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DESCENDING CHAINS OF SUBMODULES AND THE KRULL-DIMENSION OF NOETHERIAN MODULES

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0. Introduction

Throughout, R denotes an associative ring with identity, modules are unitary left R-modules. A descending chain C of non-zero submodules of a noetherian module M is well-ordered under reverse inclusion, and we denote by $o(M)$ the supremum of the ordinal types $\mathrm{ord}(C)$ of all such chains C in M.

In [1], Bass proved that if R is a commutative noetherian ring with countable Krull-ordinal $\kappa(R)$ (called classical Krull-dimension in [2, 4]), then $\omega^{\kappa(R)} \leqslant o(R)$ with equality in case R is a domain. The purpose of this note is to extend this result to noetherian modules over arbitrary (not necessarily commutative) rings and to sharpen it at the same time. Our proofs have been inspired by the methods used in [1]; we use, however, the Krull-dimension K-dim(M) of the module M (see below for a definition) instead of the Krull-ordinal $\kappa(R)$.

For any noetherian module M with Krull-dimension α there exists a sequence of ordinals

$$\alpha = \alpha(1) > \alpha(2) > \ldots > \alpha(k) \geqslant 0,$$

a sequence of natural numbers

$$(n(1), n(2), \ldots, n(k)),$$

and a finite descending chain of submodules

$$M = M_0 \supset M_1 \supset \ldots \supset M_{n-1} \supset M_n = 0$$

of submodules M_i such that the first $n(1)$ factor modules of this chain are $\alpha(1)$-critical, the next $n(2)$ factors are $\alpha(2)$-critical, and so forth, and the last $n(k)$ factors are $\alpha(k)$-critical. Here a module X is called α-critical if K-dim$(X) = \alpha$ but K-dim$(X/Y) < \alpha$ for every non-zero submodule Y of X. As a partial generalization of the classical Jordan–Hölder Theorem we show that the type $(\alpha(1), n(1), \ldots, \alpha(k), n(k))$ is an invariant of the module M (Theorem 2.6) and that for countable $\alpha(1) = $ K-dim(M) we get (Theorem 4.6)

$$o(M) = \omega^{\alpha(1)}n(1) + \ldots + \omega^{\alpha(k)}n(k)$$

Since K-dim(R) and $\kappa(R)$ coincide for a commutative noetherian ring (see [4, Theorem 13]), our result yields [1, Theorem 2.12] as a special case. We should like to point out that Theorem 4.6 is no longer true if K-dim(M) is uncountable: there exist even commutative noetherian domains of arbitrarily large Krull-dimension (see [2, Theorem 9.8]), but every descending chain of non-zero ideals of a commutative noetherian ring is countable by [1, Theorem 1.1].

Some of the results in Sections 2 and 3 have also been proved by Jategaonkar [3] for finitely generated modules over fully bounded noetherian rings. Our hypotheses, however, are much less restrictive, and the proofs are more translucent.

1. Definitions and notations

A submodule N of a module M is *essential* if $N \cap X \neq 0$ for all non-zero submodules X of M; M is *uniform* if all its non-zero submodules are essential. Two modules are *subisomorphic* if each has a monomorphism into the other one, and a module M is *compressible* if it is subisomorphic with each of its non-zero submodules. The *injective hull* of a module M is denoted by $E(M)$.

The *Krull-dimension* K-dim(M) of the module M is defined by transfinite recursion as follows: K-dim(M) = -1 iff $M = 0$, and for an ordinal α, K-dim(M) = α if K-dim(M) $\nless \alpha$ and there is no infinite descending chain $M = M_0 \supset M_1 \supset \ldots$ of submodules M_i such that K-dim(M_{i-1}/M_i) $\nless \alpha$ for $i = 1, 2, \ldots$. It is possible that there is no ordinal α with K-dim(M) = α; in that case M is said to have no Krull-dimension. We note that K-dim(M) = 0 iff M is a non-zero artinian module, and that K-dim($\mathbf{Z}$) = 1, $\mathbf{Z}$ the ring of integers.

A module M is *α-critical* if K-dim(M) = α and K-dim(M/N) $< \alpha$ for every submodule $N \neq 0$ of M. Non-zero submodules of α-critical modules are again α-critical, and it is easy to verify that compressible modules with Krull-dimension are critical, and that critical modules are uniform.

A ring R is *left (right) bounded* if every essential left (right) ideal of R contains a non-zero two-sided ideal. R is *fully left (right) bounded* if R/P is left (right) bounded for every two-sided prime ideal P of R. R is *fully bounded noetherian* if it is a fully left and right bounded ring with maximum condition on left and right ideals.

Finally, we recall that the classical *Krull-dimension* cl.K-dim(R) (*Krull-ordinal* $\kappa(R)$ in [1]) of the ring R is the smallest ordinal α for which spec(R) = $\mathrm{spec}_\alpha(R)$, where the subsets $\mathrm{spec}_\alpha(R)$ of the set spec(R) of all two-sided prime ideals of the ring R are defined as follows: $\mathrm{spec}_0(R)$ is the set of all maximal ideals of R, and for an ordinal $\alpha > 0$ we put

$$\mathrm{spec}_\alpha(R) = \{P \in \mathrm{spec}(R)\colon\ P \subset Q \text{ implies } Q \in \textstyle\bigcup_{\beta < \alpha} \mathrm{spec}_\beta(R)\}.$$

If no such ordinal exists, we say that R has no classical Krull-dimension.

2. Critical composition series

Let M be a left R-module. A finite properly descending chain

$$M = M_0 \supset M_1 \supset M_2 \supset \dots \supset M_{n-1} \supset M_n = 0$$

of submodules with critical factors M_{i-1}/M_i is called a *critical composition series*. If there is a sequence of ordinals

$$\alpha(1) > \alpha(2) > \dots > \alpha(k) \geqslant 0$$

such that the first $n(1)$ factors are $\alpha(1)$-critical, the next $n(2)$ factors are $\alpha(1)$-critical, the next $n(2)$ factors are $\alpha(2)$-critical, and so on, then we call the series a *decreasing composition series* of type

$$(\alpha(1), n(1), \alpha(2), n(2), \dots, \alpha(k), n(k)).$$

The objective of this section is to show that if a module M has two decreasing critical composition series of types $(\alpha(1), n(1), \dots, \alpha(k), n(k))$ and $(\beta(1), m(1), \dots, \beta(l), n(l))$, then $k = l$, and $\beta(i) = \alpha(i)$ and $n(i) = m(i)$ for all $i = 1, 2, \dots, k = l$. A critical composition series of type (α, n) will also be called an *α-critical composition series* for short.

It is fairly easy to show that every noetherian module has a decreasing critical composition series (of some type); in fact, noetherian modules can be characterized by the fact that every epimorphic image has such a series. We note that the Krull-dimension of a module with a decreasing critical composition series of type $(\alpha(1), n(1), \dots, \alpha(k), n(k))$ is equal to $\alpha(1)$ by [4, Lemma 7].

2.1. Lemma. *If the left R-module M has a decreasing critical composition series of type* $(\alpha(1), n(1), \dots, \alpha(k), n(k))$, *then* K-dim$(N) \geqslant \alpha(k)$ *for each submodule* $N \neq 0$ *of M.*

Proof. Let $M = M_0 \supset M_1 \supset \dots \supset M_{n-1} \supset M_n = 0$ be a decreasing critical composition series of the type mentioned. If $0 \neq N \subset M$, then $N \subseteq M_i$ but $N \not\subseteq M_{i+1}$ for some i, $0 \leqslant i \leqslant n-1$, and hence $M_{i+1} \subset N + M_{i+1} \subseteq M_i$, so that

$$\text{K-dim}(N/N \cap M_{i+1}) = \text{K-dim}(N + M_{i+1}/M_{i+1}) = \text{K-dim}(M_i/M_{i+1}) \geqslant \alpha(k).$$

Thus K-dim$(N) \geqslant \alpha(k)$ by [4, Lemma 7].

2.2. Lemma. *If for a submodule N of M the module M/N has a decreasing critical composition series of type* $(\dots, \alpha(k), n(k))$ *and if* K-dim$(N) < \alpha(k)$, *then N is the unique maximal submodule of M with* K-dim$(N) < \alpha(k)$.

Proof. Let X be any submodule of M with K-dim$(X) < \alpha(k)$. Then

$$\text{K-dim}(X + N/N) = \text{K-dim}(X/X \cap N) \leq \text{K-dim}(X) < \alpha(k),$$

so $X + N/N = 0$ by Lemma 2.1, and hence $X \subseteq X + N = N$.

2.3. Lemma. *If M has a decreasing critical composition series of type $(\alpha(1), n(1), \ldots, \alpha(k), n(k))$, then a non-zero submodule N of M has a decreasing critical composition series of type $(\beta(1), m(1), \ldots, \beta(l), m(l))$, where for each i, $1 \leq i \leq l$, there exists a j, $1 \leq j \leq k$, such that $\alpha(j) = \beta(i)$ and $m(i) \leq n(j)$.*

Proof. Let $M = M_0 \supset M_1 \supset \ldots \supset M_{n-1} \supset M_n = 0$ be a decreasing critical composition series for M of the indicated type, and assume $N \subset M_h$ but $N \not\subset M_{h+1}$ for some h, $0 \leq h \leq n-1$. Consider the series

$$(*) \qquad N = N \cap M_h \supset N \cap M_{h+1} \supseteq N \cap M_{h+2} \supseteq \ldots \supseteq N \cap M_{n-1} \supseteq N \cap M_n = 0.$$

For $i = 0, 1, \ldots, n-h-1$, we have

$$N \cap M_{h+i}/N \cap M_{h+i+1} = N \cap M_{h+i}/N \cap M_{h+i} \cap M_{h+i+1}$$

$$\simeq M_{h+i+1} + (N \cap M_{h+i})/M_{h+i+1} \subseteq M_{h+i}/M_{h+i+1}.$$

As M_{h+i}/M_{h+i+1} is $\alpha(j)$-critical for some j, $1 \leq j \leq k$, the series $(*)$ is of the desired type if we delete those factors which are equal to zero.

2.4. Remark. An analogue of Lemma 2.3 for epimorphic images of M instead of submodules is generally not true. In fact, if M is a non-noetherian α-critical module, then there exists at least one submodule $0 \neq N \subset M$ such that M/N has no critical composition series at all (see [2, Corollary 9.5]).

2.5. Proposition. *If M is a module with an α-critical composition series, then any two α-critical composition series have the same length.*

Proof. Let $l(M)$ be the least possible length of an α-critical composition series of M. We proceed by induction on $l(M)$. If $l(M) = 1$, then M is α-critical, so M cannot have an α-critical composition series of length > 1. Let $l = l(M) > 1$, and let

$$M = M_0 \supset M_1 \supset \ldots \supset M_{l-1} \supset M_l = 0,$$

$$M = N_0 \supset N_1 \supset \ldots \supset N_{m-1} \supset N_m = 0$$

be two α-critical composition series. Assume $m > l$. Since $l(M_1) = l-1$, every α-critical composition series of M_1 has length $l-1$. Assume $M_1 \cap N_{m-l} = 0$. Then

$$0 \neq N_{m-l} = N_{m-l}/M_1 \cap N_{m-l} \simeq M_1 + N_{m-l}/M_1,$$

so N_{m-l} is α-critical, contradicting the fact that N_{m-l} has an α-critical composition series of length $l > 1$. Thus $M_1 \cap N_{m-l} \neq 0$. Assume $M_1 = M_1 \cap N_{m-l}$. Then $M_1 \subseteq N_{m-l}$, and hence $M_1 = N_{m-l}$, for otherwise K-dim$(M/N_{m-l}) < \alpha$, contradicting the fact that M/N_{m-l} has an α-critical composition series of length $m - l \geqslant 1$. Thus $M_1 = N_{m-l}$ has an α-critical composition series of length l, and this contradiction shows that $M_1 \cap N_{m-l} \subset M_1$. Now

$$0 \neq M_1/M_1 \cap N_{m-l} \simeq M_1 + N_{m-l}/N_{m-l},$$

so $M_1/M_1 \cap N_{m-l}$ has an α-critical composition series of length $\geqslant 1$ by Lemma 2.3. Also by Lemma 2.3 the module $M_1 \cap N_{m-l} \neq 0$ has an α-critical composition series which must be of length $\leqslant l - 2$, because no α-critical composition series of M_1 has length $> l - 1$. Now $N_{m-l}/N_{m-l} \cap M_1 \simeq M_1 + N_{m-l}/M_1$, so $N_{m-l}/N_{m-l} \cap M_1$ is either zero or α-critical. In the first case this would yield an α-critical composition series in N_{m-l} of length $\leqslant l - 2$, in the second case it would give one of length $\leqslant (l-2) + 1 = l+1$, so $l(N_{m-l}) \leqslant l - 1$ in any case. But then N_{m-l} could not have an α-critical composition series of length l by induction hypothesis. This contradiction shows that $m = l$.

2.6. Theorem. *Let M be a left R-module with two decreasing critical composition series of types* $(\alpha(1), n(1), \ldots, \alpha(k), n(k))$ *and* $(\beta(1), m(1), \ldots, \beta(l), m(l))$, *respectively. Then $k = l$, and $\beta(i) = \alpha(i)$ and $m(i) = n(i)$ for $i = 1, \ldots, k = l$.*

Proof. Since $\alpha(1) > \alpha(2) > \ldots > \alpha(k)$ and $\beta(1) > \beta(2) > \ldots > \beta(l)$, we have $\beta(1) =$ K-dim$(M) = \alpha(1)$ by [4, Lemma 7]. We proceed by induction on $\alpha(1)$. If $\alpha(1) = 0$, then the assertion follows by the classical Jordan–Hölder Theorem because 0-critical modules are simple. Assume $\alpha(1) \geqslant 1$, and let K be the largest submodule in the decreasing critical composition series of type $(\alpha(1), n(1), \ldots, \alpha(k), n(k))$ for which K-dim$(K) = \alpha(2)$, and let L be the largest submodule in the other decreasing critical composition series for which K-dim$(L) = \beta(2)$. Then $K = L$ by Lemma 2.2 and K-dim$(K) = \alpha(2) = \beta(2) =$ K-dim$(L) < \beta(1) = \alpha(1)$. Thus $k = l$, and $\alpha(i) = \beta(i)$ and $n(i) = m(i)$ for $i = 2, 3, \ldots, k = l$ by induction hypothesis. Finally, $M/K = M/L$ has an $\alpha(1)$-critical composition series of length $n(1)$ and one of length $m(1)$, so $n(1) = m(1)$ by Proposition 2.5.

3. Basic series

Following [3], we call a non-zero submodule N of a module M with Krull-dimension *basic* if N is maximal among the α-critical submodules of M, where $\alpha = \min\{K\text{-dim}(X): 0 \neq X \subseteq M\}$. A basic series of M is a chain

$$0 = B_0 \subset B_1 \subset \ldots \subset B_{n-1} \subset B_n = M$$

of submodules of M where B_i/B_{i-1} is a basic submodule of M/B_{i-1} for $1 \leqslant i \leqslant n$. The purpose of this section is to show that basic series and decreasing critical composition series are the same. This will establish the validity of [3, Theorem 3.4(1)] for modules over any ring. The following two results appear in [3], where they are proved under the assumption that the ring R is fully bounded noetherian.

3.1. Lemma. *The following properties of the module M are equivalent:*
(i) *M is critical.*
(ii) *M is uniform and contains a critical submodule L such that*

$$\text{K-dim}(M/L) < \text{K-dim}(L).$$

Proof. (i) ⇒ (ii). If M is critical, then M is clearly uniform. If L is any non-zero submodule of M, then L is critical and

$$\text{K-dim}(M/L) < \text{K-dim}(M) = \text{K-dim}(L).$$

(ii) ⇒ (i). Since by [4, Lemma 7]

$$\text{K-dim}(M) = \max(\text{K-dim}(L), \text{K-dim}(M/L)),$$

we get $\text{K-dim}(M) = \text{K-dim}(L)$. Assume $\text{K-dim}(M/X) = \text{K-dim}(M)$ for some submodule $X \neq 0$. Since M is uniform, $L \cap X \neq 0$ and we have

$$\text{K-dim}(L + X/X) = \text{K-dim}(L/L \cap X) < \text{K-dim}(L).$$

Also

$$\text{K-dim}(M/L + X) \leqslant \text{K-dim}(M/L) < \text{K-dim}(L) = \text{K-dim}(M),$$

so we obtain

$$\text{K-dim}(M) = \text{K-dim}(M/X) = \max\{\text{K-dim}(M/L + X), \text{K-dim}(L + X/X)\} < \text{K-dim}(M),$$

and this contradiction shows that $\text{K-dim}(M/X) < \text{K-dim}(M)$ for all non-zero submodules X of M. Thus M is critical.

3.2. Lemma. *If the submodule L of the module M with Krull-dimension contains a basic submodule B of M properly, then* $\text{K-dim}(B) \leqslant \text{K-dim}(L/B)$.

Proof. Since (1) and (6) of [3, Theorem 2.5] are equivalent by Lemma 3.1 without the restriction that M is a finitely generated module over a fully bounded noetherian ring, the original proof of [3, Lemma 3.2] can be used.

3.3. Proposition. *The following properties of the series*

$$C\colon \qquad M = M_0 \supset M_1 \supset \ldots \supset M_i \supset M_{i+1} \supset \ldots \supset M_{n-1} \supset M_n = 0$$

of submodules M_i of the left R-module M with Krull-dimension are equivalent:
(i) *C is a basic series.*
(ii) *C is a decreasing critical composition series.*

Proof. (i) ⇒ (ii). If C is a basic series, then every factor M_i/M_{i+1}, $0 \leqslant i \leqslant n-1$, is critical, so C is a critical composition series. It remains to show that K-dim$(M_i/M_{i+1}) \geqslant$ K-dim(M_{i+1}/M_{i+2}). Since $L = M_i/M_{i+2}$ contains the basic submodule $B = M_{i+1}/M_{i+2}$ properly, we have

$$\text{K-dim}(M_{i+1}/M_{i+2}) = \text{K-dim}(B) \leqslant \text{K-dim}(L/B) = \text{K-dim}(M_i/M_{i+1})$$

by Lemma 3.2.

(ii) ⇒ (i). We have to show that M_i/M_{i+1} is a basic submodule of M/M_{i+1}. By Lemma 2.1 we have

$$\text{K-dim}(X/M_{i+1}) \geqslant \text{K-dim}(M_i/M_{i+1}) = \beta$$

for every submodule $X/M_{i+1} \neq 0$ of M/M_{i+1}, so the ordinal β is minimal among the Krull-dimensions of non-zero submodules of M/M_{i+1}. Assume now that there exists a submodule $X \supset M_i$ of M such that X/M_{i+1} is β-critical. Again by Lemma 2.1 we get

$$\beta \geqslant \text{K-dim}(X/M_i) \geqslant \text{K-dim}(M_{i-1}/M_i) \geqslant \text{K-dim}(M_i/M_{i+1}) = \beta,$$

so K-dim$(X/M_i) = \beta$, which is impossible if X/M_{i+1} is β-critical. Therefore M_i/M_{i+1} is a basic submodule of M/M_{i+1}.

In [3], Jategaonker called two basic series

$$M = M_0 \supset M_1 \supset \ldots \supset M_{n-1} \supset M_n = 0,$$

$$M = N_0 \supset N_1 \supset \ldots \supset N_{m-1} \supset N_m = 0$$

to be *equivalent* if $m = n$ and if for some permutation π of the set $(1, 2, \ldots, n)$ each M_{i-1}/M_i is subisomorphic with $N_{\pi(i)-1}/N_{\pi(i)}$. [3, Theorem 3.1] states that any two basic series of a finitely generated module M over a fully bounded noetherian ring are equivalent, and in view of Proposition 3.3 our Theorem 2.6 shows that they have at least the same length, with no restrictions on M and R whatsoever. Although we are presently unable to obtain the second part of [3, Theorem 3.1] in general, we obtain an alternate generalization of the Jordan–Hölder Theorem by showing that any two basic series of M are *similar* in the sense of Definition 3.4 below. It is clear that in the case of an artinian and noetherian module M both concepts reduce to the usual concept of equivalence of two composition series. Furthermore, as M_{i-1}/M_i is uniform, two equivalent basic series are clearly similar.

3.4. Definition. Two basic (or decreasing critical composition) series

$$M = M_0 \supset M_1 \supset \ldots \supset M_n = 0,$$

$$M = N_0 \supset N_1 \supset \ldots \supset N_m = 0$$

are *similar* if $m = n$ and if for some permutation π of the set $(1, 2, \ldots, n)$ the modules $E(M_{i-1}/M_i)$ and $E(N_{\pi(i)-1}/N_{\pi(i)})$ are isomorphic.

3.5. Theorem. *Any two basic series of a left R-module are similar.*

Proof. Let

$$M = M_0 \supset M_1 \supset \ldots \supset M_n = 0,$$

$$M = N_0 \supset N_1 \supset \ldots \supset N_m = 0$$

be two such series. By Theorem 2.6 they have the same length $m = n$. In view of Lemma 2.2 we may assume that both series are in fact α-critical composition series for some ordinal $\alpha \geqslant 0$. We proceed by induction on n. If $n = 1$, then M is α-critical, so there is nothing to show. Let $n = 2$ and assume $M_1 \cap N_1 \neq 0$. Then

$$\alpha > \text{K-dim}(M_1/M_1 \cap N_1) = \text{K-dim}(M_1 + N_1/N_1),$$

and hence $M_1 = M_1 \cap N_1$, since both M/N_1 and M_1 are α-critical. Similarly, $N_1 = M_1 \cap N_1$, and the two series are identical. If $M_1 \cap N_1 = 0$, then

$$N_1 = N_1/N_1 \cap M_1 \simeq M_1 + N_1/M_1,$$

and hence

$$E(N_1) \simeq E(M_1 + N_1/M_1) \simeq E(M/M_1)$$

since M/M_1 is uniform. Similarly

$$E(M_1) \simeq E(M/N_1).$$

Let now $n > 2$. If $M_1 \cap N_1 = 0$, we would have

$$M_1 = M_1/M_1 \cap N_1 \simeq M_1 + N_1/N_1,$$

so M_1 would be α-critical, contradicting the fact that it has a decreasing critical composition series of length $n - 1 \geqslant 2$. Thus $M_1 \cap N_1 \neq 0$. If $M_1 \cap N_1 = M_1$, then $M_1 \subseteq N_1$, and hence $M_1 = N_1$, because both M/M_1 and M/N_1 are α-critical, and we are done by induction hypothesis. Therefore, we may assume that $M_1 \cap N_1 \subset M_1$ and similarly $M_1 \cap N_1 \subset N_1$. Now

$$M_1/M_1 \cap N_1 \simeq M_1 + N_1/N_1,$$

so $M_1/M_1 \cap N_1$ is α-critical, and the same is true for $N_1/M_1 \cap N_1$. By Lemma 2.3 and Proposition 2.5 the submodule $M_1 \cap N_1$ possesses an α-critical composition series

$$M_1 \cap N_1 \supset X_1 \supset X_2 \supset \ldots \supset X_{n-3} \supset X_{n-2} = 0$$

of length $n - 2$. Using the induction hypothesis and the case $n = 2$, we get

$$M = M_0 \supset M_1 \supset M_2 \supset M_3 \supset \ldots \supset M_{n-1} \supset M_n = 0$$

similar to

$$M = M_0 \supset M_1 \supset M_1 \cap N_1 \supset X_1 \supset \ldots \supset X_{n-2} = 0$$

similar to

$$M = N_0 \supset N_1 \supset M_1 \cap N_1 \supset X_1 \supset \dots \supset X_{n-2} = 0$$

similar to

$$M = N_0 \supset N_1 \supset N_2 \supset N_3 \supset \dots \supset N_{n-1} \supset N_{n-1} \supset N_n = 0.$$

Since two compressible left R-modules with Krull-dimension and isomorphic injective hulls are obviously subisomorphic, we get the following corollary of Theorem 3.5.

3.6. Corollary. *Let C be a class of left R-modules which is closed with respect to the formation of submodules and epimorphic images. If every critical module in C with Krull-dimension is compressible, then any two basic series of a module M in C are equivalent.*

By [3, Theorem 2.5] every finitely generated critical module over a fully bounded noetherian ring R is compressible, so Corollary 3.6 is a generalization of [3, Theorem 3.1].

4. The ordinals $o(M)$ and K-dim(M)

Every descending chain of non-zero submodules of a noetherian module is well-ordered under reverse inclusion, and we denote by $o(M)$ the supremum of the order types of all such chains. The purpose of this section is to establish a relationship between the ordinals $o(M)$ and K-dim(M) for the case where the second one is countable.

4.1. Lemma. *Let M be a noetherian left R-module with* K-dim$(M) = \alpha \geq 0$. *For any ordinal $\beta < \alpha$ there exists a submodule N of M such that M/N is β-critical.*

Proof. The set of submodules X of M with K-dim$(M/X) \geq \beta$ is certainly not empty, so it contains a maximal element N. Since K-dim$(M/X) < \beta$ for all submodules $X \supset N$ of M, K-dim$(M/N) \leq \beta$ by [4, Proposition 8], so M/N is β-critical.

4.2. Proposition. *Let M be a noetherian α-critical left R-module. If α is countable, then there exists a descending chain of non-zero submodules of M of ordinal type ω^α and hence $\omega^\alpha \leq o(M)$.*

Proof. We proceed by induction on $\alpha =$ K-dim(M). If $\alpha = 0$, then M is simple, and the assertion is trivially true. Let $\alpha > 0$.

Case 1: $\alpha = \beta + 1$. Let N_0 be equal to M, and for $i \geq 1$ let N_i be a non-zero submodule of N_{i-1} such that N_{i-1}/N_i is β-critical. Since the non-zero submodule N_{i-1} of the α-critical module M is again α-critical, such a submodule N_i does exist by Lemma 4.1. Because of [4, Lemma 7], we get an infinite descending chain of submodules

$$C: \qquad M = N_0 \supset N_1 \supset \dots \supset N_{i-1} \supset N_i \supset \dots \supset 0$$

with β-critical factors N_{i-1}/N_i. By induction hypothesis there is a descending chain of submodules $\supset N_i$ between N_{i-1} and N_i which is of ordinal type ω^β. Using these chains to refine the chain C, we obtain a chain of type

$$\omega^\beta + \omega^\beta + \ldots = \omega^\beta \cdot \omega = \omega^{\beta+1} = \omega^\alpha.$$

Case 2: α is a limit ordinal. Since α is countable, we can find a countable sequence of ordinals $\beta(i) < \alpha$ such that $\alpha = \sup_i(\beta(i))$. Define $M = N_0$, and for $i \geqslant 1$ let N_i be a non-zero submodule of N_{i-1} such that N_{i-1}/N_i is $\beta(i)$-critical. By [4, Lemma 7] we obtain an infinite descending chain

$$C: \qquad M = N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_{i-1} \supset N_i \supset \ldots \supset 0$$

with $\beta(i)$-critical factors N_{i-1}/N_i for $i = 1, 2, \ldots$. By the inductive hypothesis there exists a chain of submodules $\supset N_i$ between N_{i-1} and N_i which is of order type $\omega^{\beta(i)}$. Using these chains to refine the chain C yields a chain of type

$$\omega^{\beta(1)} + \omega^{\beta(2)} + \ldots + \omega^{\beta(i)} + \ldots = \omega^{\sup(\beta(i))} = \omega^\alpha.$$

4.3. Remark. If K-dim$(M) = \alpha$ is not countable, Proposition 4.2 fails to be true in general. By [2, Theorem 9.8] there exists, for example, a commutative noetherian integral domain R with K-dim$(R) = \Omega + 1$, where Ω is the first uncountable ordinal. By [4, Theorem 10] R is $(\Omega + 1)$-critical, and it follows from [1, Remark 2.13] that $o(R) = \Omega < \omega^{\Omega+1}$.

4.4. Corollary. *If M is a noetherian left R-module with a basic series of type $(\alpha(1), n(1), \ldots, \alpha(k), n(k))$, $\alpha(1) < \Omega$, then there exists a descending chain of non-zero submodules of M of order type $\omega^{\alpha(1)}n(1) + \ldots + \omega^{\alpha(k)}n(k)$.*

Proof. This is an immediate consequence of Theorem 2.6 and Proposition 4.2.

4.5. Proposition. *Let M be a left R-module with a basic series of type $(\alpha(1), n(1), \ldots, \alpha(k), n(k))$, and let $N \neq 0$ be a submodule such that M/N has a basic series of type $(\beta(1), m(1), \ldots, \beta(l), m(l))$. Then*

$$\omega^{\beta(1)}m(1) + \ldots + \omega^{\beta(l)}m(l) < \omega^{\alpha(1)}n(1) + \ldots + \omega^{\alpha(k)}n(k).$$

Proof. We proceed by induction on K-dim$(M) = \alpha(1)$, the case $\alpha(1) = 0$ being true by the classical Jordan–Hölder Theorem. Let $\alpha(1) > 0$ and assume K-dim$(M/N) = \beta(1) < \alpha(1)$. Then

$$\omega^{\beta(1)}m(1) + \ldots + \omega^{\beta(l)}m(l) \leqslant \omega^{\beta(1)}(m(1) + m(2) + \ldots + m(l))$$

$$< \omega^{\beta(1)}\omega = \omega^{\beta(1)+1} \leqslant \omega^{\alpha(1)},$$

and we are done. Let now K-dim$(M/N) = \alpha(1) = \beta(1)$. If $m(1) > n(1)$, then M would have a critical composition series of type $(\alpha(1), r, \ldots)$ with $r \geqslant m(1) > n(1)$ by Lemma 2.3, but this is impossible in view of Theorem 2.6. Thus $m(1) \leqslant n(1)$. If $m(1) < n(1)$, then

$$\begin{aligned}\omega^{\beta(1)} m(1) + \ldots + \omega^{\beta(l)} m(l) &= \omega^{\alpha(1)} m(1) + \omega^{\beta(2)} m(2) + \ldots + \omega^{\beta(l)} m(l)\\ &\leqslant \omega^{\alpha(1)} m(1) + \omega^{\beta(2)} (m(2) + \ldots + m(l))\\ &< \omega^{\alpha(1)} m(1) + \omega^{\beta(2)} \cdot \omega = \omega^{\alpha(1)} m(1) + \omega^{\beta(2)+1}\\ &\leqslant \omega^{\alpha(1)} m(1) + \omega^{\alpha(1)} = \omega^{\alpha(1)} (m(1) + 1)\\ &\leqslant \omega^{\alpha(1)} n(1) \leqslant \omega^{\alpha(1)} n(1) + \ldots + \omega^{\alpha(k)} n(k),\end{aligned}$$

and we are done. If $m(1) = n(1)$, let

$$M = M_0 \supset M_1 \supset \ldots \supset M_{n(1)} \supset \ldots \supset N \neq 0$$

be a decreasing critical composition series between M and N of type $(\alpha(1), n(1), \beta(2), m(2), \ldots, \beta(l), m(l))$. By Lemma 2.3, the submodule $M_{n(1)}$ has a decreasing critical composition series, and by Theorem 2.6 it must be of type $(\alpha(2), n(2), \ldots, \alpha(k), n(l))$. Since $M_{n(1)}/N$ has a decreasing critical composition series of type $(\beta(2), m(2), \ldots, \beta(l), m(l))$ and since K-dim$(M_{n(1)}) = \alpha(2) < \alpha(1)$, we get

$$\omega^{\beta(2)} m(2) + \ldots + \omega^{\beta(l)} m(l) < \omega^{\alpha(2)} n(2) + \ldots + \omega^{\alpha(k)} n(k)$$

by the inductive hypothesis, and the assertion follows since $\omega^{\alpha(1)} n(1) = \omega^{\beta(1)} m(1)$ in this case.

4.6. Theorem. *Let $M \neq 0$ be a noetherian left R-module with a decreasing critical composition series of type $(\alpha(1), n(1), \ldots, \alpha(k), n(k))$ with countable* K-dim$(M) = \alpha(1)$. *Then*

$$o(M) = \omega^{\alpha(1)} n(1) + \ldots + \omega^{\alpha(k)} n(k).$$

Proof. By Corollary 4.4 we know that

$$\lambda = \omega^{\alpha(1)} n(1) + \ldots + \omega^{\alpha(k)} n(k) \leqslant o(M),$$

so we only have to show that any descending chain of non-zero submodules of M is of order type $\leqslant \lambda$. We proceed by induction on λ. If $\lambda = 1$, then $k = 1$, $\alpha(1) = 0$, and $n(1) = 1$, so M is simple and the statement is trivial. Let $\lambda > 1$, and assume that there exists a descending chain

$$C: \qquad M \supset \ldots \supset L_i \supset \ldots \supset L \neq 0$$

of non-zero submodules of M of ordinal type $\lambda + 1$. Since M/L is noetherian, it has a decreasing critical composition series of type $(\beta(1), m(1), \ldots, \beta(l), m(l))$, say. By Proposition 4.5 we have

$$\omega^{\beta(1)} m(1) + \ldots + \omega^{\beta(l)} m(l) = \kappa < \lambda.$$

But since M/L contains a descending chain of non-zero submodules of ordinal type λ we get $\lambda \leqslant o(M/L) = \kappa$ by the inductive hypothesis. This contradiction shows that no descending chain of non-zero submodules of M can have order type $> \lambda$, whence $o(M) = \lambda$.

Since a left noetherian ring R has no zero-divisors iff ${}_RR$ is a critical module (see [4, Theorem 10]), and since K-dim $({}_RR)$ = cl.K-dim(R) for any fully left bounded left noetherian ring R (see [5, Theorem 2.4]), Theorem 4.6 generalizes and extends the first half of [1, Theorem 2.12]. The second half of that result can be obtained as well: let $O({}_RR)$ denote the supremum of the ordinals $o(M)$, where M varies over all finitely generated left R-modules. We have:

4.7. Corollary. *Let R be a left noetherian ring with countable left Krull-dimension* K-dim$({}_RR) = \alpha(1)$. *Then*

$$O({}_RR) = o({}_RR)\cdot\omega = \omega^{\alpha(1)+1}.$$

Proof. By [1, Lemma 2.11] we have $O({}_RR) \leqslant o({}_RR)\cdot\omega$. Let ${}_RR$ be of type $(\alpha(1), n(1), \ldots, \alpha(k), n(k))$. Then R contains a descending chain of non-zero left ideals of order type $\omega^{\alpha(1)}$ by Theorem 4.6, and the free left R-module R^m contains a chain of type $\omega^{\alpha(1)}m$. Therefore

$$\begin{aligned}\omega^{\alpha(1)+1} &\leqslant O({}_RR) \leqslant o({}_RR)\cdot\omega = (\omega^{\alpha(1)}n(1) + \ldots + \omega^{\alpha(k)}n(k))\cdot\omega \\ &\leqslant (\omega^{\alpha(1)}(n(1) + \ldots + n(k)))\cdot\omega \\ &= \omega^{\alpha(1)}((n(1) + \ldots + n(k))\cdot\omega) = \omega^{\alpha(1)}\cdot\omega = \omega^{\alpha(1)+1}.\end{aligned}$$

4.8. Corollary. *Let R be a fully bounded noetherian ring without zero divisors. If* cl.K-dim$(R) = \alpha$ *is countable, then*

$$o({}_RR) = o(R_R) = \omega^{\alpha}.$$

Proof. By [5, Theorem 2.4] we have

$$\text{K-dim}({}_RR) = \text{cl.K-dim}(R) = \text{K-dim}(R_R),$$

and by [4, Theorem 10] both ${}_RR$ and R_R are α-critical. Thus the result follows from Theorem 4.6.

4.9. Remark. If R has zero divisors then Corollary 4.8 fails to be true in general. There exist even left and right artinian rings for which the length of a composition series of left ideals is different from the length of a composition series of right ideals.

Added in proof. Using a different approach, some of the results in Section 4 have also been obtained by Gulliksen [2a].

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