Generalized Variational and Quasi-Variational
Inequalities with Discontinuous Operators

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We study the existence of solutions to generalized variational and generalized
quasi-variational inequalities with discontinuous operators. We obtain results which
generalize and extend previously known theorems. We also compare our new
continuity condition on the operator in the variational inequality to previously used
continuity conditions. We then apply our results to generalized variational inequal-
ities which involve pseudo-monotone operators. © 1997 Academic Press

I. INTRODUCTION

In the last thirty years the study of variational inequalities has gained
importance in analysis from both the theoretical and the applications
points of view. The classical variational inequality problem, VI(K, f), can
be stated as follows:

Given a linear topological vector space X, with dual X*, and duality
pairing ⟨·, ·⟩: X* × X → ℝ, K ⊂ X, and f: K → X*, find an
x₀ ∈ K such that ⟨f(x₀), u − x₀⟩ ≥ 0 for all u ∈ K [14].
If the set K varies with x, i.e., there is a mapping G: K → 2^K, then we
have what is called the quasi-variational inequality problem QVI(K, f, G):

Find x₀ ∈ G(x₀) such that ⟨f(x₀), u − x₀⟩ ≥ 0 for all u ∈ G(x₀).

The quasi-variational inequality was first studied as a problem of evolution
associated with control theory by Bensoussan and Lions in 1973 [3]. More
recently authors have considered the mapping $f$ in the above formulations as a set valued map \cite{7, 10, 13, 22} denoted by $F$. These problems are called the generalized variational inequality $GVI(K, F)$.

Find an $x_0 \in K$ and $y_0 \in F(x_0)$

such that $\langle y_0, u - x_0 \rangle \geq 0$ for all $u \in K$,

and the generalized quasi-variational inequality $GQVI(K, F, G)$,

Find $x_0 \in G(x_0)$ and $y_0 \in F(x_0)$

such that $\langle y_0, u - x_0 \rangle \geq 0$ for all $u \in G(x_0)$.

Note that if $G(x) = K$ for each $x \in K$ then $GQVI(K, F, G)$ reduces to the generalized variational inequality, $GVI(K, F)$, and $QVI(K, F, G)$ reduces to $VI(K, F)$.

Much of the study of variational inequalities centers upon finding conditions on the space $X$, the set $K$, and the mappings $F$ and $G$ which will ensure existence of a solution to the inequality. In this paper we will be proving existence results for generalized variational and generalized quasi-variational inequalities where the operator $F$ is not continuous. We will then compare our more general condition on the operator $F$ to continuity conditions which have been used by other authors. Lastly, we will apply our results to generalized variational inequalities involving pseudo-monotone operators.

II. PRELIMINARIES

If $A$ is a set of a topological vector space $X$ then we will denote the (closed) convex hull of $A$ by $\text{co}(A)$ ($\overline{\text{co}}(A)$), the interior (interior with respect to $K \subset X$) of $A$ by $\text{int}(A)$ ($\text{int}_K(A)$), the closure in $X$ of $A$ by $\overline{A}$, and the closure of $A$ with respect to a subset $K \subset X$ by $\overline{K}$. If $X$ is a metric space then we will denote the (closed) ball about $x \in X$ of radius $\epsilon$ by $B(x; \epsilon) (\overline{B}(x; \epsilon))$. A set $K$ is called almost $\sigma$-compact if $K = \overline{\bigcup_{n=1}^{\infty} T_n}$ where each $T_n$ is a compact subset of $X$. If $X$ is a locally convex topological vector space and $K$ is a convex almost $\sigma$-compact set then without loss of generality we can assume that the $T_n$ are increasing, i.e., $T_n \subset T_{n+1}$ for all $n \in \mathbb{N}$, and that each $T_n$ is also convex. This follows because $\overline{\bigcup_{1}^{\infty} T_n} \subset K$ is compact and convex \cite{21}.

Let $X$ and $Y$ be topological spaces and $F$ be a set valued map from $X$ into $Y$. The domain of $F$ is defined by $\text{dom} F = \{ x \in X : F(x) \neq \emptyset \}$. We will denote a set valued map from $X$ into $Y$ with $\text{dom} F = X$ by writing $F : X \rightarrow 2^Y$. We say $F$ is upper semicontinuous at $x_0 \in X$ if for every open set $N$ in $Y$ such that $F(x_0) \subset N$, there is an open neighborhood of $x_0$ in
Let $X$, say $U(x_0)$, such that for all $x \in U(x_0)$ we have that $F(x) \subset N$. A set valued map is lower semicontinuous at $x_0 \in X$ if for each $y_0 \in F(x_0)$ and each neighborhood of $y_0$ in $Y$, say $N(y_0)$, there is a neighborhood $U(x_0)$ in $X$, such that for all $x \in U(x_0)$, we have $F(x) \cap N(y_0) \neq \emptyset$. If $F$ is upper (lower) semicontinuous for each $x \in \text{dom } F$ then we say $F$ is upper (lower) semicontinuous. The graph of $F: X \to 2^Y$ is defined by $\text{Gr}(F) = \{(x, y) : y \in F(x), x \in X\}$. We say a set valued map is closed or has a closed graph if $\text{Gr}(F)$ is closed as a subset of $X \times Y$. If for each $x \in X$, we have that $F(x)$ is a convex (or closed, or compact, etc.) subset of $Y$, then we say that $F$ has convex (or closed, or compact, etc.) values. The following facts about set valued maps can be found in [1, 2, 18].

**Lemma 2.1.** Suppose $F: X \to 2^Y$ is a set valued map with closed graph. Then $F$ has closed values.

**Lemma 2.2.** Let $X$ and $Y$ be Hausdorff topological spaces and $G: X \to 2^Y$ be lower semicontinuous and $T: X \to 2^Y$. If $G(x) \subset \overline{T(x)} \subset \overline{G(x)}$ for each $x \in X$, then $T$ is also lower semicontinuous.

**Lemma 2.3.** Let $X$ and $Y$ be Hausdorff topological spaces, $G: X \to 2^Y$ be lower semicontinuous and $L: \text{dom } L \subset X \to 2^Y$ be such that $\text{Gr}(L)$ is an open subset of $X \times Y$. Then the mapping $Q(x) = G(x) \cap L(x)$ is lower semicontinuous on its domain and $\text{dom } Q$ is an open (possibly empty) subset of $X$.

**Lemma 2.4.** Let $X$ be a normed space, $C$ and $D$ be convex subsets of $X$, and $G: D \to 2^C$ be lower semicontinuous. Then if $\epsilon > 0$, the mapping $H_\epsilon: D \to 2^C$ defined by

$$H_\epsilon(x) = B_\epsilon(G(x); \epsilon) = B(G(x); \epsilon) \cap C$$

is lower semicontinuous on $D$.

Following Browder [4–6], we say a mapping $F: K \to 2^{X^*}$ is pseudo-monotone if for any sequence $\{x_n\}$ in $K$ converging weakly to an element $x_0 \in K$ and for any sequence $\{y_n\}$ in $X^*$ with $y_n \in F(x_n)$, for each $n \in \mathbb{N}$, for which

$$\limsup_{n \to \infty} \langle y_n, x_n - x_0 \rangle \leq 0,$$

then for each $u \in K$ there exists an element $y_u \in F(x_0)$ such that

$$\langle y_u, x_0 - u \rangle \leq \liminf_{n \to \infty} \langle y_n, x_n - u \rangle.$$
It is important to note that the element \( y_0 \) in \( F(x_0) \) depends on the choice of \( u \), and is not in general uniform for all \( u \in K \). Certain quasi-linear operators [14, 18] are examples of pseudo-monotone mappings.

Lastly we state Kneser’s Minimax Inequality [16].

**Lemma 2.5 (Kneser’s Minimax Inequality).** Let \( K \) be a nonempty convex set in a vector space \( X \) and \( D \) a nonempty compact convex subset of a Hausdorff topological vector space \( Y \). Suppose \( f \) is a real-valued function on \( K \times D \) such that

(i) For each fixed \( x \in K \), \( f(x, y) \) is lower semicontinuous and convex on \( D \).

(ii) For each fixed \( y \in D \), \( f(x, y) \) is concave on \( K \).

Then \( \sup_{x \in K} \inf_{y \in D} f(x, y) = \inf_{y \in D} \sup_{x \in K} f(x, y) \).

### III. MAIN RESULTS

Our first theorem is an existence result for the GQVI \((K, F, G)\) with \( F \) noncontinuous. Some of the ideas in this proof can be found in [7].

**Theorem 3.1.** Let \( X \) be a separable Banach space and \( K \) be a nonempty compact convex subset of \( X \). Suppose \( F : K \to 2^{X^*} \) and \( G : K \to 2^K \) such that the following hold:

(i) The mapping \( L : K \to 2^K \) defined by

\[
L(x) = \left\{ w \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}
\]

has a closed graph in \( K \times K \).

(ii) \( G \) is a lower semicontinuous mapping with a closed graph and nonempty convex values.

Then there is an \( x_0 \in G(x_0) \) such that \( \inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0 \) for all \( u \in G(x_0) \). If in addition \( F(x_0) \) is also weak* compact and convex then there exists a \( y_0 \in F(x_0) \) such that \( \langle y_0, x_0 - u \rangle \leq 0 \) for all \( u \in G(x_0) \).

**Proof.** Consider the mapping \( \hat{L} : \text{dom } \hat{L} \subset K \to 2^K \) defined by

\[
\hat{L}(x) = \left\{ w \in K : \inf_{y \in F(x)} \langle y, x - w \rangle > 0 \right\}.
\]
Then for \((x,w) \in K \times K\) we have
\[
(x,w) \in \text{Gr} \hat{L} \iff w \in K, x \in K, \text{ and } \inf_{y \in F(x)} \langle y, x - w \rangle > 0
\]
\[
\implies (x,w) \not\in \text{Gr} L.
\]
Thus \(\text{Gr} \hat{L} = (\text{Gr} L)^c\) in \(K \times K\), and hence \(\text{Gr} \hat{L}\) is an open subset of \(K \times K\).

For \(j \in \mathbb{N}\) define \(G_j : K \to 2^K\) by
\[
G_j(x) = \left\{ w \in K : d(w, G(x)) < \frac{1}{j} \right\} = B_x \left( G(x); \frac{1}{j} \right).
\]

By Lemma 2.4, we see that \(G_j\) is lower semicontinuous on \(K\). By Lemma 2.2, we see that the mapping \(\overline{G_j}\) defined by
\[
\overline{G_j}(x) = \left\{ w \in K : d(w, G(x)) \leq \frac{1}{j} \right\} = \text{cl}_K \left( B \left( G(x); \frac{1}{j} \right) \right)
\]
is also lower semicontinuous on \(K\).

Now we define \(Q_j : \text{dom } Q_j \subset K \to 2^K\) by \(Q_j(x) = \overline{G_j}(x) \cap \hat{L}(x)\). Since \(\text{Gr} \hat{L}\) is an open subset of \(K \times K\) and \(\overline{G_j}\) is lower semicontinuous, then by Lemma 2.3, \(Q_j\) is lower semicontinuous on its domain.

Lastly, we define \(H_j : \text{dom } H_j \subset K \to 2^K\) by
\[
H_j(x) = \begin{cases} 
Q_j(x), & x \in K \text{ and } x \in \overline{G_j}(x) \\
\overline{G_j}(x), & x \in K \text{ and } x \not\in \overline{G_j}(x).
\end{cases}
\]

We will proceed by supposing that the domain of \(H_j = K\) and arrive at a contradiction. We first claim that if \(\text{dom } H_j = K\) then \(H_j\) is lower semicontinuous on \(K\). To see this let \(x_0 \in K\), \(y_0 \in H_j(x_0)\), and \(N(y_0)\) be an open neighborhood of \(y_0\) in \(K\). Since \(y_0 \in H_j(x_0)\) then either \(y_0 \in Q_j(x_0)\) or \(y_0 \not\in \overline{G_j}(x_0)\).

If \(y_0 \in Q_j(x_0)\), then since \(Q_j\) is lower semicontinuous on its domain, there exists a neighborhood of \(x_0\), say \(N_j(x_0)\), such that for all \(x \in N_j(x_0) \cap \text{dom } Q_j, Q_j(x) \cap N(y_0) \neq \emptyset\).

If \(y_0 \not\in \overline{G_j}(x_0)\), then since \(\overline{G_j}\) is lower semicontinuous on \(K\), there exists a neighborhood of \(x_0\), say \(N_j(x_0)\), such that for all \(x \in N_j(x_0)\), \(\overline{G_j}(x) \cap N(y_0) \neq \emptyset\).

Let \(N(x_0) = N_j(x_0) \cap N_j(x_0)\) and suppose \(x \in N(x_0)\). Then either \(x \in \overline{G_j}(x)\) or \(x \not\in \overline{G_j}(x)\). If \(x \in \overline{G_j}(x)\) then \(H_j(x) = \overline{G_j}(x)\) and since \(x \in N(x_0) \subset N_j(x_0)\), \(H_j(x) \cap N(y_0) = \overline{G_j}(x) \cap N(y_0) \neq \emptyset\). If \(x \not\in \overline{G_j}(x)\)
then $H_i(x) = Q_i(x) \neq \phi$ since we are assuming $\text{dom} \: H = K$. Hence $x \in N(x_0) \cap \text{dom} \: Q_i \subset N(x_0) \cap \text{dom} \: Q_j$ which implies $H_i(x) \cap N(x_0) = Q_i(x) \cap N(x_0) \neq \phi$. Thus $H_i$ is lower semicontinuous on $K$.

Next we claim that for each $x \in K$, $H_i(x)$ is a convex subset of $K$. If $H_i(x) = G_i(x)$, then $H_i(x)$ is convex since $G$ has convex values. If $H_i(x) = Q_i(x)$, then let $w_1, w_2 \in H_i(x)$ and consider $w = \alpha w_1 + (1 - \alpha)w_2$, $\alpha \in [0, 1]$.

Clearly $w \in G_j(x)$, since $G_j(x)$ is convex, and we have

$$\inf_{y \in F(x)} \langle y, x - w \rangle \geq \alpha \inf_{y \in F(x)} \langle y, x - w_1 \rangle + (1 - \alpha) \inf_{y \in F(x)} \langle y, x - w_2 \rangle > 0.$$  

Thus $w \in Q_i(x)$ and hence $H_i(x)$ is convex.

Now since $K \subset X$, $X$ is a separable Banach space, $X$ is perfectly normal (every closed subset of $X$ is a $G_d$ set), and $H_i$ has nonempty, convex values, we have via Michael’s Selection Theorem [19, Theorem 3.1.6(c)] that $H_i$ admits a continuous selection. Thus there is a continuous function $h_i: K \to K$ such that $h_i(x) \in H_i(x)$. By Schauder’s Fixed Point Theorem [21, p. 143] we have that $h_i$ has a fixed point. Thus there is an $x \in K$ such that $x = h_i(x) \in H_i(x)$. By the definition of $H_i$, we must have that $x \in Q_i(x)$ and $x \in G_j(x)$. If $x \in Q_i(x)$ then $\inf_{y \in F(x)} \langle y, x - x \rangle > 0$ which is a contradiction. Therefore, $\text{dom} \: H_i \neq K$.

Thus there is an $x \in K$ such that $H_i(x) = \phi$. Now since $G(x) \subset G_j(x)$ and $G(x) \neq \phi$ for all $x \in K$, we must have that $\phi = H_i(x) = Q_i(x)$ for some $x \in K$ and $x \in G_j(x)$. Since the $j$ above was arbitrary, we now have for each $j \in \mathbb{N}$ an $x_j \in K$ such that $Q_j(x_j) = \phi$ and $x_j \in G_j(x_j)$. Because $K$ is compact, we have a convergent subsequence, i.e., $x_{j_k} \to x_0 \in K$. Since $x_{j_k} \in G_j(x_{j_k})$ we have $d(x_{j_k}, G(x_{j_k})) \leq 1/j_k$ and since $G_j(x_j)$ is compact, there is $w_{j_k} \in G_j(x_{j_k})$ such that $d(w_{j_k}, x_{j_k}) \leq 1/j_k$. Again because $K$ is compact, $\{w_{j_k}\}$ has a convergent subsequence, $w_{j_k} \to w_0 \in K$. Hence

$$d(x_{j_k}, w_0) \leq d(x_{j_k}, w_{j_k}) + d(w_{j_k}, w_0) \to 0, \quad \text{as } k \to \infty.$$  

Thus $x_{j_k} \to w_0$ which implies $w_0 = x_0$. Lastly, since $x_{j_k} \to x_0$, $w_{j_k} \in G(x_{j_k})$, $w_{j_k} \to w_0$, and $G$ is closed, then $w_0 \in G(x_0)$ and thus $x_0 \in G(x_0)$. We claim that $x_0$ is our solution.

Let $w$ be an arbitrary member of $G(x_0)$. Using the same sequence $\{x_{j_k}\}$ above, where $x_{j_k} \to x_0$, we have via the lower semicontinuity of the mapping $G$, a sequence $w_{n} \in G(x_{j_k})$ such that $w_{n} \to w$ as $n \to \infty$. Since $w_{n} \in G(x_{j_k}) \subset G_j(x_{j_k})$ and $Q_j(x_{j_k}) = \phi$, that is,

$$\inf_{y \in F(x_{j_k})} \langle y, x_{j_k} - w \rangle \leq 0 \quad \text{for all } w \in G_j(x_{j_k}),$$

and hence $x_{j_k}$ is our solution.
then

$$\inf_{y \in F(x_n)} \langle y, x_n - w_n \rangle \leq 0.$$  

Thus \((x_n, w_n) \in \text{Gr } L\). Since \(x_n \to x_0\) and \(w_n \to w\) and \(\text{Gr } L\) is closed then \((x_0, w) \in \text{Gr } L\) and we have \(\inf_{y \in F(x_0)} \langle y, x_0 - w \rangle \leq 0\). Thus there is \(x_0 \in G(x_0)\) such that

$$\inf_{y \in F(x_0)} \langle y, x_0 - w \rangle \leq 0$$

and since \(w \in G(x_0)\) was arbitrarily chosen we have

$$\inf_{y \in F(x_0)} \langle y, x_0 - w \rangle \leq 0 \quad \text{for all } w \in G(x_0).$$

Lastly, if \(F(x_0)\) is weak* compact and convex then we define \(f(x, y) = \langle y, x_0 - x \rangle\) for \(x \in K, y \in F(x_0)\) and apply Kneser's Minimax Inequality (Lemma 2.5) to get \(y_0 \in F(x_0)\) such that \(\langle y_0, x_0 - u \rangle \leq 0\) for all \(u \in G(x_0)\).

Our next theorem is a generalization of [9, Theorem 4.1] from \(\mathbb{R}^n\) to Hausdorff topological vector spaces. The proof [18], which we will omit, is similar to that of [9, Theorem 4.1]. It is a standard application of Ky Fan's generalization [8] of the K naster–Kurotowski–Mazukiewicz Lemma [15].

**Theorem 3.2.** Let \(K\) be a nonempty compact convex subset of a real Hausdorff topological vector space \(X\) and \(F: K \to 2^{X^*}\) be nonempty for each \(x \in K\). If for each \(u \in K\), the set

$$B(u) = \left\{ x \in K : \inf_{y \in F(x)} \langle y, x - u \rangle \leq 0 \right\}$$

is closed, then there is an \(x_0 \in K\) such that \(\inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0\) for all \(u \in K\). If in addition \(F(x_0)\) is weak* compact and convex then there exists a \(y_0 \in F(x_0)\) such that \(\langle y_0, x_0 - u \rangle \leq 0\) for all \(u \in K\).

In order to relate Theorem 3.1 to Theorem 3.2 we have the following result. We note that condition (i)' of Theorem 3.3 is more restrictive than condition (i) of Theorem 3.2 since the values of a closed set valued map are always closed (Lemma 2.1) but a set valued map with closed values is not necessarily a closed set valued map.
THEOREM 3.3. If we replace assumption (i) in Theorem 3.1 above with the following assumption:

(i') the mapping \( B: K \to 2^K \) defined by
\[
B(w) = \left\{ x \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}
\]

has a closed graph in \( K \times K \), then the conclusions of Theorem 3.1 still hold.

Proof.
\[
(w, x) \in \text{Gr } B \iff w \in K, x \in K \text{ and } \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0
\]
\[
\iff (x, w) \in \text{Gr } L. \quad \blacksquare
\]

We now solve the GQVI \( (K, F, G) \) when \( K \) is a noncompact set. In Theorem 3.4, note that assumption (iii) is a condition used in [24, Theorem 3.3, 17, Theorem 6] which we will refer to as Yao's Condition.

THEOREM 3.4. Let \( X \) be a separable Banach space and \( K \) be a nonempty closed convex subset of \( X \). Suppose \( F: K \to 2^{X^*}, G: K \to 2^K \), and \( C \) is a nonempty compact convex subset of \( K \) such that the following hold:

(i) The mapping \( L: K \to 2^K \) defined by
\[
L(x) = \left\{ w \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}
\]
has a closed graph in \( K \times K \).

(ii) The mapping \( G_C: C \to 2^C \) defined by \( G_C(x) = G(x) \cap C \) is lower semicontinuous, has a closed graph on \( C \times C \), and nonempty convex values.

(iii) For each \( x \in C \), \( \text{int}_{G_C(x)} G_C(x) \neq \emptyset \) and for all \( x \in \text{dom}_{G_C} G_C(x) \) there is a \( u \in \text{int}_{G_C(x)} G_C(x) \) such that \( \langle y, x - u \rangle \geq 0 \) for all \( y \in F(x) \).

Then there is an \( x_0 \in G(x_0) \) such that \( \inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0 \) for all \( u \in G(x_0) \). If in addition \( F(x_0) \) is also weak* compact and convex then there exists a \( y_0 \in F(x_0) \) such that \( \langle y_0, x_0 - u \rangle \leq 0 \) for all \( u \in G(x_0) \).

Proof. Our approach, similar to [24], will be to solve GQVI \( (C, F, G_C) \) and then show our solution solves GQVI \( (K, F, G) \). We first note that the mapping \( L_C: C \to 2^C \) defined by
\[
L_C(x) = L(x) \cap C = \left\{ w \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\} \cap C
\]
\[
= \left\{ w \in C : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}
\]
has a closed graph in $C \times C$. Thus by Theorem 3.1 above, there is an $x_0 \in G_c(x_0) \subset G(x_0)$ such that
\[
\inf_{y \in F(x_0)} \langle y, x_0 - w \rangle \leq 0 \quad \text{for all } w \in G_c(x_0).
\]

Now we consider two cases:

**Case 1.** $x_0 \in \text{int}_{G(x_0)} G_c(x_0)$. Let $u \in G(x_0)$. If $u \in G_c(x_0)$ then we are done. If $u \notin G_c(x_0)$ then since $x_0 \in \text{int}_{G(x_0)} G_c(x_0)$ there is a $\lambda > 0$, $\lambda \in (0,1)$, such that $\lambda x_0 + (1 - \lambda)u \in \partial_{G(x_0)} G_c(x_0) \subset G_c(x_0)$. This implies
\[
\inf_{y \in F(x_0)} \langle y, x_0 - (\lambda x_0 + (1 - \lambda)u) \rangle \leq 0,
\]
which implies
\[
(1 - \lambda) \inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0,
\]
and thus we have
\[
\inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0.
\]

**Case 2.** $x_0 \in \partial_{G(x_0)} G_c(x_0)$. Then by assumption (ii) above, there exists a $u_0 \in \text{int}_{G(x_0)} G_c(x_0) \subset G_c(x_0)$ such that
\[
\inf_{y \in F(x_0)} \langle y, x_0 - u_0 \rangle \geq 0.
\]
Since $x_0$ solves $GQVI(C,F,G_c)$ we have
\[
\inf_{y \in F(x_0)} \langle y, x_0 - u_0 \rangle = 0.
\]

Now let $u \in G(x_0)$. If $u \in G_c(x_0)$ then we are done since $x_0$ solves $GQVI(C,F,G_c)$. If $u \notin G_c(x_0)$ then since $u_0 \in \text{int}_{G(x_0)} G_c(x_0) \subset G_c(x_0)$ there is a $\lambda \in (0,1)$ such that $\lambda u_0 + (1 - \lambda)u \in \partial_{G(x_0)} G_c(x_0) \subset G_c(x_0)$. This implies
\[
\inf_{y \in F(x_0)} \langle y, x_0 - (\lambda u_0 + (1 - \lambda)u) \rangle \leq 0,
\]
which implies
\[
0 \geq \inf_{y \in F(x_0)} \left[ \lambda \langle y, x_0 - u_0 \rangle + (1 - \lambda) \langle y, x_0 \rangle \right]
\geq \lambda \inf_{y \in F(x_0)} \langle y, x_0 - u_0 \rangle + (1 - \lambda) \inf_{y \in F(x_0)} \langle y, x_0 \rangle
= (1 - \lambda) \inf_{y \in F(x_0)} \langle y, x_0 - u \rangle.
\]

Thus we have
\[
\inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0.
\]

By Cases 1 and 2 we have there is an \( x_0 \in G(x_0) \) such that
\[
\inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0 \quad \text{for all } u \in G(x_0).
\]

Lastly, if we also assume \( F(x_0) \) is weak* compact and convex, then applying Kneser's Minimax Inequality as in Theorem 3.1, there exists an \( x_0 \in G(x_0) \) and \( y_0 \in F(x_0) \) such that \( \langle y_0, x_0 - u \rangle \leq 0 \) for all \( u \in G(x_0) \).

In our next theorem, condition (ii), and its generalizations, has been used by many authors [7, 9–12, 24, 25]. We will refer to this condition as the Karamardian Condition. Note that Theorem 3.5 is an extension of Theorem 3.1.

**Theorem 3.5.** Let \( X \) be a separable Banach space and \( K \) be a nonempty closed convex subset of \( X \). Suppose \( F: K \to 2^X \), \( G: K \to 2^K \) is a closed set valued map with nonempty, convex values, and \( C \) is a nonempty compact convex subset of \( K \) such that the following hold:

(i) The mapping \( L: K \to 2^K \) defined by
\[
L(x) = \left\{ w \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}
\]
has a closed graph in \( K \times K \).

(ii) For each \( x \in K \setminus C \) such that \( x \in G(x) \), there is a \( z \in G(x) \cap C \) such that \( \langle y, z - x \rangle < 0 \) for all \( y \in F(x) \).

(iii) There is a compact convex subset \( B \subset K \) such that \( C \subset \text{int}_k B \).

(iv) The mapping \( G_B: B \to 2^B \) defined by \( G_B(x) = G(x) \cap B \) is lower semicontinuous from \( B \) into \( 2^B \).
Then there is an \( x_0 \in G(x_0) \) such that \( \inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0 \) for all \( u \in G(x_0) \). If in addition \( F(x_0) \) is also weak* compact and convex then there exists a \( y_0 \in F(x_0) \) such that \( \langle y_0, x_0 - u \rangle \leq 0 \) for all \( u \in G(x_0) \).

**Proof.** To see that \( \text{int}_{G(x)} G_B(x) \) is nonempty for each \( x \in B \), consider \( y \in G_B(x) \) such that \( y \in \text{int}_{G(x)} G_B(x) \). Such a \( y \) exists by assumption. Then there is a neighborhood of \( y \), \( N(y) \), in \( X \), such that

\[
N(y) \cap K \subseteq B
\]

hence

\[
N(y) \cap G(x) = N(y) \cap K \cap G(x) \subseteq B \cap G(x) = G_B(x)
\]

and thus \( y \in \text{int}_{G(x)} G_B(x) \). This also shows that \( G(x) \cap C \subseteq \text{int}_{G(x)} G_B(x) \).

Now suppose \( x \in \partial_{G(x)} G_B(x) \). If \( x \in C \) then the argument above shows that \( x \in \text{int}_{G(x)} G_B(x) \) which is a contradiction. Thus \( x \not\in C \) and by condition (ii) there is a \( z \in G(x) \cap C \subseteq \text{int}_{G(x)} G_B(x) \) such that \( \langle y, z - x \rangle < 0 \) for all \( y \in F(x) \). Thus we have that Yao’s Condition (assumption (iii) of Theorem 3.4) is satisfied with the compact convex subset \( B \) and hence we can apply Theorem 3.4 to get the result.

**IV. COMPARISON OF CONTINUITY CONDITIONS**

Certainly the results in Section III are not the first attempts by mathematicians to remove the condition of continuity on the operator \( F \). In this section we examine various continuity conditions on the mapping \( F \) which have been used by other authors. In particular, we will be interested in how these continuity conditions compare to our assumption that the operator \( L \) has a closed graph in \( K \times K \) which is used in our theorem. We first state the conditions we will be interested in examining. Unless otherwise stated, we will assume that \( X \) is a real Hausdorff topological vector space, \( K \subseteq X \), \( F: K \to 2^{X^*} \), and \( G: K \to 2^K \).

1. The mapping \( L: K \to 2^K \) defined by

\[
L(x) = \left\{ w \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}
\]

has a closed graph in \( K \times K \).

2. The mapping \( B: K \to 2^K \) defined by

\[
B(w) = \left\{ x \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}
\]

has a closed graph in \( K \times K \).
The interaction set
\[ \Sigma = \left\{ x \in K : \sup_{w \in G(x)} \inf_{y \in F(x)} \langle y, x - w \rangle > 0 \right\} \]
is open in \( K \).

(4) For each \( z \in K - K \) the set
\[ S(z) = \left\{ x \in K : \inf_{y \in F(x)} \langle y, z \rangle \leq 0 \right\} \]
is closed in \( K \).

(5) The mapping \( B \) defined in Condition (2) has closed values.

(6) \( F \) is an upper semicontinuous set valued map from the strong topology on \( K \) to the strong topology on \( X^* \).

(7) \( F \) is an upper semicontinuous set valued map from the strong topology on \( K \) to the weak* topology on \( X^* \).

(8) The mapping \( x \mapsto \inf_{y \in F(x)} \langle y, x - w \rangle \) from \( K \) into \( \mathbb{R} \) is a lower semicontinuous function for each \( w \in K \).

(9) If \( x_n \to x \) in \( K \) then for each \( w \in K \),
\[ \inf_{y \in F(x)} \langle y, x - w \rangle \leq \limsup_{n \to \infty} \left( \inf_{y \in F(x_n)} \langle y, x_n - w \rangle \right). \]

Conditions (1) and (2) appear in this paper. All of the other conditions have been used by previous authors. For instance, conditions (3) and (8) appear in [22]. We note that in the statement of [22, Theorem 3] we can replace the assumption that \( F \) is an upper semicontinuous set valued map with the more general assumption (8) provided \( X \) is a normed space [18]. Condition (4) has appeared in [7, 20] while conditions (5) and (9) are in [9]. Lastly, conditions (6) and (7) are the classic upper semicontinuity conditions on \( F \) [17, 20]. We now compare these conditions via Theorem 4.1 which we will state without proof. For proof see [18]. We also note that some of the statements in Theorem 4.1 are true in a less restrictive (i.e., non-Banach) space \( X \). We also present a diagram of these comparisons in Fig. 4.2 for easy reference. For example, from Fig. 4.2 we can see that Condition (6) implies Condition (7) and that Condition (7) implies Condition (1) provided \( F \) is weak* compact valued.

**Theorem 4.1.** Let \( X \) be a Banach space, \( K \subset X \), and \( F : K \to 2^{X^*} \) and \( G : K \to 2^K \) be two set valued maps. Then

(i) \( (1) \Rightarrow (2) \Rightarrow (5) \).

(ii) If \( G \) is lower semicontinuous then \( (1) \Rightarrow (3) \).

(iii) \( (6) \Rightarrow (7) \) and if in addition, \( F \) has weak* compact values, then \( (7) \Rightarrow (1) \).
(iv) If $F$ has weak* compact values then (6) ⇒ (8) ⇒ (5).
(v) (9) ⇒ (5).
(vi) (4) ⇒ (5) and (4) ⇒ (3).
(vii) If $F$ is sequentially bounded, i.e., if $x_n \to x \in K$ then there is an $M > 0$ ($M$ depending on $(x_n)$ and $x$) such that if $y \in F(x) \cup (\bigcup_{n=1}^\infty F(x_n))$ then $\|y\| \leq M$, then (7) ⇒ (1) (and thus (2) by (i)).
(viii) If $F$ is sequentially bounded then (8) ⇒ (1) (and thus (2) by (i)).

An open question for us is under what minimal criteria, if any, does condition (5) above imply condition (2). Certainly under these conditions Theorem 3.1 would be a direct generalization of Theorem 3.2.

V. APPLICATIONS TO PSEUDO-MONOTONE OPERATORS

We begin with the following lemma which ties our results to variational inequalities with $F$ being a pseudo-monotone mapping.

**Lemma 5.1.** Let $K$ be a closed subset of a Banach space $X$ and $F: K \to 2^{X^*}$ be a sequentially bounded (i.e., if $x_n \to x_0$, then $\bigcup_{n \in \mathbb{N}} F(x_n)$ is bounded in $X^*$), pseudo-monotone mapping. Then the set valued map $B: K \to 2^K$ de-
 fined by

\[ B(w) = \left\{ x \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\} \]

has a closed graph in \( K \times K \).

**Proof.** Suppose \((w_n, x_n) \in \text{Gr}(B)\), \(w_n \to w_0\), and \(x_n \to x_0\). Then for each \(n \in \mathbb{N}\) we have

\[ \inf_{y \in F(x_n)} \langle y, x_n - w_n \rangle \leq 0. \]

Thus for each \(n \in \mathbb{N}\) there is a \(y_n \in F(x_n)\) such that

\[ \langle y_n, x_n - w_n \rangle \leq \frac{1}{n}. \]

Since \(F\) is sequentially bounded, we have that \(\{y_n\}\) is bounded. Thus \(\limsup_{n \to \infty} \langle y_n, x_n - x_0 \rangle = 0\) and since \(F\) is pseudo-monotone there is a \(y_0 \in F(x_0)\) such that

\[ \langle y_0, x_0 - w_0 \rangle \leq \liminf_{n \to \infty} \langle y_n, x_n - w_n \rangle \]

\[ = \liminf_{n \to \infty} \left[ \langle y_n, x_n - w_n \rangle + \langle y_n, w_n - w_0 \rangle \right] \]

\[ = \liminf_{n \to \infty} \langle y_n, x_n - w_n \rangle \leq 0. \]

Thus \(\inf_{y \in F(x_0)} \langle y, x_0 - w_0 \rangle \leq 0\) which implies that \((w_0, x_0) \in \text{Gr}(B)\). \(\blacksquare\)

Theorem 5.2 below extends a special case of a result of Browder [4, Theorem 7.8, with \(T \equiv \{0\}\)] by dropping the continuity on finite dimensional subspaces condition and adding the condition that the set \(K\) is almost \(\sigma\)-compact. Since in a separable Banach space any closed set is an almost \(\sigma\)-compact set, then our Theorem 5.2 is a direct generalization of Browder’s theorem in the case that \(X\) is separable. Theorem 5.2 also extends a theorem [23, Theorem 2] for monotone operators which are continuous from line segments to the weak* topology on \(X^*\) to bounded (i.e., mapping bounded sets to bounded sets) noncontinuous pseudo-monotone operators in the case that \(X\) is separable.

**Theorem 5.2.** Let \(X\) be a reflexive Banach space, \(K\) a convex almost \(\sigma\)-compact subset of \(X\), and \(F: K \to 2^{X^*}\) such that the following hold:

(i) \(F\) is a pseudo-monotone mapping which maps bounded sets to bounded sets and has convex, weak* compact values.
(ii) $F$ is coercive, i.e., there is an $z_0 \in K$ such that

$$\lim_{||x|| \to \infty, x \in K} \left\{ \inf_{y \in F(x)} \langle y, x - z_0 \rangle \right\} > 0.$$ 

Then there is an $x_0 \in K$ and a $y_0 \in F(x_0)$ such that $\langle y_0, x_0 - u \rangle \leq 0$ for all $u \in K$.

Proof. By Lemma 5.1 we have that the graph of $B: K \to 2^K$ defined by

$$B(w) = \left\{ x \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}$$

has a closed graph in $K \times K$. Since $K$ is convex and almost $\sigma$-compact, we can write $K = \overline{\text{cl}(\bigcup_{n=1}^{\infty} T_n)}$ where the $T_n$ are compact, convex, and $z_0 \in T_n$ for all $n \in \mathbb{N}$. If we define $B_n: T_n \to 2^{T_n}$ by

$$B_n(w) = \left\{ x \in T_n : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}$$

then it is easy to see that $B_n$ has a closed graph in $T_n \times T_n$. Thus by Theorem 3.2 (or Theorem 3.1 by letting $G(x) = T_n$ for each $x \in T_n$) there is an $x_n \in T_n$ and a $y_n \in F(x_n)$ such that $\langle y_n, x_n - u \rangle \leq 0$ for all $u \in T_n$.

Via assumption (ii) the sequence $\{x_n\}$ is bounded. Thus there is a subsequence of $\{x_n\}$ which converges weakly to some $x_0 \in K$. Since the sets $T_n, n = 1, \ldots, \infty$, are increasing, we can without loss of generality refer to our weakly convergent subsequence as $\{x_n\}$.

Since $x_0 \in K$ there are $q_n \in T_n$ such that $q_n \to x_0$. We also have that $\langle y_n, x_n - q_n \rangle \leq 0$ for each $n \in \mathbb{N}$. Thus

$$\langle y_n, x_n - x_0 \rangle = \langle y_n, x_n - q_n \rangle + \langle y_n, q_n - x_0 \rangle \leq \langle y_n, q_n - x_0 \rangle$$

and hence

$$\limsup_{n \to \infty} \langle y_n, x_n - x_0 \rangle \leq \limsup_{n \to \infty} \langle y_n, q_n - x_0 \rangle = 0$$

since $F$ maps bounded sets to bounded sets.

Let $u \in K$. Then since $F$ is pseudo-monotone on $K$ there is a $y_u \in F(x_0)$ such that

$$\langle y_u, x_0 - w \rangle \leq \liminf_{n \to \infty} \langle y_n, x_n - w \rangle.$$
There are also \( u_n \in T_n \) such that \( u_n \to u \). Thus we have
\[
\liminf_{n \to \infty} \langle y_n, x_n - u \rangle = \liminf_{n \to \infty} \left[ \langle y_n, x_n - u_n \rangle + \langle y_n, u_n - u \rangle \right] \\
= \liminf_{n \to \infty} \langle y_n, x_n - u_n \rangle \leq 0.
\]

The last equality follows from the fact that \( F \) maps bounded sets to bounded sets. Thus we have that

\[
\langle y_u, x_0 - u \rangle \leq 0
\]

and hence
\[
\inf_{y \in F(x)} \langle y, x_0 - u \rangle \leq 0 \quad \text{for all } u \in K.
\]

Lastly, using Kneser’s Minimax Inequality (Lemma 2.5), we get the result.

The coercivity condition in Theorem 5.2 (condition (ii)) essentially tells us that if \( x \) is outside a certain ball of radius \( R_1 > 0 \), then

\[
\inf_{y \in F(x)} \langle y, x - z_0 \rangle > 0.
\]

This implies \( x \) cannot be a solution to the variational inequality. In the case that our space \( X \) is finite dimensional or our set \( K \) is locally compact, then the coercivity condition will imply the Karamardian condition with the set \( \overline{B}(0; R_2) \cap K \) where \( R_2 = \max\{R_1, \|z_0\|\} \). In our last theorem we employ the Karamardian condition in a general Banach space. Note that we are able to drop the reflexivity requirement on the space \( X \) and that we only require the mapping \( F \) to be sequentially bounded. As an easy corollary to this theorem we can get a result, for the case when \( K \) is locally compact (or \( X \) is finite dimensional) and the operator \( F \) is coercive, which does not assume the space \( X \) is reflexive.

**Theorem 5.3.** Let \( X \) be a Banach space, \( K \) a convex, almost \( \sigma \)-compact subset of \( X \), and \( F: K \to 2^{X^*} \) such that the following hold:

(i) \( F \) is a pseudo-monotone mapping which is sequentially bounded and has weak* compact values.

(ii) \( F \) satisfies the Karamardian condition, i.e., there is compact subset \( C \) of \( K \) such that for every \( x \in K \setminus C \) there is a \( z \in C \) such that \( \langle y, x - z \rangle > 0 \) for all \( y \in F(x) \).

Then there is an \( x_0 \in K \) such that \( \inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0 \) for all \( u \in K \). If in addition \( F(x_0) \) is convex, then there is a \( y_0 \in F(x_0) \) such that \( \langle y_0, x_0 - u \rangle \leq 0 \) for all \( u \in K \).
Proof. By Lemma 5.1 we have that the graph of $B: K \to 2^K$ defined by

$$B(w) = \left\{ x \in K : \inf_{y \in F(x)} \langle y, x - w \rangle \leq 0 \right\}$$

has a closed graph in $K \times K$. Hence by Lemma 2.1, $B(w)$ is closed for each $w \in B$. Since $K$ is convex and almost $\sigma$-compact then there exists $(T_n)_{n=1}^\infty$ such that for each $n \in \mathbb{N}$, $T_n$ is compact and convex, $C \subset T_n$, the sets are increasing, i.e., $T_n \subset T_{n+1}$, and $K = \text{cl}(\bigcup_{n=1}^\infty T_n)$. For each $n \in \mathbb{N}$, let

$$B_n(u) = \left\{ x \in T_n : \inf_{y \in F(x)} \langle y, x - u \rangle \leq 0 \right\}.$$

Then $B_n(u) = B(u) \cap T_n$ which implies $B_n(u)$ is compact. Thus by Theorem 3.2 (or Theorem 3.1), there is an $x_n \in T_n$ such that

$$\inf_{y \in F(x_n)} \langle y, x_n - u \rangle \leq 0, \quad \forall u \in T_n, n \in \mathbb{N}.$$

By assumption (ii) we must have that $x_n \in C$ otherwise $x_n \in T_n \setminus C \subset K \setminus C$ which implies there is a $u \in C \subset T_n$ such that $\inf_{y \in F(x_n)} \langle y, x_n - u \rangle > 0$ which is a contradiction. Since $C$ is compact, $(x_n)_{n=1}^\infty$ must have a subsequence converging to a point, say $x_0 \in C$. Since the sets $T_n, n \in \mathbb{N}$, are increasing, we can remember the subsequence as $(x_n)_{n=1}^\infty$ and only use those $T_n$ associated with members of the subsequence. We claim that $x_0$ is our desired solution.

First we consider $u \in T_N$ for some $N \in \mathbb{N}$. Since the sets $T_n, n = 1, \ldots, \infty$, are increasing, then

$$\inf_{y \in F(x_n)} \langle y, x_n - u \rangle \leq 0, \quad \forall n \geq N.$$

Thus $x_n \in B(u)$ for all $n \geq N$. Since $B(u)$ is closed and $x_n \to x_0$, then $x_0 \in B(u)$ and thus

$$\inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0.$$

Next we consider $u \in \text{cl}(\bigcup_{n=1}^\infty T_n)$. Then there is $(u_m)_{m=1}^\infty \in \bigcup_{n=1}^\infty T_n$ such that $u_m \to u$, as $m \to \infty$. By the previous arguments, we have

$$\inf_{y \in F(x_0)} \langle y, x_0 - u_m \rangle \leq 0, \quad \forall m \in \mathbb{N}.$$

Now, since $F(x_0)$ is weak* compact, then for each $m \in \mathbb{N}$ there is a $y_m \in F(x_0)$ such that

$$\inf_{y \in F(x_0)} \langle y, x_0 - u_m \rangle = \langle y_m, x_0 - u_m \rangle.$$
Again since \( F(x_0) \) is weak* compact, then \( \{y_m\}^\infty_{m=1} \) has a weak* convergent subsequence, say \( \{y_m\}_{k=1}^\infty \to y_u \in F(x_0) \), as \( k \to \infty \). Thus we have

\[
0 \geq \inf_{y \in F(x_0)} \langle y, x_0 - u_m \rangle = \langle y_m, x_0 - u_m \rangle
\]

\[
\to \langle y_u, x_0 - u \rangle \geq \inf_{y \in F(x_0)} \langle y, x_0 - u \rangle, \quad \text{as } k \to \infty.
\]

Thus there is an \( x_0 \in K \) such that \( \inf_{y \in F(x_0)} \langle y, x_0 - u \rangle \leq 0 \) for all \( u \in K \).

Now if we also assume \( F(x_0) \) is convex then we can again use Kneser’s Minimax Inequality to show there exists an \( x_0 \in K \) and \( y_0 \in F(x_0) \) such that \( \langle y_0, x_0 - u \rangle \leq 0 \) for all \( u \in K \). \qed

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