every \(e \in E \setminus H\) and every \(x \in U(R)\), of \(s = s_{x}^{y} E \setminus H \subseteq V^\perp\) such that \(s(e) - x, \text{supp}(s) - \text{supp}(v) - E \setminus H\). Put \(V^* = \{s_{x}^{y} \mid H \in \mathcal{H}, e \in E \setminus H, x \in U(R)\}\), we have \(V^* \subseteq V^\perp, V^* = V^*(E)\) and \(V^* = U(R).V^*\). Moreover, we know that \(\{E \setminus H \mid H \in \mathcal{H}\}\) is the set of cocircuits of the C-matroid associated with \(V\). \(\square\)
References


