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Hardy inequalities for fractional integrals on general domains ☆

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Abstract

We prove a sharp Hardy inequality for fractional integrals for functions that are supported in a general domain. The constant is the same as the one for the half-space and hence our result settles a recent conjecture of Bogdan and Dyda.

Keywords: Fractional Hardy inequality; Best constant; Fractional Laplacian

1. Introduction

In this note we prove a conjecture by Bogdan and Dyda [2] concerning Hardy inequalities for fractional integrals. It was shown in [2] that for any function f supported in the half-space $\mathbb{H}^n = \{x \in \mathbb{R}^n\colon\ x = (x_1, \ldots, x_n),\ x_n > 0\}$

$$\frac{1}{2} \int\limits_{\mathbb{H}^n \times \mathbb{H}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}}\, dx\, dy \geqslant \kappa_{n,\alpha} \int\limits_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^\alpha}\, dx. \tag{1}$$

Here $0 < \alpha < 2$ and

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$$\kappa_{n,\alpha} = \pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \frac{1}{\alpha} \left[\frac{2^{1-\alpha}}{\sqrt{\pi}} \Gamma\left(\frac{2-\alpha}{2}\right) \Gamma\left(\frac{1+\alpha}{2}\right) - 1 \right] \tag{2}$$

is the sharp constant. Note that $\kappa_{n,1} = 0$ and $\kappa_{n,\alpha} > 0$ otherwise.

It was conjectured in [2] that for $1 < \alpha < 2$ this inequality continues to hold with the same constant for any convex set Ω, i.e., for functions f supported in Ω

$$\frac{1}{2} \int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}} \, dx \, dy \geqslant \kappa_{n,\alpha} \int_{\Omega} \frac{|f(x)|^2}{d_\Omega(x)^\alpha} \, dx, \tag{3}$$

where $d_\Omega(x)$ denotes the distance from the point $x \in \Omega$ to the boundary of Ω. This is a precise analogue of the Hardy inequality due to Davies [6]. For $0 < \alpha < 1$ the inequality cannot hold for compact sets. A counterexample is given in [9].

Sharp Hardy inequalities analogous to (3) but for the L^p-norms of gradients of functions are well known. The first result is due to Davies [6] for the case $p = 2$. The case for arbitrary p is derived in [14,13]. For a review the reader may consult [8]. Let us add that these results have been considerably generalized in [1].

Hardy inequalities for fractional integrals are of a more recent provenience, in particular the higher-dimensional versions were investigated by Dyda (see [9]) in great generality following previous work in [12,5]. While Hardy inequalities for fractional integrals are of interest in their own right, they deliver also spectral information on the generators of censored stable processes. The generator of a censored stable process is defined by the closure of the quadratic form on the left side of (3). Loosely speaking the censored stable process is the isotropic stable Lévy process with the jumps between Ω and its complement suppressed. Ref. [3] contains the construction of censored stable processes and a wealth of information about these. For the connection between Hardy inequalities and censored stable processes the reader may consult [5].

Since we actually prove a result stronger than (3) we need a few concepts before we can state it. Let Ω be any domain in $\mathbb{R}^n$ with non-empty boundary. The following notion is taken from Davies [7]. Fix a direction $w \in \mathbb{S}^{n-1}$ and define

$$d_{w,\Omega}(x) = \min\{|t|: \ x + tw \notin \Omega\}. \tag{4}$$

Further, define the function

$$\delta_{w,\Omega}(x) = \sup\{|t|: \ x + tw \in \Omega\}, \tag{5}$$

i.e., $\delta_{w,\Omega}(x)$ is the distance from x to the farthest point in the intersection of the line $x + tw$ and Ω. We let

$$\frac{1}{M_\alpha(x)^\alpha} := \frac{\int_{\mathbb{S}^{n-1}} dw \, [\frac{1}{d_{w,\Omega}(x)} + \frac{1}{\delta_{w,\Omega}(x)}]^\alpha}{\int_{\mathbb{S}^{n-1}} dw \, |w_n|^\alpha}. \tag{6}$$

The integral in the denominator can be easily computed to be

$$\int_{\mathbb{S}^{n-1}} dw \, |w_n|^\alpha = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}. \tag{7}$$

These definitions are analogous to the one in [7] where the estimates are expressed in terms of

$$\frac{1}{m_2(x)^2} = \frac{\int_{\mathbb{S}^{n-1}} dw \, \frac{1}{d_{w,\Omega}(x)^2}}{|\mathbb{S}^{n-1}|/n}.$$

In case the domain Ω is convex, the quantity $M_\alpha(x)$ can be bounded in terms of $d_\Omega(x)$ and $D_\Omega(x)$, the *'width of Ω at x'*, which is defined as follows. Fix $x \in \Omega$ arbitrary and pick a point z on the boundary of Ω which is closest to x, so that $d_\Omega(x) = |x - z|$. In general there may be more than one such point. Each such point defines a *unique* supporting hyper-plane P_z which is the tangent plane. This follows from the fact that the vector $x - z$ must be perpendicular to any supporting hyper-plane, for otherwise $z \in \partial\Omega$ would not be the point of closest distance to x. Thus, for each $x \in \Omega$ we obtain a family of hyper-planes $\mathcal{P}_x$. For $P \in \mathcal{P}_x$, we denote by $S(P)$ the slab of smallest width $D_{S(P)}$ that contains Ω and is bounded by P on one side and by a hyper-plane parallel to P on the other. Such a slab might be a half space if Ω is unbounded in which case we set $D_{S(P)} = \infty$. Now we define

$$D_\Omega(x) = \inf_{P \in \mathcal{P}_x} D_{S(P)}. \tag{8}$$

We have

$$\frac{1}{M_\alpha(x)^\alpha} \geqslant \left[\frac{1}{d_\Omega(x)} + \frac{1}{D_\Omega(x) - d_\Omega(x)}\right]^\alpha. \tag{9}$$

Indeed, for a given supporting hyper-plane P pick coordinates such that the standard vector e_n is normal to the plane P. Clearly $d_{w,\Omega}(x) \leqslant d_{w,S(P)}(x)$ and $\delta_{w,\Omega}(x) \leqslant \delta_{w,S(P)}(x)$. Further, note that $d_{w,S(P)}(x) + \delta_{w,S(P)}(x)$ is the length of the segment given by intersecting the slab $S(P)$ defined by P with the line $x + tw$. Projecting this segment onto the line normal to the slab yields

$$d_{w,S(P)}(x)|w_n| = d_\Omega(x), \qquad \delta_{w,S(P)}(x)|w_n| = D_{S(P)} - d_\Omega(x).$$

Note that there may exist directions w where the length of this segment is not finite which is the case when $D_{S(P)} = \infty$. Thus,

$$\left[\frac{1}{d_{w,\Omega}(x)} + \frac{1}{\delta_{w,\Omega}(x)}\right]^\alpha \geqslant |w_n|^\alpha \left[\frac{1}{d_\Omega(x)} + \frac{1}{D_{S(P)} - d_\Omega(x)}\right]^\alpha$$

holds for all $P \in \mathcal{P}_x$. Taking the supremum over $\mathcal{P}_x$ and integrating with respect to w over the unit sphere yields

$$\int_{\mathbb{S}^{n-1}} dw \left[\frac{1}{d_{w,\Omega}(x)} + \frac{1}{\delta_{w,\Omega}(x)}\right]^\alpha \geqslant \int_{\mathbb{S}^{n-1}} dw \, |w_n|^\alpha \left[\frac{1}{d_\Omega(x)} + \frac{1}{D_\Omega(x) - d_\Omega(x)}\right]^\alpha. \tag{10}$$

With these preparations we can state our main theorem.

Theorem 1.1. *Let Ω be a domain with non-empty boundary and $1 < \alpha < 2$. For any $f \in C_c^\infty(\Omega)$*

$$\frac{1}{2} \int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}}\, dx\, dy \geqslant \kappa_{n,\alpha} \int_{\Omega} \frac{|f(x)|^2}{M_\alpha(x)^\alpha}\, dx. \tag{11}$$

In particular, if Ω is a convex region then for any $f \in C_c^\infty(\Omega)$

$$\frac{1}{2} \int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}}\, dx\, dy \geqslant \kappa_{n,\alpha} \int_{\Omega} \left|f(x)\right|^2 \left[\frac{1}{d_\Omega(x)} + \frac{1}{D_\Omega(x) - d_\Omega(x)}\right]^\alpha dx \tag{12}$$

where $d_\Omega(x)$ is the distance of $x \in \Omega$ to the boundary of Ω and $D_\Omega(x)$ is defined in (8). *The constant $\kappa_{n,\alpha}$ is best possible.*

It was pointed out to us by Rupert Frank and Robert Seiringer that Theorem 1.1 can be generalized, albeit in a weaker form, by replacing the powers 2 by $p > 1$. More precisely we have,

Theorem 1.2. *Let $1 < \alpha < p < \infty$. Then for any domain $\Omega \subset \mathbb{R}^n$ and any $f \in C_c^\infty(\Omega)$*

$$\int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha}}\, dx\, dy \geqslant \mathcal{D}_{n,p,\alpha} \int_{\Omega} \frac{|f(x)|^p}{m_\alpha(x)^\alpha}\, dx \tag{13}$$

where

$$\frac{1}{m_\alpha(x)^\alpha} := \frac{\int_{\mathbb{S}^{n-1}} dw\, \frac{1}{d_{w,\Omega}(x)^\alpha}}{\int_{\mathbb{S}^{n-1}} dw\, |w_n|^\alpha}, \tag{14}$$

and

$$\mathcal{D}_{n,p,\alpha} = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \int_0^1 \frac{|1 - r^{\frac{\alpha-1}{p}}|^p}{(1-r)^{1+\alpha}}\, dr \tag{15}$$

is sharp. In particular, for Ω convex

$$\int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha}}\, dx\, dy \geqslant \mathcal{D}_{n,p,\alpha} \int_{\Omega} \frac{|f(x)|^p}{d_\Omega(x)^\alpha}\, dx. \tag{16}$$

The constant $\mathcal{D}_{n,p,s}$ has been computed before in [11] as the sharp constant for the Hardy inequality for the half-space. For $0 < p \leqslant 1$ the inequality continues to hold (see [9]), however, the sharp constant is not known.

In the next section we establish the analogous one-dimensional inequalities and then show how an averaging argument leads to the general result. At the end of Section 2 we indicate how to obtain the result for general values of p. We are grateful to Rupert Frank and Robert Seiringer to allow us to include their arguments in our work. We present them at the end of our paper.

2. The one-dimensional problem

The proof of Theorem 1.1 will rely heavily on the following one-dimensional inequality.

Theorem 2.1. *Let $f \in C_c^\infty((a,b))$. For all $1 < \alpha < 2$ we have*

$$\frac{1}{2} \int_{(a,b)\times(a,b)} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy \geqslant \kappa_{1,\alpha} \int_a^b \left|f(x)\right|^2 \left(\frac{1}{x-a}+\frac{1}{b-x}\right)^\alpha dx. \tag{17}$$

The idea of proving Theorem 2.1 is to reduce the problem on the interval to a problem on the half-line via a fractional linear mapping. The reader may consult [4] for further examples where inversion symmetry is used to obtain sharp functional inequalities.

Lemma 2.2 *(Invariance under fractional linear transformations). Let f be any function in $C_c^\infty(\mathbb{R}\setminus\{0\})$. Consider the inversion $x \to 1/x$ and set*

$$g(x) = I(f)(x) := |x|^{\alpha-1} f\left(\frac{1}{x}\right).$$

Then $g \in C_c^\infty(\mathbb{R})$ and

$$\int_{\mathbb{R}\times\mathbb{R}} \frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy = \int_{\mathbb{R}\times\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy. \tag{18}$$

Proof. For fixed ε consider the regions

$$R_1 := \left\{(x,y)\in\mathbb{R}^2\colon \left|\frac{x}{y}\right| > 1+\varepsilon\right\},$$

and likewise,

$$R_2 := \left\{(x,y)\in\mathbb{R}^2\colon \left|\frac{y}{x}\right| > 1+\varepsilon\right\}.$$

By changing variables $x \to 1/x$ and $y \to 1/y$ we find that

$$\int_{R_1\cup R_2} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy$$
$$= \int_{R_1\cup R_2} \frac{|f(1/x)-f(1/y)|^2}{|x-y|^{1+\alpha}} |x|^{\alpha-1}|y|^{\alpha-1}\,dx\,dy$$

$$= \int\limits_{R_1 \cup R_2} \frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy$$

$$+ \int\limits_{R_1 \cup R_2} \frac{|f(1/x)|^2(|x|^{\alpha-1}|y|^{\alpha-1} - |x|^{2(\alpha-1)}) + |f(1/y)|^2(|x|^{\alpha-1}|y|^{\alpha-1} - |y|^{2(\alpha-1)})}{|x-y|^{1+\alpha}}\,dx\,dy$$

which, by symmetry under exchange of x and y,

$$= \int\limits_{R_1 \cup R_2} \frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy + 2\int\limits_{R_1 \cup R_2} \frac{|f(1/x)|^2(|x|^{\alpha-1}|y|^{\alpha-1} - |x|^{2(\alpha-1)})}{|x-y|^{1+\alpha}}\,dx\,dy.$$

The second integral can be written as

$$\int\limits_{\mathbb{R}} \left|f(1/x)\right|^2 |x|^{\alpha-2}\,dx \int\limits_{\{|s|>1+\varepsilon\}\cup\{\frac{1}{|s|}>1+\varepsilon\}} \frac{|s|^{\alpha-1}-1}{|1-s|^{1+\alpha}}\,ds.$$

We have

$$\int\limits_{\{|s|>1+\varepsilon\}\cup\{\frac{1}{|s|}>1+\varepsilon\}} \frac{|s|^{\alpha-1}-1}{|1-s|^{1+\alpha}}\,ds = \int\limits_{\{|s|>1+\varepsilon\}} \frac{|s|^{\alpha-1}-1}{|1-s|^{1+\alpha}}\,ds + \int\limits_{\{\frac{1}{|s|}>1+\varepsilon\}} \frac{|s|^{\alpha-1}-1}{|1-s|^{1+\alpha}}\,ds,$$

and by changing the variable $s \to 1/s$ in the last integral we find that this sum vanishes. Letting $\varepsilon \to 0$ yields (18). □

Proof of Theorem 2.1. By translation and scaling it suffices to prove the result for the interval $(0,1)$. Let $f \in C_c^\infty((0,1))$. We have to show that

$$\frac{1}{2}\int\limits_{(0,1)\times(0,1)} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy \geqslant \kappa_{1,\alpha}\int_0^1 \left|f(x)\right|^2\left(\frac{1}{x}+\frac{1}{1-x}\right)^\alpha dx. \tag{19}$$

Set

$$g(x) = |x+1|^{\alpha-1} f\left(\frac{1}{1+x}\right).$$

Clearly, $g \in C_c^\infty((0,\infty))$. Note that

$$g(x) = I(f)(x+1)$$

and if we set $g(x) := 0$, $x < 0$, we may use Lemma 2.2 and find that

$$\frac{1}{2}\int_0^1\int_0^1 \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy + \int_0^1 dx\big|f(x)\big|^2 \int_{\mathbb{R}\setminus(0,1)} \frac{1}{|x-y|^{1+\alpha}}\,dy$$

$$= \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} \frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy$$

$$= \frac{1}{2}\int_0^\infty\int_0^\infty \frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy + \int_0^\infty dx\big|g(x)\big|^2 \int_{-\infty}^0 \frac{1}{|x-y|^{1+\alpha}}\,dy. \tag{20}$$

Some of the integrals are easily evaluated and yield

$$\frac{1}{2}\int_0^1\int_0^1 \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy = \frac{1}{2}\int_0^\infty\int_0^\infty \frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy + \frac{1}{\alpha}\int_0^\infty \frac{|g(x)|^2}{x^\alpha}\,dx$$
$$- \frac{1}{\alpha}\int_0^1 \big|f(x)\big|^2\big(x^{-\alpha} + (1-x)^{-\alpha}\big)\,dx. \tag{21}$$

Using the sharp Hardy inequality of Bogdan and Dyda [2] on the half-line yields

$$\frac{1}{2}\int_0^1\int_0^1 \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy \geqslant \kappa_{1,\alpha}\int_0^\infty \frac{|g(x)|^2}{x^\alpha}\,dx + \frac{1}{\alpha}\int_0^\infty \frac{|g(x)|^2}{x^\alpha}\,dx$$
$$- \frac{1}{\alpha}\int_0^1 \big|f(x)\big|^2\big(x^{-\alpha} + (1-x)^{-\alpha}\big)\,dx, \tag{22}$$
$$\geqslant \kappa_{1,\alpha}\int_0^1 \big|f(x)\big|^2\left(\frac{1}{x(1-x)}\right)^\alpha dx$$
$$+ \frac{1}{\alpha}\int_0^1 \big|f(x)\big|^2 \frac{1-x^\alpha-(1-x)^\alpha}{(x(1-x))^\alpha}\,dx. \tag{23}$$

Finally, we note that for $1<\alpha<2$

$$1-x^\alpha-(1-x)^\alpha \geqslant 0,$$

which proves the inequality (19). □

Theorem 2.1 generalizes easily to open sets on the real line.

Corollary 2.3. *Let $J \subset \mathbb{R}$ be open and $1 < \alpha < 2$. For every $f \in C_c^\infty(J)$*

$$\frac{1}{2}\int\limits_{J\times J} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy \geqslant \kappa_{1,\alpha}\int\limits_J \left|f(x)\right|^2\left(\frac{1}{d_J(x)}+\frac{1}{\delta_J(x)}\right)^\alpha dx, \tag{24}$$

where $\delta_J(x)$ is defined in (5).

Proof. Since any open set $J \subset \mathbb{R}$ is a countable union of disjoint intervals I_k we find, using Theorem 2.1, that

$$\begin{aligned}\frac{1}{2}\int\limits_J\int\limits_J \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy &\geqslant \frac{1}{2}\sum_{k=1}^\infty \int\limits_{I_k}\int\limits_{I_k} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy \\ &\geqslant \sum_{k=1}^\infty \kappa_{1,\alpha}\int\limits_{I_k} \left|f(x)\right|^2\left(\frac{1}{d_{I_k}(x)}+\frac{1}{\delta_{I_k}(x)}\right)^\alpha dx \\ &\geqslant \kappa_{1,\alpha}\int\limits_J \left|f(x)\right|^2\left(\frac{1}{d_J(x)}+\frac{1}{\delta_J(x)}\right)^\alpha dx. \qquad \square\end{aligned} \tag{25}$$

Lemma 2.4 *(Reduction to dimension one). Let Ω be a region in $\mathbb{R}^n$ and assume that $f \in C_c^\infty(\Omega)$, $p > 0$. Then*

$$\begin{aligned}\int\limits_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\,dx\,dy = \frac{1}{2}\int\limits_{\mathbb{S}^{n-1}} dw \int\limits_{\{x:\, x\cdot w=0\}} d\mathcal{L}_w(x) \int\limits_{\{x+sw\in\Omega\}} ds \\ \times \int\limits_{\{x+tw\in\Omega\}} dt \frac{|f(x+sw)-f(x+tw)|^p}{|s-t|^{1+\alpha}}\end{aligned} \tag{26}$$

where $\mathcal{L}_w$ denotes the $(n-1)$-dimensional Lebesgue measure on the plane $x \cdot w = 0$.

Proof. We write the expression

$$I_\Omega(f) := \int\limits_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\,dx\,dy$$

in the form

$$\int\limits_\Omega dx \int\limits_{\{x+z\in\Omega\}} dz \frac{|f(x)-f(x+z)|^p}{|z|^{n+\alpha}}$$

and using polar coordinates $z = rw$ we arrive at the expression

$$I_\Omega(f) = \int_\Omega dx \int_{\mathbb{S}^{n-1}} dw \int_{\{x+rw\in\Omega,\ r>0\}} dr \frac{|f(x)-f(x+rw)|^p}{r^{1+\alpha}},$$

$$= \frac{1}{2}\int_{\mathbb{S}^{n-1}} dw \int_\Omega dx \int_{\{x+hw\in\Omega\}} dh \frac{|f(x)-f(x+hw)|^p}{|h|^{1+\alpha}}. \tag{27}$$

Thus, the domain of integration in the innermost integral is the line $x+hw$ intersected with the domain Ω. Splitting the variable x into a component perpendicular to w and parallel to w, i.e., replacing x by $x+sw$, where $x\cdot w=0$, we arrive at

$$\frac{1}{2}\int_{\mathbb{S}^{n-1}} dw \int_{\{x:\ x\cdot w=0\}} d\mathcal{L}_w(x) \int_{\{x+sw\in\Omega\}} ds \int_{\{x+(s+h)w\in\Omega\}} dh\, \frac{|f(x+sw)-f(x+(s+h)w)|^p}{|h|^{1+\alpha}}.$$

The change of variable $t=s+h$ yields (26). □

Proof of Theorem 1.1. By Lemma 2.4 and Corollary 2.3 we find that

$$\frac{1}{2}\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}\,dx\,dy$$

$$= \frac{1}{4}\int_{\mathbb{S}^{n-1}} dw \int_{\{x:\ x\cdot w=0\}} d\mathcal{L}_w(x) \int_{\{x+sw\in\Omega\}} ds \int_{\{x+tw\in\Omega\}} dt\, \frac{|f(x+sw)-f(x+tw)|^2}{|s-t|^{1+\alpha}}$$

$$\geqslant \kappa_{1,\alpha}\frac{1}{2}\int_{\mathbb{S}^{n-1}} dw \int_{\{x:\ x\cdot w=0\}} d\mathcal{L}_w(x) \int_{\{x+sw\in\Omega\}} ds\, \left|f(x+sw)\right|^2$$

$$\times\left[\frac{1}{d_{w,\Omega}(x+sw)}+\frac{1}{\delta_{w,\Omega}(x+sw)}\right]^\alpha$$

$$= \kappa_{1,\alpha}\frac{1}{2}\int_{\mathbb{S}^{n-1}} dw \int_\Omega \left|f(x)\right|^2\left[\frac{1}{d_{w,\Omega}(x)}+\frac{1}{\delta_{w,\Omega}(x)}\right]^\alpha dx = \kappa_{n,\alpha}\int_\Omega \frac{|f(x)|^2}{M_\alpha(x)^\alpha}\,dx, \tag{28}$$

where we have used (7) in the last equation.

It remains to show that the constant $\kappa_{n,\alpha}$ in the inequality (12) is best possible. Pick a hyperplane H that is tangent to Ω at a point P. Such hyper-planes exist since Ω is convex. It was shown in [2] that the constant for the half-space problem, $\kappa_{n,\alpha}$, is best possible by constructing a sequence of trial functions. Transplanting these trial functions to Ω near P one can show that $\kappa_{n,\alpha}$ is also optimal for (12). The actual proof is an imitation of the proof of Theorem 5 in [13] and we omit the details. □

We finally come to the proof of Theorem 1.2. We thank Rupert Frank and Robert Seiringer for allowing us to present their argument.

Theorem 2.5. *Let $1 < \alpha < p < \infty$. Then for all smooth functions f with $f(0) = 0$,*

$$\int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha}} \, dx \, dy \geqslant \mathcal{D}_{1,p,\alpha} \int_0^1 \frac{|f(x)|^p}{x^\alpha} \, dx.$$

Proof. Let $\omega(x) = x^{(\alpha-1)/p}$. Then by [11, Lemma 2.4]

$$2 \int_0^\infty \big(\omega(x) - \omega(y)\big) \big|\omega(x) - \omega(y)\big|^{p-2} \frac{dy}{|x - y|^{1+\alpha}} = \frac{\mathcal{D}_{1,p,\alpha}}{x^\alpha} \omega(x)^{p-1}, \quad 0 < x < 1$$

where the integral is understood in principal value sense. Since

$$\int_1^\infty \big(\omega(x) - \omega(y)\big) \big|\omega(x) - \omega(y)\big|^{p-2} \frac{dy}{|x - y|^{1+\alpha}} \leqslant 0 \quad \text{for } x \in [0, 1],$$

we conclude that

$$V(x) := \frac{2}{\omega(x)^{p-1}} \int_0^1 \big(\omega(x) - \omega(y)\big) \big|\omega(x) - \omega(y)\big|^{p-2} \frac{dy}{|x - y|^{1+\alpha}} \geqslant \frac{\mathcal{D}_{1,p,\alpha}}{x^\alpha} \quad \text{for } x \in [0, 1].$$

Now [10, Proposition 2.2] implies that

$$\int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha}} \, dx \, dy \geqslant \int_0^1 V(x) \big|f(x)\big|^p \, dx,$$

which proves the claim. □

An easy consequence is

Theorem 2.6. *Let $f \in C_c^\infty((a, b))$. Then for all $1 < \alpha < p < \infty$ we have*

$$\int_{(a,b)\times(a,b)} \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha}} \, dx \, dy \geqslant \mathcal{D}_{1,p,\alpha} \int_a^b \frac{|f(x)|^p}{\min\{(x - a), (b - x)\}^\alpha} \, dx. \tag{29}$$

Exactly the same proof as the one of Corollary 2.3 yields

Corollary 2.7. *Let $1 < \alpha < p < \infty$. Let $J \subset \mathbb{R}$ be open and f a function on J with $f \in C_c^\infty(J)$, then*

$$\int_J \int_J \frac{|f(x) - f(y)|^p}{|x-y|^{1+\alpha}}\, dx\, dy \geqslant \mathcal{D}_{1,p,\alpha} \int_J \frac{|f(x)|^p}{d_J(x)^\alpha}\, dx.$$

Proof of Theorem 1.2. The proof is a repetition of the arguments in the proof of Theorem 1.1. □

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