Rearrangement and Extremal Results for Hermitian Matrices*†

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ABSTRACT

This paper contains extreme value results for concave and convex symmetric functions of the eigenvalues of $B+P^*AP$ as functions of the partial isometry P. The matrices A and B are Hermitian.

1. INTRODUCTION AND RESULTS

Let A and B be respectively n-square and k-square hermitian matrices. One of the central results in a recent paper [1] is the following extremal theorem:

$$\min_{P^*P=I_k} E_m(B+P^*AP)=E_m(\beta_1+\alpha_{n-k+1},\ldots,\beta_k+\alpha_n) \tag{1}$$

where $\alpha_1 \geqslant \cdots \geqslant \alpha_n$, $\beta_1 \geqslant \cdots \geqslant \beta_k$ are the eigenvalues of A and B respectively, $1 \leqslant m \leqslant k \leqslant n$, and P is an $n\times k$ partial isometry satisfying $P^*P=I_k$. The notation $E_m(X)$ designates the elementary symmetric function (e.s.f.) of degree m of the eigenvalues of the matrix X, while $E_m(x_1,\ldots,x_k)$ is the e.s.f. of the variables $x_1,\ldots,x_k$. The proof of (1) in [1] requires that the matrices A and B be positive definite, and the argument depends on computing the gradient of $E_m(B+P^*AP)$, in which the partial isometries P are parametrized in some way.

In this note we show that a number of extremal results, including an extension of (1), can in fact be quite easily proved using results of Lidskii and Wielandt [12], the Cauchy interlacing inequalities [5], the Birkhoff theorem for doubly stochastic (d.s.) matrices [2], certain convexity results for symmetric functions [6, 11], and some rearrangement theorems proved in the

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sequel. Moreover, the hypotheses that A and B are positive definite can be weakened to $\alpha_n + \beta_k \geqslant 0$. The extent to which non-negativity assumptions about the eigenvalues of A and B can be abandoned altogether is discussed in the second part of the paper.

THEOREM 1. *Assume that* $\alpha_n + \beta_k \geqslant 0$, *and let* $f(x_1, \ldots, x_k)$ *be a symmetric function, concave and monotone non-decreasing in each variable for* $x_j \geqslant 0$, $j = 1, \ldots, k$. *Define*

$$\varphi(P) = f(B + P^*AP).$$

Then

$$\min_{P^*P = I_k} \varphi(P) \geqslant \min_{\sigma} f(\beta + \alpha^\sigma), \tag{2}$$

where $\beta = (\beta_1, \ldots, \beta_k)$ *and* $\alpha^\sigma = (\alpha_{n-k+\sigma(1)}, \ldots, \alpha_{n-k+\sigma(k)})$, $\sigma \in S_k$.

Proof. For a fixed partial isometry P, denote the eigenvalues of P^*AP by $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_k$, $\nu = (\nu_1, \ldots, \nu_k)$. The Lidskii-Wielandt result states that if $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\gamma_1 \geqslant \cdots \geqslant \gamma_k$, are the eigenvalues of $B + P^*AP$, then

$$\gamma = \beta + S_0\nu$$

where S_0 is k-square d.s. Then

$$\begin{aligned} \varphi(P) &= f(\gamma) \\ &= f(\beta + S_0\nu) \end{aligned}$$

and we can define a function

$$\psi(S) = f(\beta + S\nu)$$

whose domain is the polyhedron Ω_k of k-square d.s. matrices. The vertices of Ω_k are precisely the k-square permutation matrices [2]. The Cauchy inequalities are

$$\alpha_s \geqslant \nu_s \geqslant \alpha_{n-k+s}, \qquad s = 1, \ldots, k, \tag{3}$$

and hence the kth component of $\beta + S\nu$ satisfies

$$\begin{aligned} (\beta + S\nu)_k &= \beta_k + \sum_{j=1}^{k} s_{kj}\nu_j \\ &\geqslant \beta_k + \nu_k \\ &\geqslant \beta_k + \alpha_n \\ &\geqslant 0. \end{aligned}$$

It follows that $\psi(S)$, $S \in \Omega_k$, is well defined, and for any S and T in Ω_k and $0 \leqslant \theta \leqslant 1$ we compute

$$\begin{aligned}\psi(\theta S+(1-\theta)T) &= f(\beta+(\theta S+(1-\theta)T)\nu)\\ &= f(\theta(\beta+S\nu)+(1-\theta)(\beta+T\nu))\\ &\geqslant \theta f(\beta+S\nu)+(1-\theta)f(\beta+T\nu)\\ &= \theta\psi(S)+(1-\theta)\psi(T). \qquad (4)\end{aligned}$$

The inequaltiy (4) is a consequence of the concavity of f. Thus $\psi : \Omega_k \to \mathbf{R}$ is concave, and hence from Birkhoff's theorem [2],

$$\begin{aligned}\psi(S_0) &\geqslant \min_Q \psi(Q)\\ &= \min_Q f(\beta+Q\nu),\end{aligned}$$

where the minimum is over all k-square permutation matrices Q. Thus from (3) and the monotonicity of f,

$$\begin{aligned}f(B+P^*AP) &= \varphi(P)\\ &= \psi(S_0)\\ &\geqslant \min_{\sigma\in S_k} f(\beta_1+\nu_{\sigma(1)},\ldots,\beta_k+\nu_{\sigma(k)})\\ &\geqslant \min_{\sigma\in S_k} f(\beta_1+\alpha_{n-k+\sigma(1)},\ldots,\beta_k+\alpha_{n-k+\sigma(k)})\\ &= \min_{\sigma\in S_k} f(\beta+\alpha^\sigma).\end{aligned}$$

■

As a first application take

$$f(x_1,\ldots,x_k)=E_m^{1/m}(x_1^\theta,\ldots,x_k^\theta),$$

$0 \leqslant \theta \leqslant 1$. This function is concave [6] and non-decreasing in each x_j, $x_j \geqslant 0$. We have

COROLLARY 1. *If* $\alpha_n+\beta_k \geqslant 0$ *and* $0 \leqslant \theta \leqslant 1$, *then*

$$\min_{P^*P=I_k} E_m\big((B+P^*AP)^\theta\big)=E_m\big((\beta_1+\alpha_{n+k+1})^\theta,\ldots,(\beta_k+\alpha_n)^\theta\big).$$

Proof. Let U be a k-square unitary matrix diagonalizing B. Then

$$E_m\left((B+P^*AP)^\theta\right)=E_m\left((\mathrm{diag}(\beta_1,\ldots,\beta_k)+U^*P^*APU)^\theta\right).$$

Clearly PU runs over the set of partial isometries as P does, so that by Theorem 1

$$\min_{P^*P=I_k} E_m^{1/m}\left((B+P^*AP)^\theta\right)=\min_{P^*P=I_k} E_m^{1/m}\left((\mathrm{diag}(\beta_1,\ldots,\beta_k)+P^*AP)^\theta\right)$$

$$\geqslant \min_{\sigma\in S_k} E_m^{1/m}\left((\beta_1+\alpha_{n-k+\sigma(1)})^\theta,\ldots,(\beta_k+\alpha_{n-k+\sigma(k)})^\theta\right). \quad (5)$$

The right side of (5) is achievable by simply choosing the columns of P to be a suitable selection of k orthonormal eigenvectors of A. It is easy to confirm that the minimum in (5) is taken on for σ the identity permutation [1,3]. ■

Consider next the symmetric rational function

$$f(x_1,\ldots,x_k)=\frac{E_m(x_1,\ldots,x_k)}{E_{m-1}(x_1,\ldots,x_k)} \quad (6)$$

in which $1\leqslant m-1<k$. A somewhat more difficult rearrangement theorem is required to deal with this function.

THEOREM 2. *The function* (6) *is concave and monotone non-decreasing for non-negative variables* ($E_{m-1}\neq 0$). *If* $c_1\geqslant\cdots\geqslant c_k$, $d_1\geqslant\cdots\geqslant d_k$, *and at least* $m-1$ *of the numbers* c_i+d_i *are positive* ($c_k+d_k\geqslant 0$), *then for the function* (6),

$$\min_{\sigma\in S_k} f(c_1+d_{\sigma(1)},\ldots,c_k+d_{\sigma(k)})=f(c_1+d_1,\ldots,c_k+d_k). \quad (7)$$

Proof. The proof that f is concave is found in [6]. Differentiating (6) with respect to x_j, we have

$$\frac{\partial f}{\partial x_j}=\frac{E_{m-1}(x)E_{m-1}(\hat{x}_j)-E_m(x)E_{m-2}(\hat{x}_j)}{E_{m-1}^2(x)} \quad (8)$$

where $E_{m-1}(\hat{x}_j)$, etc., means that x_j is omitted. The numerator in (8) simplifies to

$$E_{m-1}^2(\hat{x}_j)-E_m(\hat{x}_j)E_{m-2}(\hat{x}_j),$$

which is always non-negative for non-negative variables.

To prove the final assertion of the theorem, we observe that unless σ is the identity permutation, the arrangement of the arguments in $f(c_1 + d_{\sigma(1)},\ldots,c_k + d_{\sigma(k)})$ has the following form:

$$f(c_1+d_1,c_2+d_2,\ldots,c_{r-1}+d_{r-1},c_r+d_s,\ldots,c_t+d_r,\ldots,c_k+d_{\sigma(k)}), \tag{9}$$

in which $s>r\geqslant 1$ and $t>r\geqslant 1$. That is, there is a first d which does not appear as a summand with the same subscripted c. We will show that if d_s and d_r are interchanged in (9) and the positions of the remaining d's are left unaltered, then the value of (9) is not increased. Write (9) as

$$f(\ldots,c_r+d_s,\ldots,c_t+d_r,\ldots) \tag{10}$$

and compute that (10) is equal to

$$\frac{E_m(\ldots,c_r+d_s,\ldots,c_t+d_r,\ldots)}{E_{m-1}(\ldots,c_r+d_s,\ldots,c_t+d_r,\ldots)}$$

$$=\frac{(c_r+d_s)(c_t+d_r)E_{m-2}(\cdot)+(c_r+d_s+c_t+d_r)E_{m-1}(\cdot)+E_m(\cdot)}{(c_r+d_s)(c_t+d_r)E_{m-3}(\cdot)+(c_r+d_s+c_t+d_r)E_{m-2}(\cdot)+E_{m-1}(\cdot)}, \tag{11}$$

where $E_{m-2}(\cdot)$, etc., is the e.s.f. of those arguments other than the ones occurring in positions r and t The same computation shows that

$$f(\ldots,c_r+d_r,\ldots,c_t+d_s,\ldots)$$

$$=\frac{(c_r+d_r)(c_t+d_s)E_{m-2}(\cdot)+(c_r+d_r+c_t+d_s)E_{m-1}(\cdot)+E_m(\cdot)}{(c_r+d_r)(c_t+d_s)E_{m-3}(\cdot)+(c_r+d_r+c_t+d_s)E_{m-2}(\cdot)+E_{m-1}(\cdot)}. \tag{12}$$

Let $p=(c_r+d_s)(c_t+d_r)$, $q=(c_r+d_r)(c_t+d_s)$, $\gamma=c_r+d_r+c_t+d_s$ so that (11) and (12) can be respectively rewritten as

$$\frac{pE_{m-2}(\cdot)+\gamma E_{m-1}(\cdot)+E_m(\cdot)}{pE_{m-3}(\cdot)+\gamma E_{m-2}(\cdot)+E_{m-1}(\cdot)} \tag{13}$$

and

$$\frac{qE_{m-2}(\cdot)+\gamma E_{m-1}(\cdot)+E_m(\cdot)}{qE_{m-3}(\cdot)+\gamma E_{m-2}(\cdot)+E_{m-1}(\cdot)}. \tag{14}$$

Now regard (13) as a function $h(p)$ of p above, and observe that to within a positive multiple the derivative of $h(p)$ is

$$\begin{aligned}&(pE_{m-3}(\cdot)+\gamma E_{m-2}(\cdot)+E_{m-1}(\cdot))E_{m-2}(\cdot)\\&\quad-(pE_{m-2}(\cdot)+\gamma E_{m-1}(\cdot)+E_m(\cdot))E_{m-3}(\cdot)\\&=\gamma[E_{m-2}^2(\cdot)-E_{m-1}(\cdot)E_{m-3}(\cdot)]+[E_{m-1}(\cdot)E_{m-2}(\cdot)-E_m(\cdot)E_{m-3}(\cdot)].\end{aligned}\tag{15}$$

Both of the square bracketed quantities in (15) are non-negative [4], so that $h'(p)\geqslant 0$. But

$$\begin{aligned}p-q&=c_rc_t+c_rd_r+c_td_s+d_rd_s-c_rc_t-c_rd_s-c_td_r-d_rd_s\\&=(c_r-c_t)(d_r-d_s)\\&\geqslant 0.\end{aligned}$$

Hence $h(p)\geqslant h(q)$. The preceding argument requires that $m\geqslant 3$. However, if $m=2$, then $m-1=1$ and the denominator in $f(c+d^\sigma)$ is independent of σ. The proof then depends on the result for $E_2(c+d^\sigma)$ found in [3]. ■

By exactly the same argument used to prove Corollary 1 we use Theorems 1 and 2 to obtain

COROLLARY 2. *If at least $m-1$ of the sums $\beta_j+\alpha_{n-k+j}$, $j=1,\dots,k$, are positive ($\beta_k+\alpha_n>0$), then*

$$\min_{P^*P=I_k}\frac{E_m(B+P^*AP)}{E_{m-1}(B+P^*AP)}=\frac{E_m(\beta_1+\alpha_{n-k+1},\dots,\beta_k+\alpha_n)}{E_{m-1}(\beta_1+\alpha_{n-k+1},\dots,\beta_k+\alpha_n)}.$$

It is obvious that the result in Theorem 1 remains intact if "convex" replaces "concave" and "max" replaces "min". Thus we have

THEOREM 3. *If $\alpha_n+\beta_k>0$, and f is symmetric, convex, and monotone non-decreasing in each x_j, $x_j\geqslant 0$, then*

$$\max_{P^*P=I_k}f(B+P^*AP)\leqslant\max_{\sigma\in S_k}f(\beta+\alpha^\sigma)$$

where $\alpha^\sigma=(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k)})$.

Some interesting choices of f are

$$f(x_1,\dots,x_k)=\sum_{j=1}^k x_j^\rho,\qquad \rho\geqslant 1,\tag{16}$$

and

$$f(x_1,\dots,x_k)=h_m^{1/m}(x_1,\dots,x_k) \tag{17}$$

where h_m is the mth completely symmetric function of the indicated variables, i.e., the sum of all $\binom{m+k-1}{m}$ homogeneous products of degree m in $x_1,\dots,x_k$ [11].

From Theorem 3 applied to (16) we have

Corollary 3. *If* $\alpha_n+\beta_k\geq 0$ *and* $\rho\geq 1$, *then*

$$\max_{P^*P=I_k}\operatorname{tr}\left((B+P^*AP)^\rho\right)=(\beta_1+\alpha_1)^\rho+\cdots+(\beta_k+\alpha_k)^\rho.$$

In order to prove a corresponding extremal result for (17), we must determine

$$\max_{\sigma\in S_k}h_m(c_1+d_{\sigma(1)},\dots,c_k+d_{\sigma(k)}).$$

Theorem 4. *If* $c_1\geq\cdots\geq c_k$, $d_1\geq\cdots\geq d_k$, *and* $c_k+d_k\geq 0$, *then*

$$\max_{\sigma\in S_k}h_m(c+d^\sigma)=h_m(c+d). \tag{18}$$

Proof. The argument begins precisely as the proof of Theorem 2 by considering

$$h_m(\dots,c_r+d_s,\dots,c_t+d_r,\dots). \tag{19}$$

We expand (19) to obtain

$$\sum_{\nu=0}^{m}h_\nu(c_r+d_s,c_t+d_r)h_{m-\nu}(\cdot). \tag{20}$$

Thus from (20) the question comes down to proving (18) for arbitrary m and $k=2$. We do this by induction on m, with nothing to prove for $m=1$. Now

$$h_m(x_1,x_2)=x_1h_{m-1}(x_1,x_2)+x_2^m,$$

and hence

$$\begin{aligned} &h_m(c_1+d_1,c_2+d_2)-h_m(c_1+d_2,c_2+d_1)\\ &=(c_1+d_1)h_{m-1}(c_1+d_1,c_2+d_2)-(c_1+d_2)h_{m-1}(c_1+d_2,c_2+d_1)\\ &\quad+(c_2+d_2)^m-(c_2+d_1)^m\\ &\geqslant(d_1-d_2)[h_{m-1}(c_1+d_1,c_2+d_2)-h_{m-1}(c_2+d_2,c_2+d_1)]\\ &\geqslant 0. \end{aligned} \tag{21}$$

The first inequality in (21) is the induction step; the second is due to the monotonicity of h_m for non-negative variables.

If we return to (20) and apply the $k=2$ case to each of the summands, we have

$$\begin{aligned} h_m(\dots,c_r+d_s,\dots,c_t+d_r,\dots) &\leqslant \sum_{\nu=0}^{m} h_\nu(c_r+d_r,c_t+d_s)h_{m-\nu}(\cdot)\\ &= h_m(\dots,c_r+d_r,\dots,c_t+d_s,\dots). \end{aligned}$$

In other words, if $s>r$ (so that $d_s\leqslant d_r$), then

$$\begin{aligned} &h_m(c_1+d_1,\dots,c_{r-1}+d_{r-1},c_r+d_s,\dots,c_t+d_r,\dots)\\ &\leqslant h_m(c_1+d_1,\dots,c_{r-1}+d_{r-1},c_r+d_r,\dots,c_t+d_s,\dots), \end{aligned}$$

and (18) follows. ■

Combining Theorems 3 and 4, we have

COROLLARY 4. *If* $\alpha_n+\beta_k\geqslant 0$, *then*

$$\max_{P^*P=I_k} h_m(B+P^*AP)=f(\beta_1+\alpha_1,\dots,\beta_k+\alpha_k).$$

2. REMARKS

For $B=0$ the extreme values of

$$\varphi(P)=f(P^*AP), \qquad P^*P=I_k$$

have been determined for a wide variety of symmetric functions f and matrices A. Perhaps the most general result of this kind appears in [7], in which A is assumed to be normal, f is a Schur function, and it is proved that $\varphi(P)$ is in the convex hull of the values that f takes on certain k-samples of eigenvalues of A. In particular it is proved that

$$E_m(P^*AP) \in \mathcal{H}\{E_m(\alpha_{\omega(1)},\dots,\alpha_{\omega(k)}),\ \omega \in Q_{k,n}\} \tag{22}$$

where $\mathcal{H}$ denotes convex hull and $Q_{k,n}$ is the totality of sequences ω, $1 \leqslant \omega(1) < \cdots < \omega(k) \leqslant n$. Of course, the techniques of the present paper are all inapplicable even for A indefinite Hermitian, much less normal, in proving a result like (22). The proof of (22) in fact depends on the representation of $E_m(P^*AP)$ as the trace of the mth exterior power of P^*AP [9].

In an elegant paper [3], M Fiedler proves that if A and B are indefinite hermitian n-square, then

$$\min_{\sigma \in S_n} \prod_{i=1}^{n} (\alpha_i + \beta_{\sigma(i)}) \leqslant \det(A+B) \leqslant \max_{\sigma \in S_n} \prod_{i=1}^{n} (\alpha_i + \beta_{\sigma(i)}). \tag{23}$$

The essential idea of the expansion lemmas appearing in [1], as well as the rearrangement theorems for

$$E_m(c + d^\sigma),$$

also are found in [3]. In a paper presented at Gatlinburg V [10], the present author showed that for a class of symmetric polynomials including E_m, h_m, and $E_m + h_m$ a precise extension of (23) is available. These results will also appear elsewhere.

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