On the spectra of certain rooted trees

Oscar Rojo * ,1

Departamento de Matemáticas, Universidad Católica del Norte, Casilla 1280, Antofagasta, Chile

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Abstract

Let \( T \) be an unweighted tree with vertex root \( v \) which is the union of two trees \( T_1 = (V_1, E_1) \), \( T_2 = (V_2, E_2) \) such that \( V_1 \cap V_2 = \{v\} \) and \( T_1 \) and \( T_2 \) have the property that the vertices in each of their levels have equal degree. We characterize completely the eigenvalues of the adjacency matrix and of the Laplacian matrix of \( T \). They are the eigenvalues of symmetric tridiagonal matrices whose entries are given in terms of the vertex degrees. Moreover, we give some results about the multiplicity of the eigenvalues. Applications to some particular trees are developed.

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1. Introduction

Let \( G \) be a simple undirected graph on \( n \) vertices. The Laplacian matrix of \( G \) is the \( n \times n \) matrix \( L(G) = D(G) - A(G) \) where \( A(G) \) is the adjacency and \( D(G) \) is the diagonal matrix of vertex degrees. It is well known that \( L(G) \) is a positive semidefinite
matrix and that \((0, e)\) is an eigenpair of \(L(\mathcal{G})\) where \(e\) is the all ones vector. In [4], some of the many results known for Laplacian matrices are given. Fiedler [2] proved that \(\mathcal{G}\) is a connected graph if and only if the second smallest eigenvalue of \(L(\mathcal{G})\) is positive. This eigenvalue is called the algebraic connectivity of \(\mathcal{G}\).

We recall that a tree is a connected acyclic graph. Let \(\mathcal{T}\) be an unweighted rooted tree \(\mathcal{T}\) of \(k\) levels such that in each level the vertices have equal degree. Let \(d_{k-j+1}\) and \(n_{k-j+1}\) be the degree of the vertices and the number of them in the level \(j\), respectively. Let

\[
P_0(\lambda) = 1, \quad P_1(\lambda) = \lambda - 1
\]

and, for \(j = 2, 3, \ldots, k\), let

\[
P_j(\lambda) = (\lambda - d_j)P_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} P_{j-2}(\lambda).
\]

In [6], we proved that

\[
det(\lambda I - L(\mathcal{T})) = \prod_{j=1}^{k-1} P_j^{n_j-n_{j+1}}(\lambda).
\]

\[
\Omega = \{ j : 1 \leq j \leq k - 1, \ n_j > n_{j+1} \}.
\]

We derived that the eigenvalues of the adjacency matrix and of the Laplacian matrix of \(\mathcal{T}\) are the eigenvalues of leading principal submatrices of two nonnegative symmetric tridiagonal matrices of order \(k \times k\). The codiagonal entries for both matrices are \(\sqrt{d_j - 1}, 2 \leq j \leq k - 1\), and \(\sqrt{d_k}\), while the diagonal entries are zeros, in the case of the adjacency matrix, and \(d_j, 1 \leq j \leq k\), in the case of the Laplacian matrix. Moreover, we obtained some results concerning to the multiplicity of such eigenvalues.

Our goal is to extend the above mentioned results to an unweighted tree \(\mathcal{T}\) with vertex root \(v\) which is the union of two trees \(\mathcal{T}_1 = (V_1, E_1)\) and \(\mathcal{T}_2 = (V_2, E_2)\) such that \(V_1 \cap V_2 = \{v\}\) and \(\mathcal{T}_1\) and \(\mathcal{T}_2\) have the property that the vertices in each of their levels have equal degree. Below we have an example of such a tree.
We agree that the root vertex is at level 1. Let $k_1$ and $k_2$ be the numbers of levels of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. There is no loss of generality assuming $k_1 \leq k_2$. For $j = 1, 2, 3, \ldots, k_i$, the numbers $d_{i,k_j-j+1}$ and $n_{i,k_j-j+1}$ denote the degree of the vertices and the number of vertices in the level $j$ of $\mathcal{T}_i$. Then

$$n_{i,k_j-j} = (d_{i,k_j-j+1} - 1)n_{i,k_j-j+1}, \quad 2 \leq j \leq k_i - 1. \quad (4)$$

Observe $n_{1,k_1-1} + n_{2,k_2-1}$ is the degree of the root vertex and $n_{1,k_1} = n_{2,k_2} = 1$. The total number of vertices in $\mathcal{T}$ is

$$n = \sum_{j=1}^{k_1-1} n_{1,j} + \sum_{j=1}^{k_2-1} n_{2,j} + 1.$$

We introduce following notations:
- $d$ is the degree of the vertex root, that is, $d = n_{1,k_1-1} + n_{2,k_2-1} - 1$.
- $0$ is the all zeros matrix. The order of $0$ will be clear from the context in which it is used.
- $m_{i,j} = \frac{n_{i,j}}{n_{i,j+1}}$ for $1 \leq j \leq k_i - 1$.
- $I_m$ is the identity matrix of order $m \times m$.
- $e_m$ is the all ones column vector of dimension $m$.
- For $j = 1, 2, \ldots, k_i - 1$, $B_{i,j}$ is the block diagonal matrix defined by

$$B_{i,j} = \begin{bmatrix} e_{m_{i,j}} & e_{m_{i,j}} & \cdots & e_{m_{i,j}} \end{bmatrix}$$

with $n_{i,j+1}$ diagonal blocks. Thus, the order of $B_{i,j}$ is $n_{i,j} \times n_{i,j+1}$. Observe that $B_{i,k_j-1} = e_{n_{i,k_j-1}}$.

In general

1. Using the labels $1, 2, \ldots, \sum_{j=1}^{k_1-1} n_{1,j}$, we label the vertices of $\mathcal{T}_1$ from the bottom to level 2 and, in each level, from the left to the right, and
2. Using the labels $\sum_{j=1}^{k_1-1} n_{1,j} + 1, \ldots, \sum_{j=1}^{k_1-1} n_{1,j} + \sum_{j=1}^{k_2-1} n_{2,j}$, we label the vertices of $\mathcal{T}_2$ from the bottom to level 2 and, in each level, from the left to the right.
3. Finally, we use the label $n$ for the root vertex.

We illustrate the above notations and our labeling for the vertices with the following example.
Example 1. For the tree in (3) our labelling is

We have $k_1 = 3$ and $k_2 = 4$. For the tree $T_1$

$$n_{1,1} = 6, \quad d_{1,1} = 1, \quad n_{1,2} = 2, \quad d_{1,2} = 4,$$

and for the tree $T_2$

$$n_{2,1} = 6, \quad d_{2,1} = 1, \quad n_{2,2} = 3, \quad d_{2,2} = 3, \quad n_{2,3} = 3, \quad d_{2,3} = 2.$$

The degree of the vertex root is $d = 5$, the total number of vertices is $n = 21$ and the matrices defined in (5) are

$$B_{1,1} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \text{diag}\{e_3, e_3\}, \quad B_{1,2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e_2,$$

$$B_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \text{diag}\{e_2, e_2, e_2\},$$

$$B_{2,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{diag}\{e_1, e_1, e_1\}, \quad B_{2,3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e_3.$$
For the above mentioned labelling the adjacency matrix \( A(\mathcal{T}) \) and Laplacian matrix \( L(\mathcal{T}) \) become

\[
A(\mathcal{T}) = 
\begin{bmatrix}
A_1 & 0 & a_1 \\
0 & A_2 & a_2 \\
a_1^T & a_2^T & 0
\end{bmatrix}
\] (6)

and

\[
L(\mathcal{T}) = 
\begin{bmatrix}
L_1 & 0 & -a_1 \\
0 & L_2 & -a_2 \\
-a_1^T & -a_2^T & d
\end{bmatrix},
\] (7)

where, for \( i = 1, 2 \), \( A_i \) and \( L_i \) are the following block tridiagonal matrices

\[
A_i = 
\begin{bmatrix}
0 & B_{i,1} & & & \\
B_{i,1}^T & 0 & B_{i,2} & & \\
& B_{i,2}^T & \ddots & \ddots & \\
& & \ddots & \ddots & \ddots \\
& & & B_{i,k_i-2} & 0
\end{bmatrix}
\] (8)

\[
L_i = 
\begin{bmatrix}
I_{n_{i,1}} & -B_{i,1} & & & \\
-B_{i,1}^T & d_{i,2}I_{n_{i,2}} & -B_{i,2} & & \\
& -B_{i,2}^T & \ddots & \ddots & \\
& & \ddots & \ddots & -B_{i,k_i-2} \\
& & & -B_{i,k_i-2}^T & d_{i,k_i-1}I_{n_{i,k_i-1}}
\end{bmatrix}
\] (9)

and

\[
a_{i}^T = 
\begin{bmatrix}
0 & \cdots & \cdots & 0 & e_{n_{i,k_i-1}}^T
\end{bmatrix}.
\] (10)

The following lemma plays a fundamental role in this paper.

**Lemma 1.** Let

\[
M = 
\begin{bmatrix}
M_1 & 0 & \pm a_1 \\
0 & M_2 & \pm a_2 \\
\pm a_1^T & \pm a_2^T & \alpha
\end{bmatrix},
\]

where \( M_i, i = 1, 2 \), is the block tridiagonal matrix

\[
M_i = 
\begin{bmatrix}
\alpha_i,1I_{n_{i,1}} & B_{i,1} \\
B_{i,1}^T & \alpha_i,2I_{n_{i,2}} & B_{i,2} \\
& B_{i,2}^T & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & B_{i,k_i-2}^T & \alpha_i,k_i-1I_{n_{i,k_i-1}}
\end{bmatrix},
\]

and \( a_1, a_2 \) as in (10).
Let
\[ a_{i,1} = \alpha_{i,1}, \]
\[ a_{i,j} = \alpha_{i,j} - n_{i,j-1} \frac{1}{n_{i,j}} a_{i,j-1}, \quad j = 2, 3, \ldots, k_i - 1, \quad a_{i,j-1} \neq 0, \]
\[ a = \alpha - n_{1,k_1-1} \frac{1}{a_{1,k_1-1}} - n_{2,k_2-1} \frac{1}{a_{2,k_2-1}}. \]

If \( a_{i,j} \neq 0 \) for all \( j = 1, 2, \ldots, k_i - 1 \) then
\[ \det M = a_{1,1} a_{1,2} \ldots a_{1,k_1-2} a_{2,1} a_{2,2} \ldots a_{2,k_2-2} a. \] (11)

**Proof.** Suppose \( a_{1,j} \neq 0 \) for all \( j = 1, 2, \ldots, k_1 - 1 \). After some steps of the Gaussian elimination procedure, without row interchanges, we reduce \( M \) to the intermediate matrix below
\[
\begin{bmatrix}
R_1 & 0 & \pm a_1 \\
0 & M_2 & \pm a_2 \\
0 & \pm a_2^T & \alpha - n_{1,k_1-1} \frac{1}{a_{1,k_1-1}}
\end{bmatrix},
\] (12)
where \( R_1 \) is the upper triangular matrix
\[
R_1 = \begin{bmatrix}
a_{1,1} I_{n_{1,1}} & B_{1,1} \\
& a_{1,2} I_{n_{1,2}} & \ddots \\
& & \ddots & B_{1,k_1-2} \\
& & & a_{1,k_1-1} I_{n_{1,k_1-1}}
\end{bmatrix}.
\]

Suppose, in addition, \( a_{2,j} \neq 0 \) for all \( j = 1, 2, \ldots, k_2 - 1 \). We continue with the Gaussian elimination to obtain finally the upper triangular matrix
\[
\begin{bmatrix}
R_1 & 0 & \pm a_1 \\
0 & R_2 & \pm a_2 \\
0 & 0 & a
\end{bmatrix},
\] (13)
where
\[
R_2 = \begin{bmatrix}
a_{2,1} I_{n_{2,1}} & B_{2,1} \\
& a_{2,2} I_{n_{2,2}} & \ddots \\
& & \ddots & B_{2,k_2-2} \\
& & & a_{2,k_2-1} I_{n_{2,k_2-1}}
\end{bmatrix}.
\]
Thus, (11) is proved. \( \square \)
2. The spectrum of the Laplacian matrix

We will use several times the fact
\[
\det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \det A \det B.
\]

Lemma 2. Let
\[
M = \begin{bmatrix} A & 0 & a \\ 0 & B & b \\ a^T & b^T & c \end{bmatrix},
\]
where \( A \) is an \( n \times n \) matrix, \( B \) is an \( m \times m \) matrix and
\[
a^T = [0 \ \cdots \ \cdots \ 0 \ a],
\]
\[
b^T = [0 \ \cdots \ \cdots \ 0 \ b].
\]

Then
\[
\det M = c \det A \det B - b^2 \det A \det B_{m-1} - a^2 \det A_{n-1} \det B.
\]
Here \( A_{n-1} \) and \( B_{m-1} \) denote the submatrix obtained from \( A \) and \( B \) by deleting the last row and the last column, respectively.

Proof. We expand about the last row of \( M \). Clearly the cofactor for \( c \) is \( \det A \det B \). The cofactor for \( b \) in the last row is
\[
(-1)^{2n+2m+1} \det \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 0 & \cdots & 0 & 0 \\ a_{2,1} & a_{2,n} & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & 0 & \cdots & 0 & a \\ 0 & \cdots & \cdots & 0 & b_{1,1} & \cdots & b_{1,m-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & b_{2,1} & \cdots & b_{2,m-1} & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & b_{m,1} & \cdots & b_{m,m-1} & b \end{bmatrix} = - \det A \det \begin{bmatrix} b_{1,1} & \cdots & b_{1,m-1} & 0 \\ b_{2,1} & \cdots & b_{2,m-1} & \vdots \\ \vdots & \vdots & \vdots & 0 \\ b_{m,1} & \cdots & b_{m,m-1} & b \end{bmatrix} = -b \det A \det B_{m-1}.
\]
The cofactor for \( a \) in the last row is
\[
\begin{vmatrix}
  a_{1,1} & \cdots & a_{1,n-1} & 0 & 0 & \cdots & 0 & 0 \\
  a_{2,1} & a_{2,n-1} & 0 & : & : & : & : & : \\
  a_{n,1} & \cdots & a_{n,n-1} & 0 & 0 & \cdots & 0 & a \\
  0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,m} & 0 \\
  : & : & b_{2,1} & b_{2,m} & : & : & : & : \\
  0 & \cdots & 0 & b_{m,1} & b_{m,2} & \cdots & b_{m,m} & b
\end{vmatrix}
\]

\((-1)^{2n+m+1}\) \[\det \begin{vmatrix}
  a_{1,1} & \cdots & a_{1,n-1} & 0 & 0 & \cdots & 0 & 0 \\
  a_{2,1} & a_{2,n-1} & 0 & : & : & : & : & : \\
  a_{n,1} & \cdots & a_{n,n-1} & a & 0 & \cdots & 0 & 0 \\
  0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,m} & 0 \\
  : & : & 0 & b_{2,1} & \cdots & b_{2,m} & : & : \\
  0 & \cdots & 0 & b_{m,1} & b_{m,2} & \cdots & b_{m,m} & b
\end{vmatrix}\]

After \( m \) column interchanges this cofactor becomes
\[
\begin{vmatrix}
  a_{1,1} & \cdots & a_{1,n-1} & 0 & 0 & \cdots & 0 & 0 \\
  a_{2,1} & a_{2,n-1} & 0 & : & : & : & : & : \\
  a_{n,1} & \cdots & a_{n,n-1} & a & 0 & \cdots & 0 & 0 \\
  0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,m} & 0 \\
  : & : & 0 & b_{2,1} & \cdots & b_{2,m} & : & : \\
  0 & \cdots & 0 & b_{m,1} & b_{m,2} & \cdots & b_{m,m} & b
\end{vmatrix}
\]

\((-1)^{2n+2m+1}\) \[\det \begin{vmatrix}
  a_{1,1} & \cdots & a_{1,n-1} & 0 \\
  a_{2,1} & a_{2,n-1} & 0 \\
  : & : & : \\
  a_{n,1} & \cdots & a_{n,n-1} & a
\end{vmatrix}\] \[\det B = -a \det A_{n-1} \det B.\]

Therefore
\[\det M = c \det A \det B - b^2 \det A \det B_{m-1} - a^2 \det A_{n-1} \det B.\]
\[\square\]

Let
\[
\Phi_1 = \{1, 2, 3, \ldots, k_1 - 1\},
\]
\[
\Phi_2 = \{1, 2, 3, \ldots, k_2 - 1\}.
\]

We consider the following subsets of \( \Phi_1 \) and \( \Phi_2 \),
\[
\Omega_1 = \{ j \in \Phi_1 : n_{1,j} > n_{1,j+1}\},
\]
\[
\Omega_2 = \{ j \in \Phi_2 : n_{2,j} > n_{2,j+1}\}.
\]

Observe that if \( j \in \Phi_i - \Omega_i \) then \( n_{i,j} = n_{i,j+1} \) and thus, from (5), \( B_{i,j} = I_{n_{i,j}} \).
Theorem 1. Let \( i = 1, 2 \). Let

\[
P_{i,0}(\lambda) = 1, \quad P_{i,1}(\lambda) = \lambda - 1
\]

and

\[
P_{i,j}(\lambda) = (\lambda - d_{i,j}) P_{i,j-1}(\lambda) - \frac{n_{i,j-1}}{n_{i,j}} P_{i,j-2}(\lambda)
\]

for \( j = 2, 3, \ldots, k_i - 1 \), \((14)\)

\[
P(\lambda) = (\lambda - d) P_{1,k_1-1}(\lambda) P_{2,k_2-1}(\lambda)
\]

\[- n_1 k_1-1 P_{1,k_1-2}(\lambda) P_{2,k_2-1}(\lambda) - n_2 k_2-1 P_{1,k_1-1}(\lambda) P_{2,k_2-2}(\lambda).
\]

Then

(a) If \( P_{i,j}(\lambda) \neq 0 \) for all \( j = 1, 2, \ldots, k_i - 1 \) then

\[
\det(\lambda I - L(\mathcal{F})) = P(\lambda) \prod_{j \in \Omega_1} P_{1,j}^{n_{1,j}-n_{1,j+1}}(\lambda) \prod_{j \in \Omega_2} P_{2,j}^{n_{2,j}-n_{2,j+1}}(\lambda).
\]

\((16)\)

(b) \[
\sigma(L(\mathcal{F})) = \left( \bigcup_{i=1}^{2} \bigcup_{j \in \Omega_i} \{ \lambda : P_{i,j}(\lambda) = 0 \} \right) \cup \{ \lambda : P(\lambda) = 0 \}.
\]

\((17)\)

Proof. (a) We apply Lemma 1 to the matrix \( M = \lambda I - L(\mathcal{F}) \).

For this matrix \( a_{1,1} = \lambda - 1, a_{1,j} = \lambda - d_{1,j} \) for \( j = 2, 3, \ldots, k_1 - 1, a_{2} = \lambda - d, \ a_{2,1} = \lambda - 1, a_{2,j} = \lambda - d_{2,j} \) for \( j = 2, 3, \ldots, k_2 - 1 \). Let \( a_{1,1}, a_{1,2}, \ldots, a_{1,k_1-1}, a_{2,1}, a_{2,2}, \ldots, a_{2,k_2-1}, a \) be as in Lemma 1. Suppose that \( \lambda \in \mathbb{R} \) is such that \( P_{i,j}(\lambda) \neq 0 \) for all \( j = 1, 2, \ldots, k_i - 1 \). For brevity, we write \( P_{i,j}(\lambda) = P_{i,j} \).

We have

\[
a_{i,1} = \lambda - 1 = \frac{P_{i,1}}{P_{i,0}} \neq 0,
\]

\[
a_{i,2} = (\lambda - d_{i,2}) - \frac{n_{i,1}}{n_{i,2}} a_{i,1} = (\lambda - d_{i,2}) - \frac{n_{i,1}}{n_{i,2}} \frac{P_{i,0}}{P_{i,1}}
\]

\[
= \frac{(\lambda - d_{i,2}) P_{i,1} - \frac{n_{i,1}}{n_{i,2}} P_{i,0}}{P_{i,1}} = \frac{P_{i,2}}{P_{i,1}} \neq 0,
\]

\(: \)

\[
a_{i,k_1-1} = (\lambda - d_{i,k_1-1}) - \frac{n_{k_1-2}}{n_{k_1-1}} a_{i,k_1-2} = (\lambda - d_{i,k_1-1}) - \frac{n_{k_1-2}}{n_{k_1-1}} \frac{P_{i,k_1-3}}{P_{i,k_1-2}}
\]

\[
= \frac{(\lambda - d_{i,k_1-1}) P_{i,k_1-2} - \frac{n_{k_1-2}}{n_{k_1-1}} P_{i,k_1-3}}{P_{i,k_1-2}} = \frac{P_{i,k_1-1}}{P_{i,k_1-2}} \neq 0.
\]
and

\[ a = (\lambda - d) - n_{1,k_1-1} P_{1,k_1-2} - n_{2,k_2-1} P_{2,k_2-2} \]

\[ = (\lambda - d) P_{1,k_1-1} P_{2,k_2-1} - n_{1,k_1-1} P_{1,k_1-2} P_{2,k_2-1} - n_{2,k_2-1} P_{1,k_1-1} P_{2,k_2-2} \]

\[ = \frac{P}{P_{1,k_1-1} P_{2,k_2-1}}. \]

From (11)

\[
\det(\lambda I - L(\mathcal{T})) = \frac{P_{1,1}^{n_{1,1}} P_{1,2}^{n_{1,2}} P_{1,3}^{n_{1,3}} \cdots P_{1,k_1-2}^{n_{1,k_1-2}} P_{1,k_1-1}^{n_{1,k_1-1}}}{P_{1,1}^{n_{1,1}} P_{1,2}^{n_{1,2}} P_{1,k_1-2}^{n_{1,k_1-2}} P_{1,k_1-1}^{n_{1,k_1-1}}} \times \frac{P_{2,1}^{n_{2,1}} P_{2,2}^{n_{2,2}} P_{2,3}^{n_{2,3}} \cdots P_{2,k_2-2}^{n_{2,k_2-2}} P_{2,k_2-1}^{n_{2,k_2-1}}}{P_{2,1}^{n_{2,1}} P_{2,2}^{n_{2,2}} P_{2,k_2-2}^{n_{2,k_2-2}} P_{2,k_2-1}^{n_{2,k_2-1}}} P
\]

\[ = P(\lambda) \prod_{j \in \Omega_1} P_{1,j}^{n_{1,j}-n_{1,j+1}(\lambda)} \prod_{j \in \Omega_2} P_{2,j}^{n_{2,j}-n_{2,j+1}(\lambda)}. \]

Thus, (16) is proved.

(b) From (16), if \( P_{i,j}(\lambda) \neq 0 \) for \( i = 1, 2 \) and for all \( j = 1, 2, \ldots, k_i - 1 \), and \( P(\lambda) \neq 0 \), then \( \det(\lambda I - L(\mathcal{T})) \neq 0 \). Hence

\[ \sigma(L(\mathcal{T})) \subseteq (\bigcup_{i=1}^2 \bigcup_{j=1}^{k_i-1} \{ \lambda : P_{i,j}(\lambda) = 0 \}) \cup \{ \lambda : P(\lambda) = 0 \}. \]  (18)

We claim that

\[ \sigma(L(\mathcal{T})) \subseteq (\bigcup_{i=1}^2 \bigcup_{j \in \Omega_i} \{ \lambda : P_{i,j}(\lambda) = 0 \}) \cup \{ \lambda : P(\lambda) = 0 \} \]  (19)

If \( \Omega_1 = \Phi_1 \) and \( \Omega_2 = \Phi_2 \) then (19) is (18) and there is nothing to prove. Suppose that \( \Omega_i \) is a proper subset of \( \Phi_i \). Clearly, (19) is equivalent to

\[ (\bigcap_{i=1}^2 \bigcap_{j \in \Omega_i} \{ \lambda : P_{i,j}(\lambda) \neq 0 \}) \cap \{ \lambda : P(\lambda) \neq 0 \} \subseteq (\sigma(L(\mathcal{T})))^c. \]

Suppose that \( \lambda \in \mathbb{R} \) is such that \( P_{i,j}(\lambda) \neq 0 \) for \( i = 1, 2 \) and all \( j \in \Omega_i \) and \( P(\lambda) \neq 0 \). If in addition \( P_{i,j}(\lambda) \neq 0 \) for \( i = 1, 2 \) and for all \( j \in \Phi_i - \Omega_i \) then (16) holds and consequently \( \det(\lambda I - L(\mathcal{T})) \neq 0 \), that is, \( \lambda \in (\sigma(L(\mathcal{T})))^c \). If \( P_{1,j}(\lambda) = 0 \) for some \( j \in \Phi_1 - \Omega_1 \) or \( P_{2,j}(\lambda) = 0 \) for some \( j \in \Phi_2 - \Omega_2 \) then the Gaussian elimination procedure with appropriate row interchanges applied to \( M = \lambda I - L(\mathcal{T}) \) allows to see that \( \det(\lambda I - L(\mathcal{T})) \neq 0 \) as we show below.

Using (7), we have

\[
M = \lambda I - L(\mathcal{T}) \]

\[
= \begin{bmatrix} \lambda I - L_1(\mathcal{T}) & 0 & a_1^T \\ 0 & \lambda I - L_2(\mathcal{T}) & a_2^T \\ a_1 & a_2 & \lambda - d \end{bmatrix},
\]
where \( L_i(\mathcal{T}) \) and \( a_i \), \( i = 1, 2 \), are given by (9) and (10) respectively. Since \( P(\lambda) \neq 0 \), from (15), we have \( P_{1,k_1-1}(\lambda) \neq 0 \) or \( P_{2,k_2-1}(\lambda) \neq 0 \). We can assume that \( P_{1,k_1-1}(\lambda) \neq 0 \). For if not, we consider the matrix

\[
N = \begin{bmatrix}
\lambda I - L_2(\mathcal{T}) & 0 & a_2 \\
0 & \lambda I - L_1(\mathcal{T}) & a_1 \\
a_2^T & a_1^T & \lambda - d
\end{bmatrix}.
\]

Observe that \( \det M = \det N \).

Let \( s \) be the first index in \( \Phi_1 - \Omega_1 \), if any, such that \( P_{1,s}(\lambda) = 0 \). Then \( a_{1,j} \neq 0 \) for all \( j = 1, 2, \ldots, s - 1 \), \( a_{1,s} = 0 \) and

\[
P_{1,s+2}(\lambda) = (\lambda - d_{1,s+2}) P_{1,s+1}(\lambda).
\]

We observe that \( P_{1,s+1}(\lambda) \neq 0 \). Otherwise a back substitution in (14) gives \( P_{1,0}(\lambda) = 0 \). Hence \( \lambda - d_{1,s+2} = P_{1,s+2}(\lambda) P_{1,s+1}(\lambda) = a_{1,s+2} \). Since \( s \in \Phi_1 - \Omega_1 \), \( n_1,s = n_1,s+1 \), \( B_{1,s} = I_{n_1,s} \). The Gaussian elimination procedure applied to \( M = \lambda I - L(\mathcal{T}) \) yields to an intermediate matrix whose first \( \sum_{j=1}^{k_1-1} n_1,j \) rows are

\[
\begin{bmatrix}
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & B_{1,s-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
I_{n_1,s} & (\lambda - d_{1,s+1}) I_{n_1,s+1} & B_{1,s+1} & a_{1,s+2} I_{n_1,s+2} & B_{1,s+2} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
B_{1,k_1-2} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{bmatrix}
\]

After a number of \( n_1,s \) row interchanges this submatrix becomes

\[
\begin{bmatrix}
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & B_{1,s-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
I_{n_1,s} & (\lambda - d_{1,s+1}) I_{n_1,s+1} & B_{1,s+1} & a_{1,s+2} I_{n_1,s+2} & B_{1,s+2} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
B_{1,k_1-2} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{bmatrix}
\]

Therefore

\[
\det(\lambda I - L(\mathcal{T})) = (-1)^{n_1,s} a_{1,1}^{n_1,1} \cdots a_{1,s-1}^{n_1,s-1} \det C.
\]
where
\[
C = \begin{bmatrix}
a_{1,s+1}I_{n_1,s+1} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & B_{1,k_1-2}^T & (\lambda - d_{1,k_1-1})I_{n_1,k_1-1} & 0 & e_{n_1,k_1-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \lambda I - L_2 & a_2 \\
& & & a_2^T & \lambda - d
\end{bmatrix}.
\]

We repeat the above procedure to \(C\) obtaining that
\[
\det(\lambda I - L(\mathcal{F})) = \gamma \det \begin{bmatrix}
\lambda I - L_2 & a_2 \\
a_2^T & \lambda - d - n_{1,k_1-1}P_{1,k_1-2}(\lambda) - n_{2,k_2-1}P_{2,k_2-2}(\lambda)
\end{bmatrix},
\]
where \(\gamma\) is a factor different from 0. If there exists \(j \in \Phi_2 - \Omega_2\) such that \(P_{2,j}(\lambda) = 0\) we continue with the Gaussian elimination procedure with row interchanges, as we show above, to finally get
\[
\det(\lambda I - L(\mathcal{F})) = \delta_1((\lambda - d) - n_{1,k_1-1}P_{1,k_1-1}(\lambda)) = \delta_1\frac{P(\lambda)}{P_{1,k_1-1}(\lambda)P_{2,k_2-1}(\lambda)}
\]
and
\[
\det(\lambda I - L(\mathcal{F})) = \delta_2 \quad \text{if } P_{2,k_2-1}(\lambda) = 0 \text{ and } k_2 - 1 \in \Phi_2 - \Omega_2
\]
or
\[
\det(\lambda I - L(\mathcal{F})) = \delta_3 \quad \text{if } P_{1,k_1-1}(\lambda) = 0 \text{ and } k_1 - 1 \in \Phi_1 - \Omega_1,
\]
where \(\delta_1, \delta_2\) and \(\delta_3\) are different from 0. Thus, we have showed that \(\det(\lambda I - L(\mathcal{F})) \neq 0\) if \(P_{1,j}(\lambda) = 0\) for some \(j \in \Phi_1 - \Omega_1\) or \(P_{2,j}(\lambda) = 0\) for some \(j \in \Phi_2 - \Omega_2\) and then (19) is proved.

Now, we claim that
\[
\left(\bigcup_{i=1}^{2} \bigcup_{j \in \Omega_i} \{\lambda : P_{i,j}(\lambda) = 0\}\right) \bigcup \{\lambda : P(\lambda) = 0\} \subseteq \sigma(L(\mathcal{F})). \tag{21}
\]

Let \(\lambda \in \bigcup_{i=1}^{2} \bigcup_{j \in \Omega_i} \{\lambda : P_{i,j}(\lambda) = 0\}\). Then \(\lambda \in \{\lambda : P_{i,j}(\lambda) = 0\}\) for some \(j \in \Omega_i\) for \(i = 1\) or \(i = 2\). Let \(s\) be the first index in \(\Omega_i\) such that \(P_{i,s}(\lambda) = 0\). Then, \(a_{i,s} = \frac{P_{i,s}(\lambda)}{P_{i,s-1}(\lambda)} = 0\). Hence the corresponding intermediate matrix in the Gaussian elimination procedure applied to the matrix \(M = \lambda I - L(\mathcal{F})\) has a zero diagonal block. Since \(s \in \Omega_i, n_{i,s} > n_{i,s+1}\), and \(B_{i,s}\) is a matrix with more rows than columns. Therefore, the corresponding intermediate matrix has at least two equal rows and thus \(\det(\lambda I - L(\mathcal{F})) = 0\). That is, \(\lambda \in (L(\mathcal{F}))\). We have proved that
\[
\bigcup_{i=1}^{2} \bigcup_{j \in \Omega_i} \{\lambda : P_{i,j}(\lambda) = 0\} \subseteq \sigma(L(\mathcal{F})). \tag{22}
\]
We now claim that

$$\{ \lambda : P(\lambda) = 0 \} \subseteq \sigma(L(\mathcal{T})). \quad (23)$$

Let $\lambda \in \{ \lambda : P(\lambda) = 0 \}$. If $P_{i,j}(\lambda) = 0$ for some $i = 1, 2$ and some $j \in \Omega_i$ then (22) gives $\lambda \in \sigma(L(\mathcal{T}))$. Then we may suppose $P_{i,j}(\lambda) \neq 0$ for $i = 1, 2$ and for all $j \in \Omega_i$.

If in addition $P_{i,j}(\lambda) \neq 0$ for $i = 1, 2$ and for all $j \in \Phi_i - \Omega_i$ then (16) holds and then $\det(\lambda I - L(\mathcal{T})) = 0$ because $P(\lambda) = 0$. Now, we consider the case $P_{i,j}(\lambda) = 0$ for some $i = 1, 2$ and some $j \in \Phi_i - \Omega_i$. From (15), we see that $P_{1,k_1-1}(\lambda) = 0$ if and only if $P_{2,k_2-1}(\lambda) = 0$ or, equivalently, $P_{1,k_1-1}(\lambda) \neq 0$ and only if $P_{2,k_2-1}(\lambda) \neq 0$. If $P_{1,k_1-1} \neq 0$ and $P_{2,k_2-1} \neq 0$ then we have the assumptions under which (20) was obtained. Hence

$$\det(\lambda I - L(\mathcal{T})) = \frac{P(\lambda)}{P_{1,k_1-1}(\lambda)P_{2,k_2-1}(\lambda)} = 0.$$ 

We observe that $P_{1,k_1-1}(\lambda) \neq 0$ and $P_{2,k_2-1}(\lambda) \neq 0$ whenever $k_1 - 1 \in \Omega_1$ or $k_2 - 1 \in \Omega_2$. If $P_{1,k_1-1}(\lambda) = 0$ and $P_{2,k_2-1}(\lambda) = 0$ with $k_1 - 1 \in \Phi_1 - \Omega_1$ and $k_2 - 1 \in \Phi_2 - \Omega_2$ then $n_{k_1-1} = n_{k_1} = 1$, $n_{k_2-1} = n_{k_2} = 1$ and

$$M = \lambda I - L(\mathcal{T}) = \begin{bmatrix} \lambda I - L_1(\mathcal{T}) & 0 & a_1 \\ 0 & \lambda I - L_2(\mathcal{T}) & a_2 \\ a_1^T & a_2^T & \lambda - d \end{bmatrix},$$

with $a_1 = a_2 = [0 \cdots \cdots 0 1]^T$. We apply Lemma 2 to obtain

$$\det(\lambda I - L(\mathcal{T})) = (\lambda - d) \det(\lambda I - L_1(\mathcal{T})) \det(\lambda I - L_2(\mathcal{T})) - \det(\lambda I - L_1(\mathcal{T})) \det F - \det E \det(\lambda I - L_2(\mathcal{T})),$$

where the matrices $E$ and $F$ are obtained from $\lambda I - L_1(\mathcal{T})$ and $\lambda I - L_2(\mathcal{T})$ by deleting the last row and last column respectively. The Gaussian elimination procedure with row interchanges applied to the matrices $\lambda I - L_1(\mathcal{T})$ and $\lambda I - L_2(\mathcal{T})$ gives

$$\det(\lambda I - L_1(\mathcal{T})) = \delta_1 a_{1,k_1-1} = \delta_1 \frac{P_{1,k_1-1}(\lambda)}{P_{1,k_1-2}(\lambda)} = 0,$$

$$\det(\lambda I - L_2(\mathcal{T})) = \delta_2 a_{2,k_2-1} = \delta_1 \frac{P_{2,k_2-1}(\lambda)}{P_{2,k_2-2}(\lambda)} = 0.$$ 

Thus (23) is proved. From (22) and (23), we obtain (21). Finally, (19) and (21) imply (17). $\square$

**Lemma 3.** For $i = 1, 2$ and for $j = 1, 2, 3, \ldots, k_i - 1$, let $T_{i,j}$ be the $j \times j$ leading principal submatrix of the $(k_i - 1) \times (k_i - 1)$ symmetric tridiagonal matrix.
\[ T_{i,k_i-1} = \begin{bmatrix}
1 & \sqrt{d_{i,2} - 1} & & & \\
\sqrt{d_{i,2} - 1} & d_{i,2} & \ddots & & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & \ddots \\
& & & \sqrt{d_{i,k_i-2} - 1} & d_{i,k_i-2} \\
& & & \sqrt{d_{i,k_i-2} - 1} & d_{i,k_i-1} - 1 \\
\end{bmatrix}. \]

Then, for \( i = 1, 2 \) and for \( j = 1, 2, 3, \ldots, k_i - 1 \),
\[
\det(\lambda I - T_{i,j}) = P_{i,j}(\lambda), \quad j = 1, 2, \ldots, k_i - 1.
\]

**Proof.** It is well known [1, p. 229] that the characteristic polynomials, \( Q_j \), of the \( j \times j \) leading principal submatrix of the \((k - 1) \times (k - 1)\) symmetric tridiagonal matrix
\[
\begin{bmatrix}
c_1 & b_1 & & & \\
b_1 & c_2 & b_2 & & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & 0 \\
& & & \ddots & b_{k-2} \\
& & & & c_{k-1} \\
\end{bmatrix},
\]
satisfy the three-term recursion formula
\[
Q_j(\lambda) = (\lambda - c_j)Q_{j-1}(\lambda) - b_j^2 Q_{j-2}(\lambda),
\]
with
\[ Q_0(\lambda) = 1 \quad \text{and} \quad Q_1(\lambda) = \lambda - c_1. \]
For the matrices \( T_{i,k_i-1} \), we have \( c_1 = 1, \ c_j = d_{i,j} \) for \( j = 2, 3, \ldots, k_i - 1 \) and \( b_j = \sqrt{\frac{n_{i,j}}{n_{i,j+1}}} \) for \( j = 1, 2, \ldots, k_i - 2 \). For these values, the above recursion formula gives the polynomials \( P_{i,j}, \ j = 0, 1, 2, \ldots, k_i - 1 \). Now, we use (4), to see that \[ \sqrt{\frac{n_{i,j}}{n_{i,j+1}}} = \sqrt{d_{i,j+1} - 1} \] for \( j = 1, 2, \ldots, k_i - 2 \). \( \square \)

**Lemma 4.** Let \( R \) be the symmetric matrix of order \((k_1 + k_2 - 1) \times (k_1 + k_2 - 1)\) defined by
\[
R = \begin{bmatrix}
T_{1,k_1-1} & 0 & p_1 \\
0 & T_{2,k_2-1} & p_2 \\
p_1^T & p_2^T & d
\end{bmatrix},
\]
where \( T_{1,k_1-1} \) and \( T_{2,k_2-1} \) are the symmetric tridiagonal matrices defined in Lemma 3 and
\[
p_i^T = [0 \ldots \ldots 0 \sqrt{n_{i,k_i-1}}].
\]
Then

\[ \det(\lambda I - R) = P(\lambda). \]  

(24)

**Proof.** From Lemma 3

\[ \det(\lambda I - T_{1,k_1-1}) = P_{1,k_1-1}(\lambda), \]
\[ \det(\lambda I - T_{1,k_1-2}) = P_{1,k_1-2}(\lambda), \]
\[ \det(\lambda I - T_{2,k_2-1}) = P_{2,k_2-1}(\lambda), \]
\[ \det(\lambda I - T_{2,k_2-2}) = P_{2,k_2-2}(\lambda). \]

We apply Lemma 2 to the matrix \( \lambda I - R \) to obtain

\[
\begin{align*}
\det(\lambda I - R) &= (\lambda - d) \det(\lambda I - T_{1,k_1-1}) \det(\lambda I - T_{2,k_2-1}) \\
& \quad - n_{2,k_2-1} \det(\lambda I - T_{1,k_1-1}) \det(\lambda I - T_{2,k_2-2}) \\
& \quad - n_{1,k_1-1} \det(\lambda I - T_{1,k_1-2}) \det(\lambda I - T_{2,k_2-1}) \\
& = (\lambda - d) P_{1,k_1-1}(\lambda) P_{2,k_2-1}(\lambda) - n_{2,k_2-1} P_{1,k_1-1}(\lambda) P_{2,k_2-2}(\lambda) \\
& \quad - n_{1,k_1-1} P_{1,k_1-2}(\lambda) P_{2,k_2-1}(\lambda).
\end{align*}
\]

Finally, we recall (15) to get (24). \( \square \)

**Theorem 2.** Let \( T \) be the symmetric tridiagonal matrix of order \((k_1 + k_2 - 1) \times (k_1 + k_2 - 1)\) defined by

\[
T = \begin{bmatrix}
T_{1,k_1-1} & \mathbf{p}_1 \\
\mathbf{p}_1^T & d & \mathbf{p}_2 \\
\mathbf{p}_2^T & J_{k_2-1} & T_{2,k_2-1} J_{k_2-1}
\end{bmatrix}.
\]

where \( J \) is the matrix of order \((k_2 - 1) \times (k_2 - 1)\) with ones along the secondary diagonal and zeros elsewhere. Then

(a)

\[ \sigma(L(\mathcal{F})) = (\cup_{j \in \Omega_1} \sigma(T_{1,j})) \cup (\cup_{j \in \Omega_2} \sigma(T_{2,j})) \cup \sigma(T). \]

(b) The multiplicity of each eigenvalue of the matrix \( T_{i,j} \), as an eigenvalue of \( L(\mathcal{F}) \), is at least \((n_i,j - n_{i,j+1})\) for \( j \in \Omega_i \), and the eigenvalues of \( T \) as eigenvalues of \( L(\mathcal{F}) \) are simple.

**Proof.** We recall that the eigenvalues of any symmetric tridiagonal matrix with non-zero codiagonal entries are simple. Let us consider the permutation matrix

\[
P = \begin{bmatrix}
I_{k_1-1} & 0 \\
0 & J_{k_2}
\end{bmatrix}.
\]
The matrix $R$ defined in Lemma 4 is similar to the matrix $T$ defined in this theorem because $P^{-1}RP = PRP = T$. Hence (a) and (b) are immediate consequences of Theorem 1, Lemma 3 and Lemma 4. □

Example 2. Let $\mathcal{T}$ be the tree in Example 1. For this tree, $k_1 = 3$, $n_{1,1} = 6$, $d_{1,1} = 1$, $n_{1,2} = 2$, $d_{1,2} = 4$, $k_2 = 4$, $n_{2,1} = 6$, $d_{2,1} = 1$, $n_{2,2} = 3$, $d_{2,2} = 3$, $n_{2,3} = 3$, $d_{2,3} = 2$ and $d = 5$. Hence

\[
T_{1,2} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 4 \end{bmatrix}, \quad T_{2,3} = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 3 \\ 3 & 1 \end{bmatrix},
\]

\[
T = \begin{bmatrix} 1 & \sqrt{3} & \sqrt{2} \\ \sqrt{3} & 4 & \sqrt{2} \\ \sqrt{2} & 5 & \sqrt{3} \\ \sqrt{3} & 2 & 1 \\ 1 & 3 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}.
\]

Since $n_{1,1} > n_{1,2} > n_{1,3} = 1$ and $n_{2,1} > n_{2,2}$, $n_{2,3} > n_{2,4} = 1$, we have $\Omega_1 = \{1, 2\}$ and $\Omega_2 = \{1, 3\}$. The eigenvalues of $L(\mathcal{T})$ are the eigenvalues of $T_{1,1}$, $T_{1,2}$, $T_{2,1}$, $T_{2,3}$ and $T$. To four decimal places these eigenvalues are

\[
\begin{align*}
T_{1,1} : & \quad 1 \\
T_{1,2} : & \quad 0.2087 \quad 4.7913 \\
T_{2,1} : & \quad 1 \\
T_{2,3} : & \quad 0.1392 \quad 1.7459 \quad 4.1149 \\
T : & \quad 0 \quad 0.1722 \quad 1.1423 \quad 3.6995 \quad 4.3456 \quad 6.6404
\end{align*}
\]

Theorem 3. Let $L(\mathcal{T})$ be the Laplacian matrix of $\mathcal{T}$. Then

(a) $\sigma(T_{i,j-1}) \cap \sigma(T_{i,j}) = \phi$ for $j = 2, 3, \ldots, k_i - 1$.

(b) The largest eigenvalue of $T$ is the largest eigenvalue of $L(\mathcal{T})$.

(c) $\det T_{i,j} = 1$ for $i = 1, 2$ and for all $j = 1, 2, \ldots, k_i - 1$.

(d) If $\lambda$ is an integer eigenvalue of $L(\mathcal{T})$ and $\lambda > 1$ then $\lambda \in \sigma(T)$.

Proof. (a) follows from the strictly interlacing property for symmetric tridiagonal matrices with nonzero codiagonal entries. Observe that $T_{1,j}$ for $j = 1, 2, \ldots, k_1 - 1$ is a leading principal submatrix of $T$ and that $T_{2,j}$ for $j = 1, 2, \ldots, k_2 - 1$ is similar to a submatrix of $T$. Then, (b) follows from the interlacing property for symmetric matrices and Theorem 2. Clearly, $\det T_{i,1} = 1$. Let $2 \leq j \leq k_i - 1$. We apply the Gaussian elimination procedure, without row interchanges, to reduce the matrix $T_{i,j}$ to the upper triangular matrix.
Then \( \det T_{i,j} = 1 \). Thus, (c) is proved. Since \( P_{1,0}(\lambda) = P_{2,0}(\lambda) = 1 \) and \( P_{1,1}(\lambda) = P_{2,1}(\lambda) = \lambda - 1 \), it follows from the recursion formulae (14) that \( P_{1,j}(\lambda) \) and \( P_{2,j}(\lambda) \) are polynomials with integer coefficients. Therefore, if \( \lambda \) is an integer zero of \( P_{1,j} \) or \( P_{2,j} \) then \( \lambda \) exactly divides \( P_{1,j}(0) \) or \( P_{2,j}(0) \). Moreover, \( P_{i,j}(0) = (-1)^j \det T_{i,j} = (-1)^j \). Consequently, no integer greater than 1 is an eigenvalue of \( T_{i,j} \). □

3. The spectrum of the adjacency matrix

We know that

\[
A(\mathcal{T}) = \begin{bmatrix} A_1 & 0 & a_1 \\ 0 & A_2 & a_2 \\ a_1^T & a_2^T & 0 \end{bmatrix},
\]

where

\[
A_i = \begin{bmatrix} 0 & B_{i,1} \\ B_{i,1}^T & 0 & B_{i,2} \\ & & \ddots & \ddots & \ddots \\ & & & B_{i,k_i-2}^T & 0 \end{bmatrix}.
\]

Let us consider the idempotent diagonal matrix defined by

\[
D = \begin{bmatrix} D_1 & \vdots & \vdots \\ \vdots & D_2 & \vdots \\ \vdots & \vdots & -1 \end{bmatrix},
\]

\[
D_i = \begin{bmatrix} -I_{n_i,1} & I_{n_i,2} & -I_{n_i,3} \\ & \ddots & \vdots \\ & \vdots & (-1)^{k_i-1} I_{n_i,k_i-1} \end{bmatrix}.
\]
We have
\[
D(\lambda I - A(\mathcal{T}))D^{-1} = D(\lambda I - A(\mathcal{T}))D = \begin{bmatrix}
D_1(\lambda I - A_1)D_1 & 0 & D_1\mathbf{a}_1 \\
0 & D_2(\lambda I - A_2)D_2 & D_2\mathbf{a}_2 \\
\mathbf{a}_1^TD_1 & \mathbf{a}_2^TD_2 & \lambda
\end{bmatrix}.
\]

Moreover
\[
D_i(\lambda I - A_i)D_i = \begin{bmatrix}
\lambda I_{n_i,1} & \mathbf{B}_{i,1} \\
\mathbf{B}_{i,1}^T & \lambda I_{n_i,2} & \mathbf{B}_{i,2} \\
& \ddots & \ddots \\
& & \mathbf{B}_{i,k_i-2} & \lambda I_{n_i,k_i-1}
\end{bmatrix}
\]

and
\[
\mathbf{a}_i^TD_i = \begin{bmatrix} 0 & \cdots & \cdots & 0 & (-1)^{k_i-1}\mathbf{e}_{n_i-1}^T \end{bmatrix} = \pm\mathbf{a}_i^T.
\]

Let \(M_1 = D_1(\lambda I - A_1)D_1\) and \(M_2 = D_2(\lambda I - A_2)D_2\). The matrix \(\lambda I - A(\mathcal{T})\) is similar to
\[
M = \begin{bmatrix}
M_1 & 0 & \pm\mathbf{a}_1 \\
0 & M_2 & \pm\mathbf{a}_2 \\
\pm\mathbf{a}_1^T & \pm\mathbf{a}_2^T & \lambda
\end{bmatrix}.
\]

This allow us to apply Lemma 1 to find \(\det(\lambda I - A(\mathcal{T})) = \det M\). Observe that for this matrix \(\alpha_{i,j} = \lambda\) for \(j = 1, 2, \ldots, k_i\) and \(\alpha = \lambda\).

**Theorem 4.** Let \(i = 1, 2\). Let
\[
Q_{i,0}(\lambda) = 1, \quad Q_{i,1}(\lambda) = \lambda - 1
\]

and
\[
Q_{i,j}(\lambda) = \lambda Q_{i,j-1}(\lambda) - \frac{n_{i,j-1}}{n_{i,j}} Q_{i,j-2}(\lambda) \text{ for } j = 2, 3, \ldots, k_i - 1,
\]

\[
W(\lambda) = \lambda Q_{1,k_i-1}(\lambda) Q_{2,k_i-1}(\lambda) - n_{1,k_i-1} Q_{1,k_i-2}(\lambda) Q_{2,k_i-1}(\lambda) - n_{2,k_i-2} Q_{1,k_i-1}(\lambda) Q_{2,k_i-2}(\lambda).
\]

Then
(a) If \(Q_{i,j}(\lambda) \neq 0\) for all \(j = 1, 2, \ldots, k_i - 1\) then
\[
\det(\lambda I - A(\mathcal{T})) = W(\lambda) \prod_{j \in \Omega_1} Q_{1,j}^{n_{1,j}-n_{1,j+1}}(\lambda) \prod_{j \in \Omega_2} Q_{2,j}^{n_{2,j}-n_{2,j+1}}(\lambda).
\]
\[ \sigma(A(\mathcal{T})) = (\cup_{i=1}^{j} \cup_{j \in \Omega_i} \{ \lambda : Q_{i,j}(\lambda) = 0 \}) \cup \{ \lambda : W(\lambda) = 0 \}. \]

**Proof.** Similar to the proof of Theorem 1. Apply Lemma 1 to the matrix \( M = D(\lambda I - A(\mathcal{T}))D^{-1} \) in (26). For this matrix \( a_{i,j} = \lambda \) for \( j = 1, 2, \ldots, k_i \) and \( \alpha = \lambda. \) □

**Lemma 5.** For \( i = 1, 2 \) and for \( j = 1, 2, 3, \ldots, k_i - 1, \) let \( A_{i,j} \) be the \( j \times j \) leading principal submatrix of the \((k_i - 1) \times (k_i - 1)\) symmetric tridiagonal matrix
\[
A_{i,k_i-1} = \begin{bmatrix}
0 & \sqrt{d_{i,2} - 1} \\
\sqrt{d_{i,2} - 1} & 0 & \ddots \\
& \ddots & \ddots & \sqrt{d_{i,k_i-2} - 1} \\
& & \sqrt{d_{i,k_i-2} - 1} & 0 & \sqrt{d_{i,k_i-1} - 1} \\
& \sqrt{d_{i,k_i-1} - 1} & 0 & & 
\end{bmatrix}.
\]
Then, for \( i = 1, 2 \) and for \( j = 1, 2, 3, \ldots, k_i - 1, \)
\[
\det(\lambda I - A_{i,j}) = Q_{i,j}(\lambda), \quad j = 1, 2, \ldots, k_i - 1.
\]

**Proof.** Similar to the proof of Lemma 3. □

**Lemma 6.** Let \( X \) be the symmetric matrix of order \((k_1 + k_2 - 1) \times (k_1 + k_2 - 1)\) defined by
\[
X = \begin{bmatrix}
A_{1,k_1-1} & 0 & p_1 \\
0 & A_{2,k_2-1} & p_2 \\
p_1^T & p_2^T & 0
\end{bmatrix},
\]
where \( A_{1,k_1-1} \) and \( A_{2,k_2-1} \) are the symmetric tridiagonal matrices defined in Lemma 5 and \( p_1, p_2 \) as in Lemma 4. Then
\[
\det(\lambda I - X) = W(\lambda).
\]

**Proof.** Similar to the proof of Lemma 4. □

**Theorem 5.** Let \( Y \) be the symmetric tridiagonal matrix of order \((k_1 + k_2 - 1) \times (k_1 + k_2 - 1)\) defined by
\[
Y = \begin{bmatrix}
A_{1,k_1-1} & p_1 \\
p_1^T & 0 \\
p_2^T & J_{k_2-1}A_{2,k_2-1}J_{k_2-1}
\end{bmatrix}.
\]
Then
(a) \[ \sigma(A(\mathcal{T})) = (\cup_{j \in \Omega_1} \sigma(A_{1,j})) \cup (\cup_{j \in \Omega_2} \sigma(A_{2,j})) \cup \sigma(Y). \]
(b) The multiplicity of each eigenvalue of the matrix \( A_{i,j} \), as an eigenvalue of \( A(\mathcal{T}) \), is at least \((n_j - n_{i,j+1})\) for \( j \in \Omega_i \), and the eigenvalues of \( Y \) as eigenvalues of \( A(\mathcal{T}) \) are simple.

(c) The largest eigenvalue of \( Y \) is the largest eigenvalue of \( A(\mathcal{T}) \).

**Proof.** The proof for (a) and (b) is similar to the proof of Theorem 2, the matrices \( X \) and \( Y \) are similar. Finally, (c) follows from (a) of this theorem and the interlacing property for symmetric matrices. □

**Example 3.** For the tree in Example 1

\[
A_{1,2} = \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}, \quad A_{2,3} = \begin{bmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix},
\]

\[
Y = \begin{bmatrix} 0 & \sqrt{3} & \sqrt{2} & \sqrt{3} \\ \sqrt{3} & 0 & \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{3} & 0 & 1 \\ 0 & 1 & \sqrt{2} & 0 \end{bmatrix},
\]

\( \Omega_1 = \{1, 2\} \) and \( \Omega_2 = \{1, 3\} \). The eigenvalues of \( A(\mathcal{T}) \) are the eigenvalues of \( A_{1,1}, A_{1,2}, A_{2,1}, A_{2,3} \) and \( Y \). To four decimal places these eigenvalues are

\[
A_{1,1} : 0 \\
A_{1,2} : -1.7321 \quad 1.7321 \\
A_{2,1} : 0 \\
A_{2,3} : -1.7321 \quad 0 \quad 1.7321 \\
Y : -2.6762 \quad -1.7321 \quad -0.9153 \quad 0.9153 \quad 1.7321 \quad 2.6762
\]

**4. Some applications**

We recall some notions from the Graph Theory. Two distinct edges in a graph \( \mathcal{G} \) are independent if they are not incident with a common vertex. A set of pairwise independent edges of \( \mathcal{G} \) is called a matching and the matching number of \( \mathcal{G} \) is the cardinality of a matching of maximum cardinality. A tree is said to have a perfect matching if there exists a spanning forest whose components are solely paths on two vertices.

In this section, we apply the previous results to obtain the spectra of the adjacency matrix and Laplacian matrix of some particular trees.
4.1. The spectra of $L(M_{2m})$ and $A(M_{2m})$

Let $\mathcal{S}_{1,m}$ be a star on $(m + 1)$ vertices. Then $M_{2m}$ is the tree on $2m$ vertices obtained by adding a pendant edge to $m - 1$ of the pendant vertices of $\mathcal{S}_{1,m}$.

The tree $M_{10}$ is given below

\[
\text{Molitierno and Neumann [5] used the tree } M_{2m} \text{ to develop an upper bound on the algebraic connectivity of trees with perfect matchings. The main result in [5, Theorem 2.10] is: If a tree } T \text{ on at least 6 vertices has a perfect matching then the algebraic connectivity of } T \text{ is bounded above by } \frac{1}{2} (3 - \sqrt{5}), \text{ that is } \mu_{2m-1}(T) \leq \frac{1}{2} (3 - \sqrt{5}). \text{ Moreover, equality holds if and only if } T = M_{2m} \text{ for some positive integer } m \geq 3. \]

We may consider $M_{2m}$ as the union of two trees $T_1$ and $T_2$ in which the vertex root has degree equal to $m$, $T_1$ is an edge and $T_2$ is a tree with three levels being the number of vertices in level 2 and level 3 equal to $m - 1$. Thus

\[
k_1 = 2, \quad d_{1,1} = 1, \quad n_{1,1} = 1, \\
k_2 = 3, \quad d_{2,1} = 1, \quad d_{2,2} = 2, \quad n_{2,1} = n_{2,2} = m - 1, \\
d = m.
\]

Hence, $\Omega_1$ is the empty set and $\Omega_2 = \{2\}$. From Theorem 2 the spectrum of $L(M_{2m})$ is the union of the spectra of

\[
T_{2,2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 & \sqrt{m-1} \\ 1 & \sqrt{m-1} & 2 \\ n_{2,1} & n_{2,2} & 1 \end{bmatrix}.
\]

Then $\frac{1}{2} (3 - \sqrt{5})$ and $\frac{1}{2} (3 + \sqrt{5})$, the eigenvalues of $T_{2,2}$, are eigenvalues of $L(M_{2m})$ for any $m$. The characteristic polynomial of $T$ is

\[
det(\lambda I - T) = \lambda (\lambda - 2)(\lambda^2 - (m + 2) \lambda + m).
\]

Thus $0$, $2$, $\frac{1}{2} (m + 2 - \sqrt{m^2 + 4})$, $\frac{1}{2} (m + 2 + \sqrt{m^2 + 4})$ are the remainder eigenvalues of $L(M_{2m})$. Since the eigenvalues of $T$ are simple eigenvalues of $L(M_{2m})$, the multiplicity of $\frac{1}{2} (3 - \sqrt{5})$ and $\frac{1}{2} (3 + \sqrt{5})$ as eigenvalues of $L(M_{2m})$ is equal to $m - 2$. It follows that $\frac{1}{2} (3 - \sqrt{5})$ and $\frac{1}{2} (m + 2 + \sqrt{m^2 + 4})$ are the algebraic connectivity
and the Laplacian spectral radius of $M_{2m}$, respectively. Now from Theorem 5 the spectrum of $A(M_{2m})$ is the union of the spectra of

$$A_{2,2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{m-1} & 0 \\ 0 & \sqrt{m-1} & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$ 

Then $-1$ and $1$ are eigenvalues of $A(M_{2m})$ for any $m$, each of them with multiplicity $m - 2$. One can easily obtain that the eigenvalues of $Y$ are $\pm \frac{1}{2}(\sqrt{m + 3} + \sqrt{m - 1})$ and $\pm \frac{1}{2}(\sqrt{m + 3} - \sqrt{m - 1})$, each of them a simple eigenvalue of $A(M_{2m})$. Then $\frac{1}{2}(\sqrt{m + 3} + \sqrt{m - 1})$ is the spectral radius of $A(M_{2m})$.

### 4.2. The spectra of $L(G^n_m)$ and $A(G^n_m)$

The tree $G^n_m$, with $2m \leq n + 1$, is a tree on $n$ vertices obtained from a star $S_{1,n-m}$ by adding a pendant edge to $m - 1$ of the pendant vertices of $S_{1,n-m}$.

For example $G_3^{10}$ is

![Diagram of $G_3^{10}$]

Guo in [3] uses $G^n_m$ to obtain a sharp upper bound for the Laplacian spectral radius of a tree in terms of the matching number and number of vertices.

We may consider $G^n_m$, with $2m \leq n + 1$, as a rooted tree which is the union of $F_1$ and $F_2$, $F_1$ a tree with two levels having $n - 2m + 1$ vertices in level 2, $F_2$ a tree with three levels having $m - 1$ vertices in level 2 and $m - 1$ vertices in level 3. Then

$$k_1 = 2, \quad d_{1,1} = 1, \quad n_{1,1} = n - 2m + 1,$$

$$k_2 = 3, \quad d_{2,1} = 1, \quad d_{2,2} = 2, \quad n_{2,1} = n_{2,2} = m - 1,$$

$$d = n - m.$$ 

Hence, $\Omega_1 = \{1\}$ and $\Omega_2 = \{2\}$. From Theorem 2 the spectrum of $L(G^n_m)$ is the union of the spectra of

$$T_{1,1} = [1], \quad T_{2,2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and}$$
\[
T = \begin{bmatrix}
1 & \sqrt{n-2m+1} & n-m & \sqrt{m-1} \\
\sqrt{n-2m+1} & n-m & 2 & 1 \\
\sqrt{n-2m+1} & n-m & 2 & 1 \\
\sqrt{m-1} & 2 & 1 & 1
\end{bmatrix}.
\]

Then \(1, \frac{1}{2}(3 - \sqrt{5})\) and \(\frac{1}{2}(3 + \sqrt{5})\), the eigenvalues of \(T_{2,2}\), are eigenvalues of \(L(\mathcal{G}_m^n)\) for any \(m\). Also from Theorem 2 the multiplicity of the eigenvalues of \(L(\mathcal{T}_m^n)\) is as follows: \(1\) with multiplicity \(n - 2m\), \(\frac{1}{2}(3 - \sqrt{5})\) and \(\frac{1}{2}(3 + \sqrt{5})\) with multiplicity \(m - 2\) and the eigenvalues of \(T\) are simple. We see that the Laplacian spectral radius of \(\mathcal{G}_m^n\) is bounded above by \(\sqrt{n-2m+1} + n - m + \sqrt{m-1}\). Now from Theorem 5 the spectrum of \(A(\mathcal{G}_m^n)\) is the union of the spectra of

\[
A_{1,1} = [0], \quad A_{2,2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\text{ and }
\]

\[
Y = \begin{bmatrix}
0 & \sqrt{m-1} & 0 & 0 \\
\sqrt{m-1} & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

Then \(0, -1\) and \(1\) are eigenvalues of \(A(\mathcal{G}_m^n)\) for any \(m\). The eigenvalues of \(Y\) are simple eigenvalues of \(A(\mathcal{G}_m^n)\), \(0\) has multiplicity \(n - 2m\), \(-1\) and \(1\) have multiplicity \(m - 2\).

### 4.3. The spectra of \(L(\mathcal{W}_m^n)\) and \(A(\mathcal{W}_m^n)\)

Let \(2 \leq m \leq n - 1\). Let \(s = \left\lfloor \frac{n-1}{m} \right\rfloor\) and \(n - 1 = ms + t\), \(0 \leq t \leq m - 1\). Then \(\mathcal{W}_m^n\) is tree on \(n\) vertices obtained from a star \(\mathcal{S}_{1,m}\) by joining to each pendant vertex \(m - t\) paths of length \(s - 1\) and \(t\) paths of length \(s\).

In particular for the tree \(\mathcal{W}_{6}^{15}\), we have \(n = 15\), \(m = 6\), \(s = 2\) and \(t = 2\). Then \(\mathcal{W}_{6}^{15}\) is the tree on 15 vertices obtained from \(\mathcal{S}_{1,6}\) by joining to each pendant vertex 4 paths of length 1 and 2 paths of length 2. This tree is given below
Wu et al. prove in [7, Theorem 2] that: Of all trees of order \( n \) with \( m \) pendants vertices, \( n \) and \( m \) fixed, the maximum spectral radius of the adjacency matrix is attained at \( \mathcal{T}_m^m \).

We may consider \( \mathcal{T}_m^m \), with \( 2m \leq n + 1 \), as a rooted tree which is the union of \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \), \( \mathcal{T}_1 \) a tree having \( s + 1 \) levels with \( m - t \) vertices in each level and \( \mathcal{T}_2 \) a tree having \( s + 2 \) levels with \( t \) vertices in each level, except for both trees the level 1. Then

\[
k_1 = s + 1, \quad d_{1,1} = 1, \quad d_{1,2} = d_{1,3} = \cdots = d_{1,s} = 2,
\]

\[
n_{1,1} = n_{1,2} = \cdots = n_{1,s} = m - t,
\]

\[
k_2 = s + 2, \quad d_{2,1} = 1, \quad d_{2,2} = d_{2,3} = \cdots = d_{2,s+1} = 2,
\]

\[
n_{2,1} = n_{2,2} = \cdots = n_{2,s+1} = t.
\]

Hence, \( \Omega_1 = \{s\} \) and \( \Omega_2 = \{s + 1\} \). From Theorem 2 the spectrum of \( L(\mathcal{T}_m^m) \) is the union of the spectra of

\[
T_{1,s} = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1 \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{bmatrix}
\text{of order } s \times s,
\]

and

\[
P_1 = \begin{bmatrix}
0 & \cdots & \cdots & 0 & \sqrt{m - t}
\end{bmatrix},
\]

\[
P_2 = \begin{bmatrix}
0 & \cdots & \cdots & 0 & \sqrt{t}
\end{bmatrix}.
\]

Then the Laplacian spectral radius is bounded above by \( \sqrt{m - t} + m + \sqrt{t} \). From Theorem 5 the spectrum of \( A(\mathcal{T}_m^m) \) is the union of the spectra of

\[
A_{1,s} = \begin{bmatrix}
0 & 1 & \cdots & 1 \\
1 & 0 & \cdots & 1 \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{bmatrix}
\text{of order } s \times s,
\]
$A_{2,s+1}$ as $A_{1,s}$ of order $(s + 1) \times (s + 1)$

and

$$Y = \begin{bmatrix}
    A_{1,s} & p_1 \\
    p_1^T & 0 \\
    p_2^T & J_{s+1}A_{s+1}J_{s+1}
\end{bmatrix}.$$

The multiplicity of the eigenvalues of these submatrices, as eigenvalues of $L(\mathcal{W}_m^n)$ and $A(\mathcal{W}_m^n)$ respectively, is as follows: the eigenvalues of $T_{1,s}$ and $A_{1,s}$ have multiplicity $m - t - 1$, the eigenvalues of $T_{2,s+1}$ and $A_{2,s+1}$ have multiplicity $t - 1$, the eigenvalues of $T$ and $Y$ are simple. We see that the spectral radius of $A(\mathcal{W}_m^n)$ is bounded above by $\sqrt{m - t + \sqrt{t}}$.

In particular, the eigenvalues $L(\mathcal{W}_{15}^6)$ are the eigenvalues of

$$T_{1,2} = \begin{bmatrix}
    1 & 1 \\
    1 & 2
\end{bmatrix}, \quad T_{2,3} = \begin{bmatrix}
    1 & 1 & 1 \\
    1 & 2 & 1 \\
    1 & 1 & 2
\end{bmatrix}$$

and

$$T = \begin{bmatrix}
    1 & 1 & 2 & 2 \\
    1 & 2 & 2 & \sqrt{2} \\
    2 & 6 & \sqrt{2} & 2 \\
    2 & \sqrt{2} & 2 & 1 \\
    \sqrt{2} & 2 & 1 & 2 \\
    1 & 2 & 1 & 1
\end{bmatrix}.$$

Finally, the eigenvalues $A(\mathcal{W}_{15}^6)$ are the eigenvalues of

$$A_{1,2} = \begin{bmatrix}
    0 & 1 \\
    1 & 0
\end{bmatrix}, \quad A_{2,3} = \begin{bmatrix}
    0 & 1 & 1 \\
    1 & 0 & 1 \\
    1 & 1 & 0
\end{bmatrix}$$

and

$$Y = \begin{bmatrix}
    0 & 1 & 2 & \sqrt{2} & \sqrt{2} \\
    1 & 0 & 2 & 0 & 0 \\
    2 & 0 & \sqrt{2} & 0 & 1 \\
    2 & \sqrt{2} & 0 & 1 & 0 \\
    \sqrt{2} & 2 & 0 & 1 & 0
\end{bmatrix}.$$

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References