Static and free vibration analysis of laminated glass beam on viscoelastic supports

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Abstract

Static behavior and free vibration analysis of laminated glass beam on viscoelastic supports are performed. For the static case, an analytical way is developed for analyzing and optimization of laminated glass beam with general restraints at the boundaries. In the case of free linear vibrations, the modal properties of the glass are determined using a finite element method which is a powerful tool in the design of support damping treatment of a sandwich glass for passive vibration control.

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1. Introduction

The problem of dissipating energy in structures so as to reduce vibration and noise, and to avoid fatigue failure, is becoming an increasingly important consideration in mechanical design. Initially reduced to aerospace field, it is now applied in almost all industrial fields. Viscoelastic materials are used in damping technique for vibration and noise control. This kind of control is called passive control. Laminated glasses are widely used in contemporary buildings as architectural glazing. They are also used in automotive industries as windscreen. Laminated glass comprises two glass layers bonded together by an elastomeric polymer called polyvinyl butyral (PVB). The PVB-material keeps the shards of broken glass plate in the frame of the glass unit after the failure and makes them safety. The elastic modulus of the PVB-material is very less than the glass one. The lower stiffness of the central layer and the difference between in-plane displacements of the faces induce an important transverse shear in the viscoelastic layer. Thus, damping is introduced by this transverse shear. The complex nonlinear behavior of the PVB-material, which is viscoelastic, temperature and frequency dependent, makes the modeling of laminated glass very delicate. Besides this material nonlinearity, one can consider...
the geometrical nonlinearity of thin plates under transverse loadings. One of the previous research on laminated glasses is the mathematical model of Vallabhan et al. (1993) for analyzing the bending behavior of laminated glass units. Based on the minimization of the total potential energy, five nonlinear partial differential equations with their associated boundary conditions are obtained and solved iteratively by using finite difference method (FDM) over relaxation. The von Karman’s nonlinear theory of plates is used to model the glass plates. The bending and membrane strain energy of the glass plates as well as the shear strain energy of the PVB-interlayer are included in the total potential energy. Experiments are also conducted with strain gauge measurements to validate the model. Aşık (2003) developed an algorithm for implicit integration of these five equations and their unconditionally stable solution in order to overcome numerical difficulties encountered in the previous model. Other works are devoted to laminated glasses among which one can note Norville et al. (1998) simple multilayer beam model, beam experiments performed by Behr et al. (1993), not forgetting Edel (1997) who studies temperature effect on the behavior of laminated glass through three-point bending experiments and finite elements (FE) model with ABAQUS software. Duser et al. (1999) use FE method to model laminated glasses under transverse loadings. 3-D solid elements are used to model the layers and their interaction. They model the PVB-interlayer as linear viscoelastic and nonlinear solution is performed. A statistical model based on two-parameters Weibull’s law of distribution is developed for the breakage and strength determination of the glass plates. Aşık and Tezcan (2005) develop a mathematical model of laminated beams, that is based on nonlinear strain–displacement relationship. The nonlinearity disappears in the model for a simply supported boundary condition and a closed-form solution is derived. They use the FDM to solve the nonlinear problem when clamped–clamped boundary condition is considered. The model is then used to investigate the linear and nonlinear behavior of symmetric laminated glass beam in comparison with the laminated glass plate investigated before by Aşık (2003). More recently, Ivelin Ivanov (2006) developed a model for analysis, modeling and optimization of laminated glasses as plane beam. The analytical solutions derived by Ivanov in the case of simply supported boundary condition and a distributed transverse loads coincide with those derived by Aşık and Tezcan (2005) for the same conditions. A parametric study is then carried out by Ivelin Ivanov (2006) to investigate the influence of the three layers thicknesses on the deflection and stresses when the glass beam is under transverse load. The analytical expressions derived are also used for the optimization of triplex glass. The objective function of this optimization problem is the areal weight of the glass laminate and the constraints are on the layers thickness, the maximum deflection (required for rigidity), the shear strain in the PVB-interlayer (to avoid delamination of the layers (Rahul-Kumar et al., 2000)), the compressive and tensile stress (for strength requirements). The results of the optimization show that the inner layer of the laminated glass unit under external transverse loads should be thinner than the external glass layers for lightweight structure design of the architectural glazing. Through this literature review, one can notice that only classical boundary conditions of laminated glass structures are analyzed. Also, there are few works on the investigation of modal properties of the laminated glass structures under various boundary supports. Generally, windshields are bond to the automotive with viscoelastic material. Strictly speaking, the laminated glass is not entirely clamped to the vehicle. Therefore, the effect of viscoelastic support on the static and damping behaviors of laminated glass beam is needed to be investigated. Thus, the main aims of this study are to develop a mathematical model for the static and free vibration analysis of laminated beams on viscoelastic supports (see Fig. 1). The two extremities of the beam are supported by a viscoelastic material. This boundary condition can be modeled by two springs (rotational, $K_R$, and translational, $K_T$) at each extremity of the beam (Haberman, 2006; Kang and Kim, 1996). Because of the viscoelastic nature of the boundary material, the springs rigidity moduli are complex numbers. In static domain, the imaginary parts of these rigidity moduli are zero but they are taken into account when dealing with dynamic analysis. To this end, in the present study, the mathematical models are introduced for laminated glass beam by using variational formulation based on the principle of virtual works (VW). For the static analysis, two coupled differential equations with six associated boundary conditions governing the behavior of the sandwich beam are obtained and the closed-form solutions are presented. The influence of the boundary parameters on the strain and stress fields is then investigated. The free vibration analysis is carried out based on formulation using FE method and the effect of the boundary damping treatment on modal properties of the laminated glass is also investigated. In terms of application, these models can be used to select the appropriate boundary parameters to obtain a desirable static or damping behavior of such laminated glass structures.
2. Model of static analysis

In this section, a model of static analysis of laminated glass beam is performed. Fig. 1 shows a pertinent approximation of viscoelastic boundary conditions as linear and rotational springs.

2.1. Assumptions and displacement fields

The glass unit is considered under the following assumptions, common to many authors (Daya and Potier-Ferry, 2002; Mead and Markus, 1969; Rao, 1978; Sainsbury and Zhang, 1999):

- plane sections initially normal to the mid surface remain plane and normal to the mid surface during the bending for each glass ply but not for the entire beam;
- the transverse normal stress $\sigma_z$ is small compared to the axial normal stress $\sigma_x$ which is used in classical beam theory;
- the three layers have the same transversal displacement $w(x)$;
- no slip occurs at the interfaces between central and elastic layers;
- the PVB-interlayer only transfers shear, and has negligible compression in transverse direction.

It’s assumed that the PVB-interlayer is homogeneous, isotropic and linear viscoelastic material. Its Poisson ratio is supposed constant real number while its shear modulus is constant complex number. The real part of the complex modulus represents the capability of the viscoelastic material to store energy and the imaginary part implies energy dissipating in the material. Note that these assumptions are often used in damped viscoelastic sandwich structures studies (Daya and Potier-Ferry, 2002; Rao, 1978; Sainsbury and Zhang, 1999). The model consists of a three layers sandwich beam with length $L$ and width $b$.

In the following, the subscript $i$ is related to layer $i$; thus $i = 1$, $i = 2$ and $i = 3$ correspond, respectively, to the top, the central and the bottom layers; $h_i$ represents the thickness of layer $i$. Each point $M$ of the beam is defined by its cartesian coordinates in the reference where the $x$-axis is supported by the mid surface of the interlayer and along the beam’s length while the $y$-axis and the $z$-axis are, respectively, along the beam’s width and the beam’s thickness. The origin $O$ of the reference is taken in the middle of the beam. Based on these assumptions, a unified displacement fields is then presented for the three layers. After the bending of the beam, the point $M_i$ of the $i$th layer is defined in the plane $(x,z)$ by Eq. (1):

$$\overrightarrow{OM_i}(x,z) = \left( u_i(x) + (z - z_i)\beta_i(x) \right)$$

$$\left( w(x) \right).$$

(1)
In Eq. (1), \( u_i(x) \) represents the axial displacement of the middle surface of the \( i \)th layer, \( z_i \), its coordinate, \( \beta_i(x) \) is the rotation of the normal to \( i \)th layer’s mid-plane, and \( w(x) \), the common transverse displacement. According to the above assumptions, one can verify that:

\[
\beta_1(x) = \beta_3(x) = -\frac{\mathrm{d}w(x)}{\mathrm{d}x}, \quad z_1 = \frac{h_1 + h_2}{2}, \quad z_2 = 0 \quad \text{and} \quad z_3 = -\frac{h_3 + h_2}{2}.
\]

(2)

Considering the continuity of the displacements at the interfaces between the central and the faces layers, one gets the following relations:

\[
\begin{align*}
\left\lbrace\begin{array}{l}
\quad u_1(x) + u_3(x) + h_1 - h_3 \frac{\mathrm{d}w(x)}{\mathrm{d}x}, \\
\quad \frac{h_1 + h_3}{2h_3} \frac{\mathrm{d}w(x)}{\mathrm{d}x}.
\end{array}\right.
\end{align*}
\]

(3)

2.2. Formulations of the static problem

The bending and membrane strains of the glass layers and the shear strain of the PVB-interlayer are considered in the following formulation. The bending and membrane effects of the PVB-interlayer are neglected as it is considered and proved in earlier works such as (Aşik and Tezcan, 2005; Ivelin Ivanov, 2006; Rao, 1978; Sainsbury and Zhang, 1999). The beam is subjected to distributed load \( q \) and to concentrated load \( P \) at its center. According to the symmetry loads and boundary conditions, only a half of the sandwich is modeled so the right portion \( x \in [0, L/2] \) is considered. In static domain, the principle of virtual works (VW) is given by:

\[
\delta P_{\text{int}} + \delta P_{\text{ext}} = 0,
\]

(4)

where \( \delta P_{\text{int}} \) denotes the VW of the internal forces and \( \delta P_{\text{ext}} \), the VW done by external forces. Their expressions are given, respectively, by:

\[
\begin{align*}
\delta P_{\text{int}} &= -\int_0^L [EIw'' \delta w' + N_1 \delta u_1' + N_3 \delta u_3' + (N_1 + N_3)w' \delta w' + G_2bh_2\gamma_2 \delta \gamma_2] \mathrm{d}x, \\
\delta P_{\text{ext}} &= \int_0^L qbdw \mathrm{d}x - KTw(L/2) \delta w(L/2) - KRw'(L/2) \delta w'(L/2) + \frac{P}{2} \delta w(0),
\end{align*}
\]

(5)

where \( \delta u_1, \delta u_3, \delta w \) and \( \delta \gamma_2 \) denotes the virtual displacements and the virtual shear strain. In Eq. (5), \( N_1, N_3 \) and \( \gamma_2 \) are the axial forces in the top and bottom layers and the shear strain in the PVB-interlayer given by:

\[
\begin{align*}
N_1 &= EA_1 \left[ u_1' + \frac{1}{2} (w')^2 \right], \\
N_3 &= EA_3 \left[ u_3' + \frac{1}{2} (w')^2 \right], \\
\gamma_2 &= \frac{\eta_3 - \eta_1}{h_3} + \frac{\eta_1}{h_2} w'.
\end{align*}
\]

(6)

The terms in Eqs. (5) and (6) are: \( E \) the elastic modulus of glass, \( G \) the shear modulus of the delayed elasticity of the PVB-interlayer, \( A_1 \) and \( A_3 \) the cross area of top and bottom glass ply, \( I \) the sum of quadratic moments of the two glass layers, \( K_T \) and \( K_R \) are, respectively, the stiffnesses of translation and rotation springs at the extremities of the beam, \( h_0 = h_2 + (h_1 + h_3)/2 \) and the notation \( (f', f'') \) represents \( (\mathrm{d}f/\mathrm{d}x, \mathrm{d}^2f/\mathrm{d}x^2) \) where \( f \) is function of \( x \). The nonlinear term in Eq. (5) is \( (N_1 + N_3)w' \). Inserting Eq. (5) in Eq. (4), one gets two equations related to the virtual quantities \( \delta u_1 \) and \( \delta u_3 \) in one hand and \( \delta w \) in the other which are expressed as follows:
\[
\int_0^\frac{L}{2} [N_1 \delta u'_1 + G_2 b'_{\gamma_2} \delta u_1] \, dx = 0, \\
\int_0^\frac{L}{2} [N_3 \delta u'_3 - G_2 b'_{\gamma_2} \delta u_3] \, dx = 0, \\
\int_0^\frac{L}{2} [E I w'' \delta w'' + (N_1 + N_3) w' \delta w' + G_2 b h_0_{\gamma_2} \delta w'] \, dx = \int_0^{\frac{L}{2}} q b \delta w \, dx + \delta U,
\]
with
\[
\delta U = - K_T w \left( \frac{L}{2} \right) \delta w \left( \frac{L}{2} \right) - K_R \left( \frac{L}{2} \right) \delta w' \left( \frac{L}{2} \right) + \frac{P}{2} \delta w(0).
\]
Integrating by parts Eq. (7), one gets two differential equations with their associated boundary conditions which are:

\[
\begin{align*}
N'_1 &= G_2 b_{\gamma_2}, \\
N'_3 &= -G_2 b_{\gamma_2}, \\
N_1 \left( \frac{L}{2} \right) &= N_3 \left( \frac{L}{2} \right) = 0, \\
N'_1(0) &= N'_3(0) = 0.
\end{align*}
\]

Adding together the two terms of Eq. (9), one obtains \( N'_1 + N'_3 = 0 \), which implies that the global normal force \( N_1 + N_3 \) is constant. Considering the natural boundary conditions, Eq. (10), it is evident that \( N_1 + N_3 = 0 \) at the support and all along the length. The consequence is that the nonlinear term in Eqs. (5) and (8) is also zero for all \( x \) along the beam. This remark implies that the translational and rotational springs at the beam’s extremities do not induce nonlinear behavior of the laminated glass unit. Therefore, the beam will always behave in linear fashion no matter how is its lateral deformation. The notation \( N = N_1 = -N_3 \) is adopted in the following. Based on the above conclusion, two successive partial integrations of Eq. (8) and the use of Eqs. (9) and (10) lead to the following set of equations that define completely the problem:

\[
\begin{align*}
N' &= G_2 b_{\gamma_2}, \\
(E I w'')'' - G_2 b h_0_{\gamma_2} = q b, \\
N \left( \frac{L}{2} \right) &= N'(0) = 0, \\
w'(0) &= 0, \\
w'' \left( \frac{L}{2} \right) &= - \frac{P}{E I}, \\
w'''(0) &= \frac{P}{E I}, \\
w'' \left( \frac{L}{2} \right) &= \frac{P}{E I} N' \left( \frac{L}{2} \right) + \frac{P}{E I} w' \left( \frac{L}{2} \right).
\end{align*}
\]

The influence of the nondimensional parameters, \( g_R = K_R L / (E I) \) and \( g_T = K_T L^3 / (E I) \), is discussed in Section 2.5.

2.3. Analytical solution of the problem

Here, an analytic solution is given for the above problem posed by Eqs. (11) and (12). Ordinary differential equation is first obtained for the function \( N(x) \) which is then used to derive the common lateral displacement \( w(x) \). It can be proved that these two functions govern the overall behavior of the sandwich beam. Using Eq. (6), one obtains:

\[
\frac{u'_i - u'_j}{E b} = \frac{N}{E b} \left( \frac{h_1 + h_3}{h_1 h_3} \right).
\]

In order to use Eq. (13), the first equation of Eq. (11) is once differentiated with respect to \( x \) and leads to Eq. (14):
\[ N'' = \frac{G_2}{E h_2} \left( \frac{h_1 + h_1}{h_1 h_3} \right) N + \frac{G_2 b h_0}{h_2} w''. \]  

(14)

Using Eq. (14) and the second equation of Eq. (11), one gets the following fourth-order ordinary differential equation:

\[ N'' = \alpha^2 N'' = \mu q, \]  

where

\[ \alpha = \sqrt{\frac{G_2 b}{E h_2} \left( \frac{h_0^2}{I} + \frac{A_1 + A_3}{A_1 A_3} \right)}, \quad I = \frac{b (h_3^2 + h_5^2)}{12} \quad \text{and} \quad \mu = \frac{G_2 b h_0^2}{E h_2}. \]  

(16)

Using the standard beam parameters \((g, \text{shear parameter})\) and \((Y, \text{geometric parameter})\), defined by Mead and Markus (1969) and Rao (1978):

\[ g = \frac{G_2 b L^2}{4 E h_2} \left( \frac{A_1 + A_3}{A_1 A_3} \right) \quad \text{and} \quad Y = \frac{h_0^2}{I} \left( \frac{A_1 A_3}{A_1 + A_3} \right), \]

one obtains:

\[ \alpha = \frac{2}{L} \sqrt{g (1 + Y)} \quad \text{and} \quad \mu = \frac{4 b}{L^2 h_0} g Y. \]

The general solution of Eq. (15) is:

\[ N(x) = - \frac{q \mu x^2}{2 \alpha x} + a_1 + a_2 x + a_3 \cosh(\alpha x) + a_4 \sinh(\alpha x), \]  

(17)

where \(a_1, a_2, a_3, a_4\) are the integration constants. The Eq. (14) can be written as follows to determine the function \(w(x)\) when the function \(N(x)\) is known:

\[ w'' = \eta N'' + \delta N, \]  

(18)

where:

\[ \eta = \frac{h_2}{G_2 b h_0} \quad \text{and} \quad \delta = - \left( \frac{A_1 + A_3}{E A_1 A_3 h_0} \right). \]  

(19)

With the expression of \(N(x)\), Eq. (17), one gets \(w(x)\) by integrating twice Eq. (18). Two new integration constants \(a_5\) and \(a_6\) are introduced. The following expression for \(w(x)\) is then obtained:

\[ w(x) = \frac{\delta + x^2 \eta f_1(x)}{2 \alpha^2} + \frac{f_2(x)}{24 \alpha^2}, \]  

(20)

with:

\[ \left\{ \begin{array}{l}
 f_1(x) = a_3 \cosh(\alpha x) + a_4 \sinh(\alpha x), \\
 f_2(x) = - q \delta \mu x^4 + 4 \alpha^2 \delta a_2 x^3 - 12 (q \eta \mu - x^2 \delta a_1) x^2 + 24 \alpha^2 (a_5 x + a_6) \end{array} \right. \]  

(21)

Then the six constants of integration \(a_1, a_2, a_3, a_4, a_5\) and \(a_6\) are determined using the six boundary conditions of Eq. (12). They are listed in Appendix A. It can be noticed through the expressions of the integration constants \(a_i\) that only \(a_6\) depends on the translational spring \(K_T\). Therefore, \(K_T\) influences only the common transverse displacement \(w(x)\). However the expression of \(w(x)\), Eq. (20), shows that \(w'(x)\) and \(w''(x)\) do not depend on \(a_6\). So the stress and the strain fields of the laminated glass beam are only affected by the rotational spring \(K_R\). The strain field is obtained from Eq. (1) and then the stress field is deduced by the classical Hooke's law. One can verify that the normal stress in the glass layers through their thickness is defined by:

\[ \sigma_i(x, z) = \frac{N_i(x)}{A_i} - E(z - z_i) w''(x). \]  

(22)
Then, the normal stresses at the top and bottom surfaces of the glass plies and the shear strain in the PVB-interlayer are given by the expressions:

\[ \sigma_{\text{top}}(x) = \frac{N(x)}{A_1} - \frac{Eh_1}{2} w''(x), \]

\[ \sigma_{\text{bot}}(x) = \frac{N(x)}{A_1} + \frac{Eh_1}{2} w''(x), \]

\[ \sigma_{3\text{top}}(x) = -\frac{N(x)}{A_3} - \frac{Eh_3}{2} w''(x), \]

\[ \sigma_{3\text{bot}}(x) = -\frac{N(x)}{A_3} + \frac{Eh_3}{2} w''(x), \]

\[ \gamma_2(x) = \frac{N'(x)}{G_2 b} = -\frac{a_2 + \alpha \sinh(\alpha x) a_3 + \alpha \cosh(\alpha x) a_4}{G_2 b}. \]

### 2.4. Validation of the model

In order to validate the present model, the closed-form solution is applied to the particular case of simply supported beam subjected to distributed load \( q \). This case has been investigated by Aşık and Tezcan (2005) and Ivelin Ivanov (2006). To represent this case in the present model, one takes \( K_R = 0, K_T = +\infty \) and \( P = 0 \). Inserting these values in Eqs. (17) and (20), one gets the following expressions for the functions \( N(x) \) and \( w(x) \) which are identical, respectively, to expressions (26) and (31) derived by Ivelin Ivanov (2006):

\[ N(x) = \frac{q \mu}{2\alpha^2} \left[ 2 \cosh(\alpha x) \right] - x^2 + \frac{L^2 x^2 - 8}{4\alpha^2}, \]

\[ w(x) = \frac{q \mu}{\alpha^6} \left[ (\delta + \alpha^2 \eta) \cosh(\alpha x) \right] - \frac{\alpha^2 \delta}{24} x^4 + \frac{A_1}{384} x^2 + \frac{A_2}{384}, \]

where:

\[ A_1 = -192 \alpha^2 \delta + 24L^2 \alpha^4 \delta - 192 \alpha^4 \eta, \]

\[ A_2 = -384 \alpha^4 \delta + 48L^2 \alpha^2 \delta - 5L^4 \alpha^4 \delta - 384 \alpha^2 \eta + 48L^2 \alpha^4 \eta. \]

The model is next validated by the work of Aşık and Tezcan (2005). Their model consists of a laminated glass beam simply supported at its ends and subjected to distributed load \( q \) and a point load \( P \) at its center. This case is represented in this present model by assigning \( K_R = 0, K_T = +\infty \). Then, one can verify, with the help of this variable change, \( X = x + L/2 \), that Eqs. (17) and (20) of this paper lead, respectively, to Eqs. (17) and (19) derived by Aşık and Tezcan (2005). These elementary validations show the effectiveness of the model derived herein.

### 2.5. Effect of the parameters \( K_T \) and \( g_R \) on the stress, strain and displacement

As it is said in Section 2.3, there is no effect of \( K_T \) on the normal stresses and strains. Thus, the effect of \( K_R \) on the normal stresses and strains is investigated in this section through the nondimensional parameter \( g_R \). The effect of \( K_T \) on the common transverse displacement \( w(x) \) is studied through the nondimensional parameter \( g_T \). The material, geometrical and load data are as: \( L = 0.8 \) m, \( b = 0.1 \) m, \( E = 70 \) GPa, \( G_2 = 0.69 \) MPa, \( q = 1 \) kPa, \( P = 0 \) N, \( Y = 3.7007 \), \( g = 0.8411 \).

Figs. 2–4(a) show, respectively, the effect of \( g_R \) on the normal stresses of the top and bottom surfaces of the glass layers and also on the shear strain in the PVB-interlayer. The chosen normal stresses and shear strain is justified by certain constraints. Indeed, for the strength requirements of the glass layers, there is allowable compressive stress \( \sigma_c \) and tensile stress \( \sigma_t \) accepted. The constraint on the shear strain in the PVB-interlayer
is necessary to avoid delamination of the layers (Jagota et al., 2000). Fig. 4(b) shows the influence of $g_R$ and $g_T$ on the maximum deflection, $\text{max } w(x)$, of the laminated glass beam.

From Fig. 2, one observes that when the laminated glass is subjected to a pressure, $q$, on its top glass layer, the top surface and bottom surface of this glass layer are entirely in compression and tension, respectively, for lower values of $g_R$. When increasing the value of $g_R$, tension and compressive regions appear from the extremities of the top glass's top surface and bottom surface, respectively. Fig. 3 shows the same behavior for the bottom glass layer.

The PVB-interlayer shear strain decreases when the value of $g_R$ increases, Fig. 4(a). The maximum deflection is not affected by $g_R$ (see Fig. 4(b)), while its value is little bit affected by the value of $g_T$, Fig. 4(b).
Thus, one remarks that these parameters have no negligible effect on the stresses and strains in the laminated glass beam unit. It is therefore very important to take into account the effect of this boundary condition when dealing with laminated glass unit.

3. Free vibrations analysis of laminated glass beam

In this section, the free vibration analysis of the laminated glass beam is performed. The beam is supported at its ends by a viscoelastic material that is modeled by two complex springs. A simplified model is proposed here to carry out the modal properties of the laminated glass beam with boundary damping treatment. With this simple model, one can investigate the effect of viscoelastic springs on damping properties of the laminated glass beam. Only linear vibrations are considered in this part. The viscoelastic behavior is introduced by a complex constant modulus as in many references (Daya and Potier-Ferry, 2001; Ferry, 1970; Rao, 1978; Sainsbury and Zhang, 1999). Dealing with the harmonic regime here, all the following functions of $x$ and $z$ (displacement, strain and stress) are mathematically complex multiplied scalarly by the term $e^{j\omega t}$ where $\omega$ is the pulsation, $t$ the time and $j = \sqrt{-1}$. This term disappears in the final equations and therefore it does not appear anywhere in the following formulation. Besides the notations adopted in the static analysis, $G_2 = G_2(1 + j\eta_2)$ is the complex shear modulus of the PVB-interlayer where $\eta_2$ is its loss factor, $K_T = K_T(1 + j\eta_T)$ and $K_R = K_R(1 + j\eta_R)$ are the complex translational and rotational springs stiffness, respectively, with their associated loss factors $\eta_T$ and $\eta_R$. As in static study, similar nondimensional parameters $\tilde{g}_R = K_R L/(EI) = g_R(1 + j\eta_R)$ and $\tilde{g}_T = K_T L^3/(EI) = g_T(1 + j\eta_T)$ are also introduced. It is well known that when material properties are viscoelastic, the stiffness constants become complex and depend nonlinearly on the vibration frequency and the temperature (Daya and Potier-Ferry, 2001; Ferry, 1970). However, the stiffness matrix is often assumed to be constant in certain frequency band. Thus, the free vibration analysis of laminated glass beam is introduced by a complex eigenvalue problem. In the following, this problem is formulated and solved using FE method.

3.1. Formulation using the FE method

The displacement field, Eq. (1), is again considered here. The differences are that the nonlinear term is neglected in the following formulation and the origin $O$ of the reference is taken at the left end of the beam. The principle of VW is expressed now as follows:

Thus, one remarks that these parameters have no negligible effect on the stresses and strains in the laminated glass beam unit. It is therefore very important to take into account the effect of this boundary condition when dealing with laminated glass unit.

3. Free vibrations analysis of laminated glass beam

In this section, the free vibration analysis of the laminated glass beam is performed. The beam is supported at its ends by a viscoelastic material that is modeled by two complex springs. A simplified model is proposed here to carry out the modal properties of the laminated glass beam with boundary damping treatment. With this simple model, one can investigate the effect of viscoelastic springs on damping properties of the laminated glass beam. Only linear vibrations are considered in this part. The viscoelastic behavior is introduced by a complex constant modulus as in many references (Daya and Potier-Ferry, 2001; Ferry, 1970; Rao, 1978; Sainsbury and Zhang, 1999). Dealing with the harmonic regime here, all the following functions of $x$ and $z$ (displacement, strain and stress) are mathematically complex multiplied scalarly by the term $e^{j\omega t}$ where $\omega$ is the pulsation, $t$ the time and $j = \sqrt{-1}$. This term disappears in the final equations and therefore it does not appear anywhere in the following formulation. Besides the notations adopted in the static analysis, $G_2 = G_2(1 + j\eta_2)$ is the complex shear modulus of the PVB-interlayer where $\eta_2$ is its loss factor, $K_T = K_T(1 + j\eta_T)$ and $K_R = K_R(1 + j\eta_R)$ are the complex translational and rotational springs stiffness, respectively, with their associated loss factors $\eta_T$ and $\eta_R$. As in static study, similar nondimensional parameters $\tilde{g}_R = K_R L/(EI) = g_R(1 + j\eta_R)$ and $\tilde{g}_T = K_T L^3/(EI) = g_T(1 + j\eta_T)$ are also introduced. It is well known that when material properties are viscoelastic, the stiffness constants become complex and depend nonlinearly on the vibration frequency and the temperature (Daya and Potier-Ferry, 2001; Ferry, 1970). However, the stiffness matrix is often assumed to be constant in certain frequency band. Thus, the free vibration analysis of laminated glass beam is introduced by a complex eigenvalue problem. In the following, this problem is formulated and solved using FE method.

3.1. Formulation using the FE method

The displacement field, Eq. (1), is again considered here. The differences are that the nonlinear term is neglected in the following formulation and the origin $O$ of the reference is taken at the left end of the beam. The principle of VW is expressed now as follows:
\[ \delta P_{\text{int}} + \delta P_{\text{ext}} = \delta P_{\text{acc}}, \quad (29) \]

where \( \delta P_{\text{int}} \) denotes the VW of the internal forces, \( \delta P_{\text{ext}} \), the VW done by external forces and \( \delta P_{\text{acc}} \) represents the resulting VW put into the system as acceleration. These terms are defined by:

\[
\delta P_{\text{int}} = - \int_0^L \left[ E I w'' \delta w'' + E A_1 u_1'' \delta u_1'' + E A_3 u_3'' \delta u_3'' + \tilde{G}_2 b h_2 \gamma_2 \delta \gamma_2 \right] \, dx, \quad (30)
\]

\[
\delta P_{\text{ext}} = - \tilde{K}_T [w(0) \delta w'(0) + w(L) \delta w'(L)]
\]

- \( \tilde{K}_R [w'(0) \delta w'(0) + w'(L) \delta w'(L)] \),

\[
\delta P_{\text{acc}} = - \Omega^2 \int_0^L \left[ \sum_{k=1}^3 \rho_i A_k w \delta w + \rho_1 (A_1 u_1 \delta u_1 + A_3 u_3 \delta u_3) \right] \, dx. \quad (32)
\]

In Eq. (32), \( \rho_i \) denotes the density of the \( i \)th layer. The other terms are defined in previous subsections. The FE discretization is used for Eqs. (30)–(32). This study employs one-dimensional elements bounded by two nodal points. Each node has four degrees of freedom (DOF) which describe the longitudinal (\( u_1, u_3 \)) and transverse displacements (\( w_n \)) and slope (\( \theta_n \)) of the sandwich beam at the node. The total set of nodal displacements for the element is:

\[
\{ q^e \} = \begin{bmatrix} q_i \\ q_j \end{bmatrix} = \begin{bmatrix} w \\ u_1 \\ u_3 \end{bmatrix} = \begin{bmatrix} W \\ U_1 \\ U_3 \end{bmatrix} \{ q^e \}. \quad (33)
\]

With the classical polynomial shape functions, the displacements field vector may be written as:

\[
\begin{bmatrix} w \\ u_1 \\ u_3 \end{bmatrix} = \begin{bmatrix} W \\ U_1 \\ U_3 \end{bmatrix} \{ q^e \}. \quad (34)
\]

\([W], [U_1], [U_3]\) are the shape functions listed in Appendix \( B \):

\[
[W] = \begin{bmatrix} N_1(\xi) & N_2(\xi) & 0 & 0 & N_3(\xi) & N_4(\xi) & 0 & 0 \end{bmatrix},\]

\[
[U_1] = \begin{bmatrix} 0 & 0 & N_5(\xi) & 0 & 0 & N_6(\xi) & 0 \end{bmatrix},\]

\[
[U_3] = \begin{bmatrix} 0 & 0 & N_5(\xi) & 0 & 0 & N_6(\xi) \end{bmatrix}.\]

Inserting Eq. (34) into Eq. (29), one gets the following eigenvalue problem:

\[
([K'] - \Omega^2[M']) \{ q^e \} = 0, \quad (35)
\]

where \([K']\) and \([M']\) are the element stiffness and mass matrices, respectively. Their expressions are given in Appendix \( B \). The element stiffness and mass matrix are then assembled to get the overall complex eigenvalue problem:

\[
([K] - \Omega^2[M]) \{ q \} = 0. \quad (36)
\]

Remember that, in general case Eq. (36) is a complex nonlinear (due to the frequency dependent of the stiffness matrix) eigenvalue problem that characterizes the free vibration of viscoelastic structures. There are great numerical difficulties in solving Eq. (36), because the components of \([K]\) are general complex valued and frequency dependent. Many approaches have been proposed to solve the problem Eq. (36): the complex eigenvalues method, the modal strain energy method (MSE), the direct frequency response method, the asymptotic approach and the order-reduction-iteration technique. A review of these methods can be found in Refs. Chen et al. (1999), Daya and Potier-Ferry (2001, 2002) and Duigou et al. (2003). In this paper, the complex linear eigenvalue problem (36) is solved using MATLAB (2006) software.

Solving Eq. (36), one gets the natural frequencies \( \omega \) and their associated loss factor \( \eta \) for the laminated glass beam from the following formula (DiTaranto and McGraw, 1969):

\[
\Omega^2 = \omega^2 (1 + j \eta). \quad (37)
\]
3.2. Validation test

To validate the present work, it is considered here a uniform symmetrical three layers damped sandwich beam with clamped-free boundary conditions analyzed by Sainsbury and Zhang (1999) using the Galerkin element method (GEM). Geometrical and material parameters are:

\[ E = 7.037 \times 10^4 \text{ MPa}, \quad G_2 = 0.7037 \text{ MPa}, \quad \rho_1 = \rho_3 = 2770 \text{ kg m}^{-3}, \quad \rho_2 = 970 \text{ kg m}^{-3}, \quad h_1 = h_3 = 1.52 \text{ mm}, \quad h_2 = 0.127 \text{ mm} \quad \text{and} \quad \eta_2 = 0.3, \quad L = 177.8 \text{ mm}, \quad b = 12.7 \text{ mm}. \]

The comparison of the results in Table 1 validates the effectiveness of the model.

3.3. Parametric study and discussion

The influence of \( g_T, g_R, \eta_T \) and \( \eta_R \) on the modal properties of the laminated glass beam is now investigated. The model is applied to laminated glass beam which geometrical and material data are defined in Table 2. For a sandwich beam with viscoelastic boundary supports, the eigenvalues (natural frequencies and modal loss factors) can be obtained from Eq. (36). Once the eigenvalues of the system are found, the mode shape corresponding to each eigenvalue can be determined from the eigenvector obtained from Eq. (36) and the shape functions given by Eq. (34). Although the model is presented for symmetric boundary supports, it is very simple to extend it to the case where the springs stiffness and loss factors at each end of the beam are different by specifying them in Eq. (31). A wide range of numerical results can be obtained. Influences of the boundary parameters on the eigenvalues are presented here only for some combinations of stiffnesses and loss factors of the boundary springs. Figs. 5–9 show the sensitivity of the eigenvalues to the translational or rotational stiffness parameters.

From Fig. 5(b), one notices that the viscoelastic boundary condition improves the modal loss factors compare to the clamped–clamped and simple–simple boundary conditions. Fig. 6 shows that the modal frequencies are not affected by the boundary springs loss factors, \( \eta_R \) and \( \eta_T \), while the modal loss factors of the first 10 modes are improved with a maximum for the fifth mode. Similar behavior is observed for the parameter, \( g_R \), which affects little bit the modal frequencies but greatly influences the modal loss factors (see Fig. 8). The sensitivity of the modal properties to \( g_T \) is shown by Fig. 7. The modal frequencies and loss factors are both affected by \( g_T \). As it is expected, the loss factor, \( \eta_2 \), of the viscoelastic interlayer does not affect the modal frequencies but improves greatly the modal loss factor, Fig. 9.

Table 1

<table>
<thead>
<tr>
<th>Modes</th>
<th>GEM 3 (Sainsbury and Zhang, 1999)</th>
<th>Present model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \omega ) (Hz)</td>
<td>( \eta )</td>
</tr>
<tr>
<td>1</td>
<td>65.02</td>
<td>0.0816</td>
</tr>
<tr>
<td>2</td>
<td>299.65</td>
<td>0.0720</td>
</tr>
<tr>
<td>3</td>
<td>750.35</td>
<td>0.0462</td>
</tr>
<tr>
<td>4</td>
<td>1405.44</td>
<td>0.0267</td>
</tr>
<tr>
<td>5</td>
<td>2279.31</td>
<td>0.0172</td>
</tr>
<tr>
<td>6</td>
<td>3370.18</td>
<td>0.0117</td>
</tr>
<tr>
<td>Error max (%)</td>
<td>Ref.</td>
<td>Ref.</td>
</tr>
</tbody>
</table>

Table 2

Geometrical and material data of the laminated glass unit for parametric studies

<table>
<thead>
<tr>
<th>Quantities</th>
<th>Elastic layers</th>
<th>PVB-interlayer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus (GPa)</td>
<td>70</td>
<td>—</td>
</tr>
<tr>
<td>Shear modulus (MPa)</td>
<td>—</td>
<td>0.69</td>
</tr>
<tr>
<td>Density (kg m(^{-3}))</td>
<td>2500</td>
<td>1100</td>
</tr>
<tr>
<td>Thickness (mm)</td>
<td>( h_1 = h_3 = 4.50 )</td>
<td>( h_2 \approx 1.0 )</td>
</tr>
<tr>
<td>Loss factor</td>
<td>—</td>
<td>0.3</td>
</tr>
<tr>
<td>Length = 1600 mm, width = 800 mm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Fig. 5. Comparison of three boundary conditions.

Fig. 6. Effect of $\eta_R$ and $\eta_T$.

Fig. 7. Effect of $g_T$. 
It can be seen from these figures that an appropriate selection of the support springs parameters can permit to obtain desirable modal properties. For instance, if a certain natural frequency and modal loss factor are desired, the procedure to select the complex translational and rotational boundary springs is as follow. The results presented herein can be used to obtain the combinations of $K_T$ and $K_R$ which satisfy the specified natural frequency and the associated modal loss factor for the assumed values of $\eta_T$ and $\eta_R$. The procedure must be then iterated by varying the values of $\eta_T$ and $\eta_R$, until the desired modal loss factor is obtained. Fig. 9(b) shows that the modal properties of the laminated glass beam unit increases with the loss factor $\eta_2$ of the central layer (PVB-material). But the effect of $\eta_2$ on the natural frequencies, Fig. 9(a), is less. It can be inferred from the above discussion that the model and results presented can be used in the design of support damping treatments of beams systems for passive vibration control.

4. Conclusions

The equivalent stiffness representation of the viscoelastic boundary supports lead, in static domain, to two coupled linear differential equations involving lateral displacement $w$ and axial displacements $u_1$ and
by the variational principle for the analysis of the laminated glass beam. Analytical solution is
derived for the displacement, strain and stress fields. The sensitivity of these fields to the rotational
spring stiffness is investigated. This provides a useful tool in engineering point of view to determine
the strength of laminated glass beam. The free vibrations of a laminated glass beam with viscoelastic
boundary support have been analyzed. Numerical results, based on the FE method, for the first fifteen
natural frequencies and modal loss factors of the beam have been presented. These figures show the
sensitivity of the modal properties of the beam to the translational or rotational stiffness parameters.
Based on these results, a procedure for selecting the support springs stiffness parameters has been indi-
cated to obtain desirable modal properties. The effects of the central viscoelastic layer damping for the
laminated glass beam is also included in this presentation. One can see that the models derived and the
results represented herein can be used in the determination of the strength and modal properties of lam-
inated glass beam and also in the design of support damping of sandwich beams for passive vibration
control.

Appendix A. Integration constants

The integration constants \( a_i \) introduced in Section 2.3 have the following expressions:

\[
\begin{align*}
  a_1 &= \frac{LP}{4EI\bar{x}^2\eta} + \frac{L^2q\mu}{8\bar{x}^2} - \frac{P\sinh\left(\frac{L\bar{x}}{2}\right)}{2EI\bar{x}^3\eta} + \frac{(C_1 + C_2) \cosh\left(\frac{L\bar{x}}{2}\right)}{C_3}, \\
  a_2 &= -\frac{P}{2EI\bar{x}^2\eta}, \\
  a_3 &= -\frac{(C_1 + C_2)}{C_3}, \\
  a_4 &= -\frac{P(\delta + \bar{x}^2\eta)}{2EI\bar{x}^2\eta}, \\
  a_5 &= -\frac{P(d + \bar{x}\eta)}{2EI\bar{x}^2\eta}, \\
  a_6 &= \frac{L^2q(-5L^2\delta + 48\eta)\mu + g_1(a_1, a_4)}{348\bar{x}^2} \\
  &+ \frac{Lq\mu(-EI\delta + h_0) + g_2(a_3, a_4)}{2\bar{x}^2K_T}. \\
\end{align*}
\]

The terms in (A.1) are the following:

\[
\begin{align*}
  C_1 &= -48(EI)^2q\bar{x}^2\eta^3\mu + 24EI\bar{x}^3\eta \sinh\left(\frac{L\bar{x}}{2}\right) \\
  &\quad + (-24P\delta + 3L^2P\bar{x}^2\delta - 24P\bar{x}^2\eta + 2EIL^3q\bar{x}^2\delta\eta\mu)K_R, \\
  C_2 &= -24EI\bar{x}^3\eta^3\mu K_R + 24P\delta \cosh\left(\frac{L\bar{x}}{2}\right)K_R \\
  &\quad + 24P\bar{x}^2\eta \cosh\left(\frac{L\bar{x}}{2}\right)K_R - 12LP\bar{x}\delta \sinh\left(\frac{L\bar{x}}{2}\right)K_R, \\
  C_3 &= 24EI\bar{x}^3\eta \left[(2EI\bar{x}^3\eta - L\bar{x}\delta K_R) \cosh\left(\frac{L\bar{x}}{2}\right)\right] \\
  &\quad + 2(\delta + x^2\eta)K_R \sinh\left(\frac{L\bar{x}}{2}\right),
\end{align*}
\]
where
\[ g_1(a_3, a_4) = 48((-8 + L^2x^2)\delta - 8x^2\eta)\cosh\left(\frac{Lx}{2}\right) + 16a_4\left[-(L^3x^3\delta) + 12Lx(\delta + x^2\eta) + 3((-8 + L^2x^2)\delta - 8x^2\eta)\sinh\left(\frac{Lx}{2}\right)\right], \]
\[ g_2(a_3, a_4) = 2x^3\left[a_3(\kappa (\delta + x^2\eta) - h_0)\sinh\left(\frac{Lx}{2}\right) + a_4\left(-E\delta + (\kappa (\delta + x^2\eta) - h_0)\cosh\left(\frac{Lx}{2}\right) + h_0\right)\right]. \]

Appendix B. Shape functions, stiffness and mass matrices

The classical shape functions used in Section 3.1 are defined as:
\[ N_1(\zeta) = \frac{(1 - \zeta)^2(2 + \zeta)}{4}, \]
\[ N_2(\zeta) = \frac{l(1 - \zeta)^2(1 + \zeta)}{8}, \]
\[ N_3(\zeta) = \frac{(1 + \zeta)^2(2 - \zeta)}{4}, \]
\[ N_4(\zeta) = -\frac{l(1 + \zeta)^2(1 - \zeta)}{8}, \]
\[ N_5(\zeta) = \frac{(1 - \zeta)}{2}, \]
\[ N_6(\zeta) = \frac{(1 + \zeta)}{2}, \]
where \( l \) is the element length, \( \zeta = 2x/l - 1, x \in [0, l], \zeta \in [-1, 1] \). If a quantity \( g \) is function of \( x \) which is also function of \( \zeta \), one gets:
\[ g' = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial \zeta} \frac{\partial \zeta}{\partial x} = \frac{2}{l} \frac{\partial g}{\partial \zeta}, \quad g'' = \frac{\partial^2 g}{\partial x^2} = \frac{4}{l^2} \frac{\partial^2 g}{\partial \zeta^2}, \quad \text{and} \quad \int_0^l g dx = \frac{l}{2} \int_{-1}^1 g d\zeta. \]

The element stiffness and mass matrices used in Section 3.1 are expressed as:
\[ [K_e] = \frac{l}{2} \int_{-1}^1 \left\{ E\kappa [W''][W'''] + EA_1^T[U'_1][U'_1] \right\} d\zeta, \]
\[ + \int_{-1}^1 \left\{ (\rho_1 A_1 + \rho_1 A_3 + \rho_2 A_2)[W][W] + \rho_1 (A_1^T[U'_1][U'_1] + A_3^T[U'_3][U'_3]) \right\} d\zeta, \]
\[ [M_e] = \frac{l}{2} \int_{-1}^1 \left\{ [\Gamma_2] - [U_3] + h_0 [W''']]/h_2. \]

References


Edel, M.T., 1997. The effect of temperature on the bending of laminated glass units. MS thesis, Department of Civil Engineering, Texas A&M University, College Station, TX.


