Generalized Cohen–Macaulay dimension

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Abstract

A new homological dimension, called GCM-dimension, will be defined for any finitely generated module $M$ over a local Noetherian ring $R$. GCM-dimension (short for Generalized Cohen–Macaulay dimension) characterizes Generalized Cohen–Macaulay rings in the sense that: a ring $R$ is Generalized Cohen–Macaulay if and only if every finitely generated $R$-module has finite GCM-dimension. This dimension is finer than CM-dimension and we have equality if CM-dimension is finite. Our results will show that this dimension has expected basic properties parallel to those of the homological dimensions. In particular, it satisfies an analog of the Auslander–Buchsbaum formula. Similar methods will be used for introducing quasi-Buchsbaum and Almost Cohen–Macaulay dimensions, which reflect corresponding properties of their underlying rings.

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1. Introduction

One of the most influential results in commutative algebra describes the local rings of finite global dimension as being the regular rings. It is a formal consequence of a result due to Auslander, Buchsbaum and Serre:

\[ R \text{ is regular} \iff \text{pd}_R M < \infty \text{ for every } R\text{-module } M \iff \text{pd}_R k < \infty, \quad (\ast) \]

where \( k = R/m \) is the residue field of the local ring \( R \), and \( \text{pd}_R M \) denotes the projective dimension of \( M \). The reasonable question is that whether it is possible to refine the classical projective dimension so that the new dimensions would characterize in the sense of \((\ast)\), other classes of rings.

The best studied non-classical homological dimension is the Gorenstein dimension, which was defined by Auslander and Bridger [1]. Several refinements of the projective dimension, namely CI, CI\(^*\),\( G^* \) and CM-dimension, was introduced afterwards. They satisfy the inequalities:

\[ \text{CM-dim}_R M \leq \text{G-dim}_R M \leq \left\{ \begin{array}{l} \text{CI\(^*\)-dim}_R M \\ \text{G\(^*\)-dim}_R M \end{array} \right\} \leq \text{CI-dim}_R M \leq \text{pd}_R M. \]

If one of these dimensions is finite, then it is equal to those to its left [10].

Their definitions involve two choices—that of a homological dimension, and that of a class of ring homomorphisms. Using the terminology of [2, Section 8], fixing a class \( h \) of surjective homomorphisms of local rings, a diagram of local homomorphisms \( R \to R' \leftarrow Q \) is called a \( h \)-quasi-deformation if \( R \to R' \) is flat extension and \( R' \leftarrow Q \) belongs to \( h \). If \( \mathcal{C}(R) \) denotes a resolving subcategory of the category \( \mathcal{F}(R) \), finite \( R \)-modules, for which \( M \) belongs to \( \mathcal{C}(R) \) if and only if \( M \otimes_R R' \) belongs to \( \mathcal{C}(R') \), where \( R \to R' \) is a flat homomorphism of local rings, then a new homological dimension can be defined as

\[ h\|\mathcal{C}\text{-dim}_R M = \inf \left\{ \mathcal{C}(Q)\text{-dim}(M \otimes_R R') - \mathcal{C}(Q)\text{-dim} R' \left| R \to R' \leftarrow Q \text{ is a } h\text{-quasi-deformation} \right. \right\}. \]

Let \( c \) denote the class of surjective homomorphisms of local rings with kernel generated by a regular sequence and \( \mathcal{P}(R) \) denotes the subcategory of projective \( R \)-modules, then \( \text{CI-dim}_R M = c\|\mathcal{P}\text{-dim} M \) [3].

The procedure of defining the other homological dimensions, mainly is an extension of the above argument for definition of CI-dimension. For example, if \( g \) (resp. \( p \)) denotes the class of surjective homomorphisms \( Q \to R' \), such that the kernel \( J \) satisfy equality \( \text{grade}(J, Q) = \text{pd}_Q Q/J \) and \( \beta^{(j)}_J(Q/J) = 1 \) for \( j = \text{grade}(J, Q) \) (resp. \( \text{grade}(J, Q) = \text{G-dim}_Q Q/J \) and \( \mathcal{P}(R) \) (resp. \( \mathcal{G}(R) \)) denotes the category of projective modules (resp. modules of G-dimension 0), then \( G^*\text{-dim}_R M = g\|\mathcal{P}\text{-dim} M \) (resp. \( \text{CM-dim}_R M = p\|\mathcal{G}\text{-dim} M \)).
In all of the above definitions, one should pay attention to an important point which is the equality of $\mathcal{C}(Q)\text{-dim } R'$ and $\text{grade}_Q R'$. This means that we can rewrite the definition as follows:

$$h\parallel \mathcal{C}\text{-dim } M = \inf \left\{ \mathcal{C}(Q)\text{-dim} (M \otimes_R R') - \text{grade}_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a } h\text{-quasi-deformation} \right\}.$$ 

We pursue this point of view in introducing a new homological dimension which characterizes generalized Cohen–Macaulay rings. To this end, we obtain the first inequality in the following displayed formula:

$$\text{grade}_R M \leq \min \left\{ \text{G-dim}_R M, \text{dim } R - f_m \left( \frac{R}{\text{Ann}_R M} \right) \right\} \leq \text{G-dim}_R M.$$ 

Here $f_m$ is the finiteness dimension relative to $m$; see Section 2.

Using the above inequality, in view of the classical definitions of perfect and G-perfect modules, we define a GF-perfect (short for G-finite perfect) module, as module $M$ such that $\text{G-dim}_R M$ is finite and

$$\text{grade}_R M = \min \left\{ \text{G-dim}_R M, \text{dim } R - f_m \left( \frac{R}{\text{Ann}_R M} \right) \right\}.$$ 

An ideal $J$ of local ring $(Q, n)$ is called GF-perfect if the $Q$-module $Q/J$ has the corresponding property.

By this terminology, taking $h$ to be the class of all surjective homomorphisms $Q \rightarrow R'$ making $R'$ into a GF-perfect $Q$-module, we define GCM-dimension, short for Generalized Cohen–Macaulay dimension, as

$$\text{GCM-dim}_R M = h\parallel \text{G-dim } M.$$ 

The results in Section 2 will show that this dimension has expected properties of homological dimensions. In particular, a ring $R$ is Generalized Cohen–Macaulay if and only if every finitely generated module has finite GCM-dimension. Moreover we prove that GCM-dimension is bounded above by CM-dimension, with equality if the latter one is finite.

Moreover, using this new point of view, we will introduce another dimension that corresponds to the quasi-Buchsbaum property of rings. It will called quasi-Buchsbaum dimension, denoted QB-dim. It is also modelled on the CI-dimension and has parallel basic properties. QB-dimension fits into the following scheme of inequalities,

$$\text{GCM-dim}_R M \leq \text{QB-dim}_R M \leq \text{CM-dim}_R M,$$

with equality to the left of any finite one.

Finally we will show that how one can define a new notion of perfectivity, called CMD-perfect (short for Cohen–Macaulay-defect perfect), and use it in order to define a dimension that characterizes almost Cohen–Macaulay rings. We denote it by ACM-dimension.
Summing up the above results, we can complete the hierarchy of homological dimensions as follows:

\[ \text{Rfd}_R M \leq \begin{cases} \text{GCM-dim}_R M \leq \text{QB-dim}_R M \\ \text{ACM-dim}_R M \leq \cdots \leq \text{pd}_R M, \end{cases} \]

where \( \text{Rfd}_R M \) denotes the restricted flat dimension of \( M \), defined in [5], as

\[ \text{Rfd}_R M = \sup \{ \text{depth}_R p - \text{depth}_{R_p} M_p \mid p \in \text{Spec}(R) \}. \]

Throughout the paper, \( R \) is a local ring with maximal ideal \( m \) and residue field \( k = R/m \). All modules are finitely generated.

2. GCM-dimension

Let \( M \) be an \( R \)-module. Recall that the grade of \( M \) was defined by Rees as the least integer \( i \geq 0 \) such that \( \text{Ext}^i_R(M, R) \neq 0 \). When \( I \) is an ideal of \( R \), the grade of the \( R \)-module \( R/I \) will be denoted by \( \text{grade}(I, R) \).

The finiteness dimension \( f_I(M) \) of \( M \) relative to \( I \), is defined by the following formula:

\[ f_I(M) = \inf \{ i \in \mathbb{N} : H^i_I(M) \text{ is not finitely generated} \}. \]

See, for instance, [4, 9.1.3]. Set \( J = \text{Ann}_R M \). By definition \( \text{grade}_R M = \text{grade}(J, R) \leq \text{ht } J \). Now since \( f_m(R/J) \leq \text{dim } R/J \), we have \( \text{grade}_R M \leq \text{dim } R - f_m(R/J) \). So in view of the inequality \( \text{grade}_R M \leq \text{G-dim}_R M \) [1, 3.14], we always have the following sequence of inequalities:

\[ \text{grade}_R M \leq \text{Min}\{ \text{G-dim}_R M, \ \text{dim } R - f_m(R/J) \} \leq \text{G-dim}_R M \leq \text{pd}_R M. \]

\( M \) is said to be perfect (resp. G-perfect) if \( \text{grade}_R M = \text{pd}_R M \) (resp. \( \text{grade}_R M = \text{G-dim}_R M \)). The notion of G-perfect module is introduced by Foxby in [6], where he called it quasi-perfect, and studied further by Golod in [8]. The following definition can be consider as a generalization of this notion.

**Definition 2.1.** An \( R \)-module \( M \) is called GF-perfect (short for G-finite perfect) if \( \text{G-dim}_R M \) is finite and

\[ \text{grade}_R M = \text{Min}\{ \text{G-dim}_R M, \ \text{dim } R - f_m(R/J) \}, \]

where \( J = \text{Ann}_R M \).

Clearly any perfect (resp. G-perfect) module is GF-perfect, but not vice versa. We will see this by the aid of Example 2.5 below.
Now let $Q$ be a local ring and $J$ be an ideal of $Q$. Abusing notation, $J$ is called GF-perfect if the $Q$-module $Q/J$ is GF-perfect.

We say that $R$ has a GF-deformation if there exists a local ring $Q$ and a GF-perfect ideal $J$ in $Q$ such that $R = Q/J$. A GF-quasi-deformation of $R$ is a diagram of local homomorphisms $R \to R' \leftarrow Q$ with $R \to R'$ a flat extension and $Q \to R'$ a GF-deformation. Set $M' = M \otimes_R R'$.

**Generalized Cohen–Macaulay dimension.** For a module $M \neq 0$ over a local ring $R$, we define

$$\text{GCM-dim}_R M = \inf \left\{ \text{G-dim}_Q M' - \text{grade}(J, Q) \left| \begin{array}{c} \text{R} \to \text{R}' \leftarrow Q \text{ is a} \\
\text{GF-quasi-deformation} \end{array} \right. \right\}$$

and complement this by $\text{GCM-dim}_R 0 = -\infty$.

We would like to add one remark here, which emphasizes on the difference between this definition with the previous ones. According to [2, 8.5], using the same terminology as introduction, if $(h\parallel C)$ equals $(c\parallel P)$ (resp. $(g\parallel G)$) and there is a h-quasi-deformation $R \to R' \leftarrow Q$ with $C(Q)\text{-dim}_R (M \otimes_R R') < \infty$, then $h\parallel C\text{-dim}_R M = C(Q)\text{-dim}(M \otimes_R R') - C(Q)\text{-dim} R'$. This means that one can ignore the “inf” in their definitions. But in the definition of GCM-dimension it has a critical role.

Our first result will determine the place of GCM-dimension in the hierarchy of homological dimensions.

**Proposition 2.2.** There are inequalities

$$\text{Rfd}_R M \leq \text{GCM-dim}_R M \leq \text{CM-dim}_R M$$

with equality if $\text{CM-dim}_R M$ is finite.

**Proof.** For the first inequality, we may assume that $\text{GCM-dim}_R M$ is finite. So there exists a GF-quasi-deformation $R \to R' \leftarrow Q$ with $J = \text{Ker}(Q \to R')$, such that $\text{GCM-dim}_R M = \text{G-dim}_Q M' - \text{grade}(J, Q)$. Let $p$ be a prime ideal of $R$. Clearly $\text{grade}(J, Q) \leq \text{grade}(J_q, Q_q) \leq \text{G-dim}_{Q_q} R_q$, where $p'$ is a prime ideal in $R'$ lying over $p$ and $q$ is the inverse image of $p'$ in $Q$. On the other hand, $\text{G-dim}_Q M' \geq \text{G-dim}_{Q_q} M'_q$. So

$$\text{G-dim}_Q M' - \text{grade}(J, Q) \geq \text{G-dim}_{Q_q} M'_q - \text{G-dim}_{Q_q} R'_q.$$

Now it is easy, using faithful flatness of $R \to R'$ and also Auslander–Buchsbaum formula for G-dimension, to see that the right hand side of the above inequality is equal to $\text{depth}_{R_p} - \text{depth}_{R_q} M_p$. This shows that $\text{GCM-dim}_R M \geq \text{Rfd}_R M$.

For the second inequality, without loss of generality we may assume that $\text{CM-dim}_R M$ is finite. Now the inequality follows because every G-quasi-deformation is GF-quasi-deformation, and so appears in the determination of $\text{GCM-dim}_R M$. For equality, note that by [5, 2.8], if $\text{CM-dim}_R M < \infty$, it is equal to the $\text{Rfd}_R M$. So the above inequalities, become equality. □
A natural question now arises: If GCM-dim$_R M$ is finite, must it equal to Rd$_R M$? The answer is “no”, as the following example shows.

Example 2.3. Let $R$ be a local ring that is generalized Cohen–Macaulay but is not Cohen–Macaulay. Choose $R$ such that depth $R = \dim R - 1$. (E.g., $R = K[[X, Y]]/(X^2, XY)$ where $K$ is a field.) Let $k$ be the residue field of $R$. By definition, one computes Rd$_R k = \dim R - 1 = \text{depth } R$. Since $R$ is generalized Cohen–Macaulay, Theorem 2.4 implies that GCM-dim$_R k$ is finite. Since $R$ is not Cohen–Macaulay, GCM-dim$_R k$ is not equal to depth $R$ by Theorem 2.8.

Note that this is also an example where CM-dimension is infinite while GCM-dimension is finite. Similarly, if one takes the residue field of a ring that is not generalized Cohen–Macaulay, this will give an example of a module with infinite GCM-dimension and finite restricted flat dimension. The above example is due to referee.

The theory of generalized Cohen–Macaulay modules, that is a generalization of the theory of Cohen–Macaulay modules, is defined in [9, p. 238], as modules $M$, such that $f_m(M) = \dim M$. Our choice of terminology for generalized Cohen–Macaulay dimension is motivated by the next result, that provides a characterization for generalized Cohen–Macaulay rings, in terms of GCM-dimension of their modules.

Theorem 2.4. The following are equivalent.

(i) $R$ is generalized Cohen–Macaulay.

(ii) GCM-dim$_R M < \infty$ for every $R$-module $M$.

(iii) GCM-dim$_R k < \infty$.

Proof. (i) $\Rightarrow$ (ii) Since $R$ is generalized Cohen–Macaulay, so is $\hat{R}$, the completion of $R$. By Cohen’s structure theorem $\hat{R}$ is isomorphic to $Q/J$, where $Q$ is a regular local ring (with maximal ideal $n$). It follows from the regularity of $Q$, that grade$(J, Q) = \dim Q - \dim Q/J$. But $\dim Q/J = f_n(Q/J)$, because $Q/J$ is Generalized Cohen–Macaulay. Moreover since $Q$ is regular, G-dim$_Q Q/J$ is finite. So $J$ is GF-perfect. Now the result follows from the fact that over regular local rings all modules have finite G-dimension.

(ii) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (i) Suppose GCM-dim$_R k < \infty$ and $R \to R' \leftarrow Q$ is the corresponding GF-quasi-deformation. So $R' = Q/J$, for some ideal $J$ of $Q$. There are two possibilities: If $\text{grade}(J, Q) = G\dim Q/J$, it follows that the above deformation is in fact a $g$-quasi-deformation, and hence CM-dim$_R k < \infty$. Therefore in this case $R$ is a generalized (even) Cohen–Macaulay ring. Now suppose $\text{grade}(J, Q) = \dim Q - f_n(Q/J)$. Since $\text{grade}(J, Q) \leq \text{ht } J$ and $f_n(Q/J) \leq \dim Q/J$, it follows that $\text{grade}(J, Q)$ has to be $\text{ht } J$ and $f_n(Q/J)$ has to be $\dim Q/J$. Therefore $R' (= Q/J)$ is generalized Cohen–Macaulay. Now it is easy (e.g., using flat base change theorem [4, 4.3.2]), to see that $R$ is generalized Cohen–Macaulay. □

The following gives an example of a GF-perfect module that is not G-perfect and is also, therefore, not perfect.
Example 2.5. Let $R$ be a generalized Cohen–Macaulay ring which is not Cohen–Macaulay. By the part (i) $\Rightarrow$ (ii) of the above proof, $\hat{R} = Q/J$ satisfying the condition that $\text{grade}(J, Q) = \dim Q - f_n(Q/J)$. Moreover $\text{G-dim}_Q Q/J$ is finite, because $Q$ is regular. So $J$ is GF-perfect. It is clear that it can not be G-perfect as $R$ is not Cohen–Macaulay.

Next theorem presents the GCM-version of the Auslander–Buchsbaum depth formula. This result should be compared to the fact that, if $M$ is generalized Cohen–Macaulay and $p$ is a non-maximal prime ideal, then $M_p$ is Cohen–Macaulay.

**Theorem 2.6.** If $\text{GCM-dim}_R M < \infty$, then for any $p \in \text{Supp}(M) \setminus \{m\}$, $\text{CM-dim}_{R_p} M_p$ is finite and we have

$$\text{GCM-dim}_{R_p} M_p = \text{depth}_{R_p} - \text{depth}_{R_p} M_p = \text{CM-dim}_{R_p} M_p.$$  

**Proof.** If $\text{CM-dim}_{R_p} M_p$ is finite, by Proposition 2.2, it is equal to $\text{GCM-dim}_{R_p} M_p$, and by [7, 3.10], it is equal to $\text{depth}_{R_p} - \text{depth}_{R_p} M_p$. So to conclude the equalities it is enough to prove that $\text{CM-dim}_{R_p} M_p$ is finite. Since $\text{GCM-dim}_R M$ is finite, there exists a GF-quasi-deformation $R \to R' \leftarrow Q$ with $J = \text{Ker}(Q \to R')$. If $\text{grade}(J, Q) = \text{G-dim}_Q Q/J$, then this deformation is a G-quasi-deformation and so $\text{CM-dim}_R M$ is finite. So the result follows from the fact that $\text{CM-dim}_{R_p} M_p \leq \text{GCM-dim}_R M$ (see [7, 3.12]).

Now suppose $\text{grade}(J, Q) = \dim Q - f_n(Q/J)$. This implies that $f_n(Q/J) = \dim Q/J$ and so $Q/J$ is generalized Cohen–Macaulay. The faithful flatness of $R \to R'$, forces $R$ to be generalized Cohen–Macaulay. Hence $R_p$ is Cohen–Macaulay and so by [7, 3.11], $\text{CM-dim}_{R_p} M_p$ is finite. The proof now is complete. $\blacksquare$

Our next two results show that in case we have Auslander–Buchsbaum formula for any finite module, the ring has to be Cohen–Macaulay.

**Proposition 2.7.** Let $M$ be an $R$-module of finite GCM-dimension such that $\text{GCM-dim}_R M = \text{depth}_R - \text{depth}_R M$. Then

$$\text{GCM-dim}_R M = \text{CM-dim}_R M.$$  

**Proof.** Let $R \to R' \leftarrow Q$ be the GF-quasi-deformation for which we have

$$\text{GCM-dim}_R M = \text{G-dim}_Q M \otimes_R R' - \text{grade}(J, Q).$$  

In particular $\text{G-dim}_Q M \otimes_R R'$ is finite. The Auslander–Buchsbaum formula for G-dimension in conjunction with the faithful flatness of $R'$, implies the equality

$$\text{G-dim}_Q M \otimes_R R' - \text{G-dim}_Q Q/J = \text{depth}_R - \text{depth}_R M.$$  

This in view of our assumption implies that $\text{grade}(J, Q) = \text{G-dim}_Q Q/J$. That is $J$ is G-perfect. So $\text{CM-dim}_R M$ is finite. Now we can deduce the result using Proposition 2.2. $\blacksquare$
Theorem 2.8. The following are equivalent.

(i) $R$ is Cohen–Macaulay.

(ii) For every $R$-module $M$, $\text{GCM-dim}_R M = \text{depth } R - \text{depth}_R M$.

(iii) $\text{GCM-dim}_R k = \text{depth } R$.

Proof. (i) $\Rightarrow$ (ii) It follows from the Cohen–Macaulayness of $R$ that, for any finite module $M$, $\text{CM-dim}_R M$ is finite and so by Proposition 2.2, it is equal to $\text{GCM-dim}_R M$. So the result follows.

(ii) $\Rightarrow$ (iii) This is clear.

(iii) $\Rightarrow$ (i) By previous proposition, $\text{GCM-dim}_R k = \text{CM-dim}_R k$. So by [7, 3.11], $R$ is Cohen–Macaulay. $\square$

Proposition 2.9. Let $R \to S$ be a local flat extension. Then

$$
\text{GCM-dim}_R M \leq \text{GCM-dim}_S (M \otimes_R S).
$$

Proof. We can (and do) assume that $\text{GCM-dim}_S (M \otimes_R S)$ is finite. Let $S \to R' \leftarrow Q$ be a GF-quasi-deformation, with $\text{G-dim}_Q (M \otimes_R S) \otimes_S R' < \infty$. Since $R \to S$ and $S \to R'$ are flat extensions, the homomorphism $R \to R'$ is also flat. So $R \to R' \leftarrow Q$ is again a GF-quasi-deformation. Moreover since $R'$ is a $S$-algebra, it follows that $\text{G-dim}_Q (M \otimes_R R') < \infty$. This concludes the inequality. $\square$

3. QB- and ACM-dimension

Another interesting class of Noetherian modules over local rings is the class of quasi-Buchsbaum modules. There is a cohomological characterization of these modules: A finite module $M$ over local ring $(R, m)$ is called quasi-Buchsbaum if $m^i H_i^m (M) = 0$ for all $i$ with $0 \leq i < \text{dim}_R M$. Equivalently we can say that $H_i^m (M)$ is semi-simple for all $0 \leq i < \text{dim}_R M$. Clearly every quasi-Buchsbaum module is generalized Cohen–Macaulay.

Related to this, we can introduce a new invariant for any finite module $M$.

Definition 3.1. Let $M$ be a finitely generated module over commutative Noetherian local ring $(R, m)$. The semi-simple dimension $s(M)$ of $M$ is defined by

$$
s(M) = \inf \{ i \in \mathbb{N} : m^i H_i^m (M) \neq 0 \}.
$$

We adopt the convention that the infimum of the empty set of integers is to be taken as $\infty$. Clearly an $R$-module $M$ is quasi-Buchsbaum if and only if $s(M) = \text{dim } M$.

For an $R$-module $M$, if we set $J = \text{Ann}_R M$ and use the facts that $\text{grade}_R M = \text{grade}(J, R) = \text{ht } J$ and $s(R/J) \leq \text{dim } R/J$, we can deduce that

$$
\text{grade}_R M \leq \text{dim } R - s(R/J).
$$
So we can record the following sequence of inequalities:

$$\text{grade}_R M \leq \min \{ \text{G-dim}_R M, \dim R - s(R/J) \} \leq \text{G-dim}_R M \leq \text{pd}_R M.$$ 

**Definition 3.2.** An $R$-module $M$ is called GS-perfect (short for G-semi-simple perfect) if $\text{G-dim}_R M$ is finite and

$$\text{grade}_R M = \min \{ \text{G-dim}_R M, \dim R - s(R/J) \},$$

where $J = \text{Ann}_R M$.

Clearly any perfect (resp. G-perfect) module is GS-perfect. An example of a GS-perfect module which is not G-perfect can construct easily, by following the argument of Example 2.5.

If $J$ is an ideal of a local ring $Q$, abusing terminology, $J$ is called GS-perfect if the $Q$-module $Q/J$ is GS-perfect.

We say that $R$ has a GS-deformation if there exists a local ring $Q$ and a GS-perfect ideal $J$ in $Q$ such that $R = Q/J$. A GS-quasi-deformation of $R$ is a diagram of local homomorphisms $R \to R' \leftarrow Q$ with $R \to R'$ a flat extension and $Q \to R'$ a GS-deformation. Set $M' = M \otimes_R R'$.

**Quasi-Buchsbaum dimension.** For a module $M \neq 0$ over a local ring $R$, we define

$$\text{QB-dim}_R M = \inf \left\{ \text{G-dim}_Q M' - \text{grade}(J, Q) \mid R \to R' \leftarrow Q \text{ is a GS-quasi-deformation} \right\}.$$ 

As usual we set $\text{QB-dim}_R 0 = -\infty$.

The next two results will determine the place of QB-dimension in the hierarchy of homological dimensions and explain motivation for the choice of terminology. Their proofs are similar to Proposition 2.2 and Theorem 2.4, respectively.

**Proposition 3.3.** There is an inequality

$$\text{GCM-dim}_R M \leq \text{QB-dim}_R M \leq \text{CM-dim}_R M$$

with equality to the left of any finite one.

**Theorem 3.4.** The following are equivalent.

(i) $R$ is quasi-Buchsbaum.
(ii) $\text{QB-dim}_R M < \infty$ for every $R$-module $M$.
(iii) $\text{QB-dim}_R k < \infty$.

We mention that one can list more results (similar to results in previous section) to show that QB-dimension is really a homological dimension with expected properties. They can
proved exactly using the same arguments as we have done in the previous section for proving the similar results.

The class of “Almost Cohen–Macaulay rings” is introduced in [5, Section 3], as rings \( R \) satisfying the condition that \( \dim R_p - \depth R_p \leq 1 \) for all prime ideals \( p \). Towards the end of the paper we are going to introduce and study a homological dimension corresponding to this class of rings.

The Cohen–Macaulay defect \( \cmd R \) of a local ring \( R \) is the (always non-negative) difference \( \dim R - \depth R \) between the Krull dimension and the depth.

**Definition 3.5.** An \( R \)-module \( M \) is called CMD-perfect (short for Cohen–Macaulay-defect perfect) if \( G \dim_R M \) is finite and

\[
\grade_R M \geq \min \{ G \dim_R M, \dim R - \depth(R/J) - 1 \},
\]

where \( J = \Ann_R M \).

As before an ideal \( I \) of \( R \) is called CMD-perfect if the \( R \)-module \( R/I \) is so.

A CMD-quasi-deformation of \( R \) is a diagram of local homomorphisms \( R \rightarrow R' \leftarrow Q \) with \( R \rightarrow R' \) a flat extension and \( R' = Q/J \) for some CMD-perfect ideal \( J \) of \( Q \). Set \( M' = M \otimes_R R' \).

**Almost Cohen–Macaulay dimension.** For a module \( M \neq 0 \) over a local ring \( R \), we define

\[
\ACM-R \dim_R M = \inf \left\{ G \dim_Q M' - \grade(J, Q) \left| \begin{array}{c} R \rightarrow R' \leftarrow Q \text{ is a CMD-quasi-deformation} \\ R \rightarrow R' \leftarrow Q \text{ is a CMD-quasi-deformation} \end{array} \right. \right\}
\]

and complement this by \( \ACM-R \dim_R 0 = -\infty \).

A modification of the arguments that we have used in section two, can be used here to show that ACM-dimension (short for almost Cohen–Macaulay dimension) has the expected properties parallel to those of homological dimensions. Over an almost Cohen–Macaulay ring \( R \), every finite module \( M \) has \( \ACM-R \dim_R M < \infty \); conversely, if the residue field has finite ACM-dimension then the ring is almost Cohen–Macaulay. The ACM-dimension fits into the following sequence of inequalities,

\[
\Rfd_R M \leq \ACM-R \dim_R M \leq \CM-R \dim_R M,
\]

with equality holding whenever \( \CM-R \dim_R M \) is finite.

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References