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Periodic Coxeter matrices

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Abstract

Let $A = kQ/I$ be a finite dimensional triangular k-algebra. Consider the *Cartan matrix* C_A and the *Coxeter matrix* $\varphi_A = -C_A^{-t} C_A$. Let $\chi_\varphi(T) = \det(T\,\mathrm{id} - \varphi_A)$ be the *Coxeter polynomial* of A. We study conditions on $\operatorname{Spec}\varphi_A$ in order that φ_A is a periodic matrix. We show that in case φ_A is periodic then the *Euler quadratic form* $q_A(x) = xC_A^{-t}x^t$ is non-negative and $q_A > 0$ if and only if $1 \notin \operatorname{Spec}\varphi_A$.

Keywords: Coxeter matrix; Euler quadratic form; Periodic matrix

0. Introduction

Coxeter matrices of finite dimensional algebras play an important role in several topics, such as Lie theory and the representation theory of associative algebras (see for example [2–5,7] for fundamental concepts and [9,11] for revisions of the use of Coxeter matrices in representation theory).

Let A be a finite dimensional associative algebra over an algebraically closed field k. We shall assume that A is basic and triangular, that is, $A = kQ/I$ for a quiver (=finite oriented graph) without oriented cycles and an admissible ideal I of the path algebra kQ (see [2]). Let $Q_0 = \{1, \dots, n\}$ be the set of vertices of Q. Then the *Cartan matrix* C_A is the $n \times n$ matrix whose (i, j)-entry is $\dim_k A(i, j)$. This matrix is invertible and defines a bilinear form $\langle x, y\rangle_A = xC_A^{-t}y^t$ with the property that

$$\langle [X], [Y]\rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}_A^i(X, Y)$$

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for the classes $[X]$, $[Y]$ in the Grothendieck group $K_0(A)$ of finite dimensional A-modules X, Y. The *Coxeter matrix* is the $n \times n$ matrix $\varphi_A = -C_A^{-t} C_A$.

In this note, we give conditions on the eigenvalues of φ_A which imply the periodicity of φ_A. In particular, we show that if φ_A is periodic, then the *Euler form* $q_A(x) = \langle x, x\rangle_A$ is non-negative. The proofs use elementary linear algebra arguments.

Section 2 of the work provides examples of algebras with periodic Coxeter transformation. Some of these examples are well-known, some were recently obtained in the study of supercanonical algebras [10]. In Section 3 we shall consider the relation of the Coxeter matrices φ_A and φ_B for a one-point extension $A = B[M]$ of a k-algebra B by a B-module M.

1. Periodicity of Coxeter matrices

1.1. Let $A = kQ/I$ be a finite dimensional triangular k-algebra. Consider the Cartan matrix C_A and the Coxeter matrix $\varphi_A = -C_A^{-t} C_A$. Let $\chi_\varphi(T) = \det(T\mathrm{id} - \varphi_A)$ be the *Coxeter polynomial* of A. The roots of $\chi_\varphi(T)$ form the set $\mathrm{Spec}\,\varphi_A$ of eigenvalues of φ_A.

Theorem. *With the above notation, the following are equivalent*:

(a) φ_A *is periodic*.
(b) $\mathrm{Spec}\,\varphi_A \subset \mathbb{S}^1$ *and* φ_A *is diagonalizable*.
(c) $\chi_\varphi(T)$ *is the product of cyclotomic polynomials and* φ_A *is diagonalizable*.

Proof. (a) ⇒(c): Assume $\varphi_A^p = I_n$. Consider the Jordan form $\bigoplus J_{n_i}(\lambda_i)$ of φ_A, then $J_{n_i}(\lambda_i)^p = I_{n_i}$. This implies that $n_i = 1$ and $\lambda_i^p = 1$.

(c) ⇒(b): is clear.

(b) ⇒(a): The polynomial $\chi_\varphi(T)$ is monic with integral coefficients. By a classical result of Kronecker (see [8] or a recent proof in [5]), if the roots of $\chi_\varphi(T)$ are in $\mathbb{S}^1$, then they are roots of unity. We may choose $p \in \mathbb{N}$ with $\lambda^p = 1$ for every $\lambda \in \mathrm{Spec}\,\varphi_A$. Since φ_A is diagonalizable, the $\varphi_A^p = I_n$. □

1.2. We recall that $q_A(x) = xC_A^{-t}x^t$ is the Euler (quadratic) form.

Proposition. *Assume* φ_A *is periodic, then the following hold*:

(a) $q_A \geqslant 0$;
(b) $1 \notin \mathrm{Spec}\,\varphi_A$ *if and only if* $q_A > 0$.

Proof. Consider the symmetrization $C_A^{-1} + C_A^{-t}$ of the matrix associated to q_A. We shall prove that the eigenvalues of $C_A^{-1} + C_A^{-t}$ are non-negative. Suppose that $x(C_A^{-1} + C_A^{-t}) = \lambda x$ with $\lambda < 0$. Then we get for $\mu = -\lambda > 0$,

$$x\varphi_A = x(I_n + \mu C_A), \quad x\varphi_A^2 = x(I_n + \mu C_A)^2, \ldots$$
$$x = x\varphi_A^p = x(I_n + \mu C_A)^p = x\left[I_n + p\mu C_A + \binom{p}{2}\mu^2 C_A^2 + \cdots\right],$$

where p is the period of φ_A. Then

$$0 = x\left[p\mu C_A + \binom{p}{2}\mu^2 C_A^2 + \cdots\right].$$

But the matrix in the parenthesis is triangular with positive diagonal entries $p\mu + \binom{p}{2}\mu^2 + \cdots$. Hence $x = 0$. Therefore $q_A \geqslant 0$.

Clearly $1 \notin \operatorname{Spec}\varphi_A$ if and only if $0 \notin \operatorname{Spec}(C_A^{-1} + C_A^{-t})$ and in that case $q_A > 0$. $\square$

1.3. The following complement to (1.2) follows a statement in [6].

Proposition. *Assume φ_A is periodic and $1 \in \operatorname{Spec}\varphi_A$. Then 1 is a multiple root of $\chi_\varphi(T)$. Equivalently* $\operatorname{corank} q_A \geqslant 2$.

Proof. Observe that $\chi_\varphi(T)$ is a reciprocal polynomial, that is, $T^n\chi_\varphi(T^{-1}) = \chi_\varphi(T)$. Indeed,

$$\begin{aligned} T^n \det(T^{-1}I_n + C_A^{-t}C_A) &= \det(C_A^t)\det(I_n + TC_A^{-t}C_A)\det(C_A^{-1}) \\ &= \det(TI_n - \varphi_A^t) = \chi_\varphi(T). \end{aligned}$$

Hence we may write $\chi_\varphi(T) = (1 + T^n) + a_1(T + T^{n-1}) + a_2(T^2 + T^{n-2}) + \cdots$ for certain integral numbers $a_1, a_2, \ldots$ Since $(T-1)^2$ divides $(T^i - 1)(T^{n-i} - 1)$, then $\chi_\varphi(T)$ and $(1 + a_1 + a_2 + \cdots)(1 + T^n)$ are congruent modulo $(T-1)^2$.

By hypothesis, $T - 1$ divides $\chi_\varphi(T)$, therefore $T - 1$ divides $(1 + a_1 + a_2 + \cdots)(1 + T^n)$. But $T - 1$ is not a divisor of $1 + T^n$, which implies that $1 + a_1 + a_2 + \cdots = 0$. Therefore $\chi_\varphi(T) \equiv 0 \bmod (T-1)^2$. $\square$

1.4. Consider the Coxeter matrix φ_A as a linear transformation

$$\varphi_A : V = K_0(A) \bigotimes_{\mathbb{Z}} \mathbb{Q} \to V.$$

Consider the spectral decomposition $V = \bigoplus_{\lambda \in \operatorname{Spec}\varphi_A} V_{\varphi_A}(\lambda)$. We get also the following characterization.

Proposition. *The matrix φ_A satisfies $\varphi_A^p = I_n$ if and only if the following hold*:

(i) $q_A \geqslant 0$;
(ii) $\operatorname{rad} q_A = V_{\varphi_A}(1)$;
(iii) $(\sum_{i=0}^{p-1}\varphi_A^i)(V_{\varphi_A}(\mu)) = 0$ *for* $1 \neq \mu \in \operatorname{Spec}\varphi_A$.

Proof. Suppose $\varphi_A^p = I_n$. By (1.2), $q_A \geqslant 0$. Always $\operatorname{rad} q_A \subset V_{\varphi_A}(1)$. Let $x \in V_\varphi(1)$, by (1.1), $x\varphi_A = x$ and therefore $q_A(x) = 0$.

Finally, if $1 \neq \mu \in \operatorname{Spec} \varphi_A$ and $x \in V_\varphi(\mu)$, then

$$(1-\mu)x\left(\sum_{i=0}^{p-1}\varphi_A^i\right) = x(I_n - \varphi_A)\left(\sum_{i=0}^{p-1}\varphi_A^i\right) = x(I_n - \varphi_A^p) = 0.$$

Suppose (i)–(iii) hold. By (iii),

$$\operatorname{Im}\left(\sum_{i=0}^{p-1}\varphi_A^i\right) \subset V_\varphi(1).$$

If $x \in V_{\varphi_A}(1)$, then by (ii), $x\varphi_A = x$. Therefore,

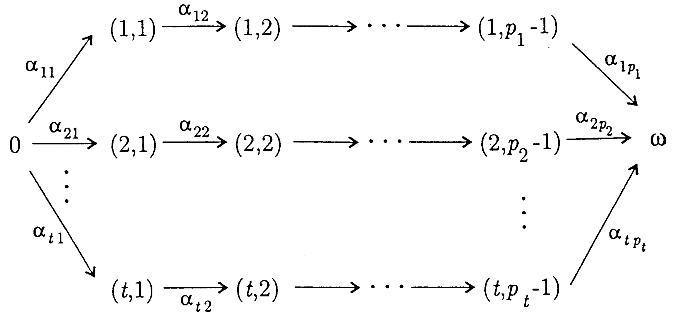
$$I_n - \varphi_A^p = \left(\sum_{i=a}^{p-1}\varphi_A^i\right)(I_n - \varphi_A) = 0. \qquad \square$$

2. Examples

2.1. Let A be an algebra tilted of Dynkin type Δ (see [12] for definitions). Then $q_A > 0$ and in particular $\operatorname{rad} q_A = \{0\}$. The Coxeter matrix ϕ_A is periodic of period $p(\Delta)$. Moreover, if $\chi_A(T)$ is the characteristic polynomial we get the following table (where $Q_n(T)$ denotes the nth cyclotomic polynomial in $\mathbb{Z}[T]$). See [1,11,12].

Δ	Factorization of $\chi_A(T)$	Period of φ_A
$\mathbb{A}_n$	$\prod_{2\leqslant m\mid n+1} Q_m(T)$	$n+1$
$\mathbb{D}_n$	$Q_2(T)\prod_{n\leqslant m\mid 2n} Q_m(T)$	$2(n-1)$
$\mathbb{E}_6$	$Q_3(T)Q_{12}(T)$	12
$\mathbb{E}_7$	$Q_2(T)Q_{18}(T)$	18
$\mathbb{E}_8$	$Q_2(T)Q_{10}(T)Q_{30}(T)$	30

2.2. Let A be a *canonical* algebra, that is, $A = kQ/I$, where Q is given as



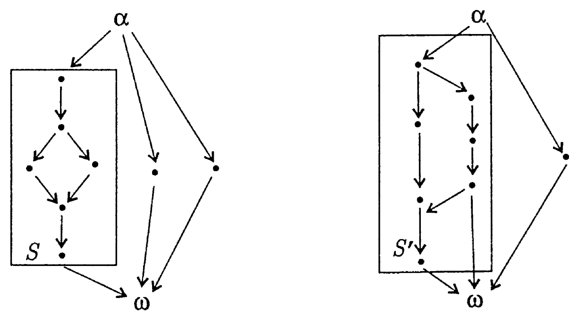
with $t \geqslant 2$, $p_1, \ldots, p_t \geqslant 2$ and I is generated by $\alpha_{i1} \cdots \alpha_{ip_i} - \alpha_{21} \cdots \alpha_{2p_2} + \lambda_i \alpha_{11} \cdots \alpha_{1p_1}$ for pairwise different $\lambda_3, \ldots, \lambda_t \in k$. The algebra A is *canonical tubular* if $t = 4$ and $p_i = 2$ $(1 \leqslant i \leqslant 4)$ or if $t = 3$ and $\sum_{i=1}^{3} 1/p_i = 1$. In this case φ_A is periodic of period $\operatorname{lcm}\{p_1, \ldots, p_t\}$. See [9].

2.3. In [10] the concept of *supercanonical algebras* was recently introduced. Let $S_1, \ldots, S_t$ be a finite family of posets $t \geqslant 2$. Define $A = A(S_1, \ldots, S_t;\ \lambda_3, \ldots, \lambda_t)$ for pairwise different $\lambda_3, \ldots, \lambda_t \in k$, as follows: $A = kQ/I$, where Q consists of the disjoint union of the vertices of S_i $(1 \leqslant i \leqslant t)$ and additionally, a minimal element α and a maximal element ω. The relations in I are those in the posets S_i $(1 \leqslant i \leqslant t)$ plus the $t - 2$ relations

$$\kappa_i - \kappa_2 + \lambda_i \kappa_1, \quad 3 \leqslant i \leqslant t,$$

where κ_i denotes any non-zero path in A from α to ω passing through S_i.

Consider the supercanonical algebras:



$A = A(S,(1),(1);1)$ $\quad$ $A' = A(S',(1))$

The algebra A is tame with $q_A \geqslant 0$ of corank $q_A = 2$. Moreover, φ_A is periodic of period 10. The algebra A' is wild but $q_{A'} \geqslant 0$ of corank $q_{A'} = 2$; moreover, $\varphi_{A'}$ is periodic of period 18.

2.4. A supercanonical algebra $A = A(S_1, \ldots, S_t;\ \lambda_3, \ldots, \lambda_t)$ is called of *Dynkin class* if for each $1 \leqslant i \leqslant t$, the incidence algebra $k[S_i]$ is tilted of Dynkin type. For these algebras it is shown in [10] that φ_A is periodic if and only if $q_A \geqslant 0$ with corank $q_A = 2$.

3. One-point extensions

3.1. Let B be a finite dimensional k-algebra and M be a finite dimensional B-module. The one-point extension $A = B[M]$ of B by M is the algebra

$$\begin{pmatrix} B & M \\ 0 & k \end{pmatrix}$$

with the usual matrix operations. The following is shown in [12]:

$$C_A = \left[\begin{array}{c|c} C_B & v \\ \hline 0 & 1 \end{array}\right] \quad \text{for } v = [M] \in K_0(B),$$

$$\varphi_A = \left[\begin{array}{c|c} \varphi_B & -C_B^{-t} v^t \\ \hline -v\varphi_B & q_B(v) - 1 \end{array}\right].$$

We start with a remark essentially shown in [4].

Lemma. *Let $A = B[M]$ and $v = [M] \in K_0(B)$. Then $q_A \geqslant 0$ if and only if the following hold*:

(i) $q_B \geqslant 0$;
(ii) $\langle v, \operatorname{rad} q_B \rangle_B = 0$;
(iii) *there exists a vector $y \in K_0(B)$ with $y(\varphi_B - 1) = v$ and $q_B(y) = 1$.*

Proof. Clearly $q_A\binom{y}{a} = q_B(y) - a\langle v, y\rangle_B + a^2$. Hence $q_A \geqslant 0$ of and only if $q_B(y) - \frac{1}{4}\langle v, y\rangle_B^2 \geqslant 0$ for every $y \in K_0(B)$. The minimum of the last inequality is reached in a vector $y_0 \in K_0(B)$ satisfying $y_0(C_B^{-t} + C_B^{-1}) = vC_B^{-t}$. Hence $q_B(y_0) \geqslant 1$ if $q_A \geqslant 0$. Observe that for $y = y_0\varphi_B^{-1}$ we get $y(\varphi_B - 1) = v$. If we had $q_B(y) = q_B(y_0) = 0$, then $y\varphi_B = y$ and $v = 0$, a contradiction. □

3.2. Proposition. *Let $A = B[M]$ and $v = [M] \in K_0(B)$. Assume that φ_A is periodic with period p. Then the following hold*:

(a) For every $x \in K_0(B)$, $x\varphi_B^p - x \in \sum_{i=1}^{p-1} \mathbb{Z}\, v\varphi_B^i$;
(b) $\#\{\lambda \in \operatorname{Spec}\varphi_B : \lambda \notin \mathbb{S}^1\} \leqslant \dim_{\mathbb{Q}} \sum_{i=1}^{p-1} \mathbb{Q}\, v\varphi_B^i$.

Proof. (a) Let $x \in K_0(B)$. Set $x_0 := x$ and $a_0 := 0$. Then

$$(x_0, a_0)\varphi_A = (x_0\varphi_B - a_0 v\varphi_B, -\langle x_0, v\rangle_B + a_0(q_B(v) - 1)).$$

Set $x_1 := x_0\varphi_B - a_0 v\varphi_B$ and $a_1 := -\langle x_0, v\rangle_B + a_0(q_B(v) - 1)$ and in general

$$(x_i, a_i)\varphi_A = (x_{i+1}, a_{i+1}),$$

where $x_{i+1} = x_i\varphi_B - a_i v\varphi_B$.

If $(x, 0)\varphi_A^p = (x, 0)$, then $x = x\varphi_B^p + \sum_{i=1}^{p-1} b_i v\varphi_B^i$ for certain $b_i \in \mathbb{Z}$ (observe that $b_p = a_0 = 0$).

(b) If $x\varphi_B = \mu x$ for some $\mu \notin \mathbb{S}^1$, then

$$(x, 0) = (x, a)\varphi_A^p = \left(\mu^p x - \sum_{i=1}^{p-1} b_i v\varphi_B^i, a_p\right).$$

Hence

$$x = \frac{1}{\mu^p - 1} \sum_{i=1}^{p-1} b_i v \varphi_B^i. \qquad \square$$

3.3. Corollary. *Let $A = B[M]$ with $v = [M] \in K_0(B)$. Assume that φ_A is periodic and $q_A(v) = 0$. Then the following happens*:

(i) Spec $\varphi_B \subset \mathbb{S}^1$;
(ii) φ_B *is not periodic*.

Proof. (i) Suppose $x\varphi_B = \mu x$ for some $\mu \notin \mathbb{S}^1$. By Section 3.2, $x = av$ for some $a \in \mathbb{Z}$ and $\mu x = x\varphi_B = av = x$, a contradiction.

(ii) By Section 3.1, there is a vector $y \in K_0(B)$ with $y\varphi_B - y = v$. Since $v\varphi_B = v$, then $y\varphi_B^p = y + pv$ for any $p \geqslant 1$. If $\varphi_B^p = 1$, then $pv = 0$, a contradiction. $\square$

3.4. In some cases, we may give conditions for a one-point extension $A = B[M]$ to get φ_A periodic.

Proposition. *Let $A = B[M]$ with $v = [M] \in K_0(B)$ be such that $q_B(v) = 0$. Then φ_A is periodic* (*of period p*) *if and only if p is even and*

$$x\varphi_B^p - x = -\frac{p}{2}\langle x, v\rangle_B \quad \textit{for every } x \in K_0(B).$$

Proof. As in Section 3.2 we get for $x \in K_0(B)$,

$$(x, 0)\varphi_A^i = \begin{cases} (x\varphi_B^i + \frac{i}{2}\langle x, v\rangle_B v, 0) & \text{if } i \text{ is even,} \\ (x\varphi_B^i + \frac{i-1}{2}\langle x, v\rangle_B v, -\langle x, v\rangle_B) & \text{if } i \text{ odd} \end{cases}$$

If $\varphi_A^p = 1$, we get p even (since otherwise $\langle x, v\rangle_B = 0$ for all $x \in K_0(B)$ which implies that $v = 0$). Conversely, if p is even and $x\varphi_B^p - x = -(p/2)\langle x, v\rangle_B v$ holds, then $(x, 0)\varphi_A^p = (x, 0)$ for every $x \in K_0(B)$. Moreover,

$$(v, 1)\varphi_A = (0, -1)$$

and

$$(0, -1)\varphi_A = (v, 1), \quad \text{which yields } \varphi_A^p = 1. \qquad \square$$

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