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Generating Systems of Groups and Reidemeister–Whitehead Torsion

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0. INTRODUCTION

Deciding whether two given generating systems $x = \{x_1, ..., x_n\}$ and $y = \{y_1, ..., y_n\}$ of a group G are Nielsen equivalent (see Definition 0.1 below) is a well known problem in combinatorial group theory, with important applications to low dimensional topology. The difficult part of this problem is the case where one has to prove that x and y are inequivalent. There are various known techniques for doing so but, apart from certain singular examples, they apply only to three special classes of groups: 2-generator groups, 1-relator groups, and groups with finite abelian quotient of equal rank as G (see the discussion given in [L1, Sect. 1]).

In this paper we present a fundamentally different approach by establishing a direct connection between the Nielsen equivalence question and the classical notion of Reidemeister–Whitehead torsion. This connection yields a new invariant for Nielsen equivalence classes of generating systems. The strength of our new invariant is that it applies directly to all finitely generated groups, and that calculating this invariant for any specific group with explicitly given generating systems (together with a system of defining relations) depends only on the computational capacities of the user.

Denote by $F(X)$ and $F(Y)$ the free groups on bases $X = \{X_1, ..., X_n\}$ and $Y = \{Y_1, ..., Y_n\}$, respectively, and let β_x, β_y be the canonical epimorphisms $F(X) \twoheadrightarrow G$, given by $X_i \to x_i$, and $F(Y) \twoheadrightarrow G$, given by $Y_i \to y_i$.

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DEFINITION 0.1. The generating systems x and y of G are called *Nielsen equivalent* if there exists an isomorphism $\alpha: F(Y) \to F(X)$ with $\beta_x \circ \alpha = \beta_y$.

For every finitely generated group G we define in Section 1 an abelian group $\mathcal{N}(G)$. This group $\mathcal{N}(G)$ is the "Whitehead group" of the quotient of the group ring $\mathbb{Z}G$ modulo the Fox ideal I of G (see Definition 1.1). If $I = \mathbb{Z}G$ we call $\mathcal{N}(G)$ *degenerate* and define $\mathcal{N}(G) = \{0\}$. We prove:

THEOREM I. *Every ordered pair x, y of generating systems of minimal cardinality for G defines an element $\mathcal{N}(y, x) \in \mathcal{N}(G)$, such that the following properties hold*:

(i) *$\mathcal{N}(y, x)$ depends only on the Nielsen equivalence classes of x and y. If x and y are Nielsen equivalent then $\mathcal{N}(y, x) = 0 \in \mathcal{N}(G)$.*

(ii) *If $z = \{z_1, ..., z_n\}$ is another generating system of minimal cardinality, then*

$$\mathcal{N}(z, y) + \mathcal{N}(y, x) = \mathcal{N}(z, x).$$

(iii) *For $H_1 = G * \langle u_1 \rangle * \cdots * \langle u_m \rangle$, $H_2 = G * [\langle v_1 \rangle \oplus \cdots \oplus \langle v_m \rangle]$ $(m \geqslant 2)$, and $H_3 = G * \langle w_1 \mid w_1^2 \rangle * \cdots * \langle w_m \mid w_m^2 \rangle$ one has $\mathcal{N}(G) = \mathcal{N}(H_3) = \mathcal{N}(H_2) \subsetneq \mathcal{N}(H_1)$ and $\mathcal{N}(y, x) = \mathcal{N}(y \cup w, x \cup w) = \mathcal{N}(y \cup v, x \cup v) \to \mathcal{N}(y \cup u, x \cup u)$.*

(iv) *For all $n \in \mathbb{N}$ the construction $\mathcal{N}$ describes a functor from the category $\mathbf{C}_n$ of groups with fixed rank n and surjective homomorphisms to the category $\mathbf{Ab}$ of abelian groups. In particular we obtain for all objects G, H of $\mathbf{C}_n$ and any morphism $f: G \twoheadrightarrow H$*

$$\mathcal{N}(f)(\mathcal{N}(y, x)) = \mathcal{N}(f(x), f(y))$$

for all generating systems x and y of G with cardinality n.

Part (i) of Theorem I provides a tool for distinguishing Nielsen inequivalent generating systems x, y of G by showing that $\mathcal{N}(y, x) \neq 0 \in \mathcal{N}(G)$. This is done using Theorem II as stated below or the more elaborate methods described in Section 2. As a result the authors have been able to exhibit non-trivial elements $\mathcal{N}(y, x)$ for many families of groups, in particular for Fuchsian groups (see [LM1]), finite groups (see [L1]), knot groups (see Section 4 and [LM2]), and one relator products of cyclics (see Section 4). For Fuchsian groups the invariant $\mathcal{N}(y, x)$ is a complete invariant of Nielsen equivalence classes of minimal generating systems, but in general $\mathcal{N}(y, x)$ is not quite as sharp.

The original notion of Reidemeister torsion can be reinterpreted as a weaker but more computable vesion of Whitehead torsion. (For a reference on Whitehead torsion see [M2, Tu].) The basic idea here is to represent the group ring in a matrix ring with commutative entries, and to evaluate an equivalence class of matrices (i.e., an element of K_1) by the determinant map. In our case this approach yields:

THEOREM II. *Let G be presented by*

$$G = \langle x_1, ..., x_n \mid R_1, R_2, ... \rangle,$$

and let a second generating system $y_1 = w_1(x_1, ..., x_n), ..., y_n = w_n(x_1, ..., x_n)$ *be given as words in the* x_i. *For any word* $w = w(x_1, ..., x_n)$ *let W denote the corresponding word* $W = w(X_1, ..., X_n) \in F(X)$. *Let* $\partial w/\partial x_i \in \mathbb{Z}G$ *denote the image of the Fox derivative* $\partial W/\partial X_i$ *under the map* $\beta_x : \mathbb{Z}F(X) \to \mathbb{Z}G$, $X_i \to x_i$.

Let A be a commutative ring with $1 \in A$, *and let* $\rho: \mathbb{Z}G \to \mathbb{M}_m(A)$, $\rho(1) = 1$, *be a ring homomorphism satisfying* $\rho(\partial R_k/\partial x_i) = 0$ *for all* R_k *and* x_i. *If the determinant of the* $(mn \times mn)$*-matrix* $\rho((\partial w_j/\partial x_i)_{j,i})$ *is not contained in the subgroup of* A^* *generated by the determinants of* $\rho(\pm x_1), ..., \rho(\pm x_n)$, *then* $x_1, ..., x_n$ *and* $y_1, ..., y_n$ *are Nielsen inequivalent generating systems of G.*

In Section 2 we investigate this concept systematically and introduce "Reidemeister type" torsion groups $\mathcal{N}(G)_m$, $m \in \mathbb{N}$, which are quotients of the group $\mathcal{N}(G)$. The groups $\mathcal{N}(G)_m$ are obtained via representations of $\mathbb{Z}G/I$ as $(m \times m)$-matrices over certain universal commutative rings. These are quotients of polynomial rings in finitely many variables and hence easily tractable. As m is taken successively larger one can calculate the value of $\mathcal{N}(y, x)$ in $\mathcal{N}(G)_m$ with increasing computational effort and decreasing information loss. Denote by $\mathcal{N}(G)^*$ the quotient of $\mathcal{N}(G)$ modulo all elements which map to zero in all of the $\mathcal{N}(G)_m$. Particular evaluation techniques of $\mathcal{N}(G)_m$ are described at the end of Section 2; the authors are currently engaged in implementing one of them on a computer.

Since $\mathcal{N}(G)$ is the analogue of the Whitehead group Wh(G), with $\mathbb{Z}G$ replaced by $\mathbb{Z}G/I$, it is natural to ask which of the properties of Wh(G) are inherited by $\mathcal{N}(G)$ (or $\mathcal{N}(G)^*$), and which are not.

A striking difference, for example, is the frequent occurence of non-trivial values: Whereas no example of a torsion free group G with Wh$(G) \neq 0$ is known, many such groups with $\mathcal{N}(G) \neq 0$ (e.g., knot groups) can be found easily (see Section 4).

Furthermore, J. Stallings proved that the Whitehead group of a free product of groups $G_1 * G_2$ is isomorphic to the direct sum of the Whitehead groups of the summands (see [St]). Unfortunately the precise analogue is not true for $\mathcal{N}(G)$ (see Example 3.1). In order to obtain an analogous result for $\mathcal{N}(G)$ one needs to pass over to coefficients in a field $\mathfrak{k}$, where one can similarily define groups $\mathcal{N}(G; \mathfrak{k})$ and $\mathcal{N}(G; \mathfrak{k})^*$ (see Section 3). We obtain:

THEOREM III. *Let G_1 and G_2 be groups and $\mathfrak{k}$ be a field, such that both $\mathcal{N}(G_1;\mathfrak{k})^*$ and $\mathcal{N}(G_2;\mathfrak{k})^*$ are non-degenerate. Then the natural embeddings $G_1 \subsetneq G_1 * G_2$ and $G_2 \subsetneq G_1 * G_2$ induce an isomorphism*

$$\mathcal{N}(G_1;\mathfrak{k})^* \oplus \mathcal{N}(G_2;\mathfrak{k})^* \xrightarrow{\cong} \mathcal{N}(G_1 * G_2;\mathfrak{k})^*.$$

It is shown in Section 3 that without the non-degeneracy assumption on the summands the above theorem fails in general. It seems likely that using deep results of Waldhausen (see [W]) the above statement can be proved for $\mathcal{N}(G;\mathfrak{k})$ rather than $\mathcal{N}(G;\mathfrak{k})^*$, but for our applications this does not make a difference since all our computations take place in $\mathcal{N}(G;\mathfrak{k})^*$ or a quotient of it.

The invariant $\mathcal{N}(G)$ presented in this paper can be generalized in a natural way to higher dimensional analogues $\mathcal{N}^q(G)$, $q \geqslant 2$. For $q=2$ it yields finite 2-dimensional complexes which are homotopy equivalent but not simple-homotopy equivalent (see [L1, L2]).

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1. THE TORSION INVARIANT

Let G be a finitely generated group with presentation

$$G = \langle x_1, ..., x_n \mid R_1, R_2, ... \rangle,$$

and let $\beta_x: F(X) \to G$, $\beta_y: F(Y) \to G$ as defined in Section 0. Denote by $\partial/\partial X_1: \mathbb{Z}F(X) \to \mathbb{Z}F(X)$ the ith Fox derivative of the integer group ring $\mathbb{Z}F(X)$; i.e., the unique $\mathbb{Z}$-linear function satisfying $\partial X_i/\partial X_j = \delta_{ij}$ and $\partial VW/\partial X_i = \partial V/\partial X_i + V\,\partial W/\partial X_i$ for any V, $W \in F(X)$ (see, e.g., [G]). By a slight abuse of notation we denote throughout this paper a group homomorphism $F \to G$ and its $\mathbb{Z}$-linear extension to a ring homomorphism $\mathbb{Z}F \to \mathbb{Z}G$ by the same symbol. Similarily we do not distinguish notationally between a ring homomorphism $A \to B$ and the induced homomorphism on the $(m \times m)$-matrix rings $\mathbb{M}_m(A) \to \mathbb{M}_m(B)$.

DEFINITION 1.1. Let $G = \langle x_1, ..., x_n \mid R_1, R_2, ... \rangle$ and $\beta_x: F(X) \to G$ be as above.

(a) Let I_x be the two sided ideal in $\mathbb{Z}G$ generated by

$$\{\beta_x(\partial R_k/\partial X_i) \mid k = 1, 2, ..., X_i \in X\},$$

and let γ_x denote the quotient map $\mathbb{Z}G \twoheadrightarrow \mathbb{Z}G/I_x$.

(b) Define the *Fox ideal* of G to be the two sided ideal I in $\mathbb{Z}G$ generated by all I_x, where x is a generating system of minimal cardinality of G. Denote the quotient map $\mathbb{Z}G \twoheadrightarrow \mathbb{Z}G/I$ by γ.

Remark 1.2. (1) An easy computation using the chain rule for Fox derivatives (see [BZ, pp. 125]) shows that the ideal $I_x < \mathbb{Z}G$ is equal to the two sided ideal generated by

$$\{\beta_x(\partial R/\partial X_i) \mid R \in \operatorname{Ker} \beta_x, X_i \in X\},$$

and hence independent of the choice of the generators R_k of the kernel of β_x.

(2) If X' is a different basis of $F(X)$ and x' its β_x-image in G then, using the chain rule again, one obtains $I_x = I_{x'}$. Thus I_x is an invariant of the Nielsen equivalence class of x.

(3) The authors do not know of any example of generating systems x and x' of G with the same cardinality where $I_x \neq I_{x'}$. In particular we obtain $I_x = I$ for all examples known to us.

(4) If we consider non-minimal generating systems with the property that the kernel of some $\beta_x : F(X) \twoheadrightarrow G$ contains a primitive element of $F(X)$ we always have $I_x = \mathbb{Z}G$, as follows directly from the definition of I_x. Hence the analogue to Definition 1.1 (b) for non-minimal generating systems leads always to an ideal which is equal to the whole group ring.

Let $\alpha : F(Y) \to F(X)$ be a homomorphism (not necessarily an isomorphism) which satisfies $\beta_x \circ \alpha = \beta_y$ (compare also with Definition 0.1). Such a homomorphism exists always, since $F(Y)$ is free, but α is in general far from being unique. The map α determines an $(n \times n)$-matrix $(\partial\alpha(Y_j)/\partial X_i)_{j,i}$ with entries in $\mathbb{Z}F(X)$. In general both $(\partial\alpha(Y_j)/\partial X_i)_{j,i}$ and its image $\beta_x((\partial\alpha(Y_j)/\partial X_i)_{j,i}) \in \mathbb{M}_n(\mathbb{Z}G)$ are different for different choices of α.

LEMMA 1.3. *Let $\alpha : F(Y) \to F(X)$ be a homomorphism which satisfies $\beta_x \circ \alpha = \beta_y$.*

(1) *For any two minimal generating systems x, y of G the matrix $(\partial y/\partial x) \in \mathbb{M}_n(\mathbb{Z}G/I)$, obtained from $\beta_x(\partial\alpha(Y_j)/\partial X_i)_{j,i}$ by applying the map $\gamma : \mathbb{Z}G \to \mathbb{Z}G/I$, is independent of the choice of α.*

(2) *The matrix $(\partial y/\partial x)$ is invertible.*

Proof. (1) Let $\alpha' : F(Y) \to F(X)$ be a second map which satisfies

$\beta_x \circ \alpha' = \beta_y$. Then for all $W \in F(Y)$ one has $\beta_x(\alpha'(W)) = \beta_y(W) = \beta_x(\alpha(W))$, and hence

$$\alpha'(W) \cdot \alpha(W)^{-1} \in \operatorname{Ker} \beta_x = \langle R_1, R_2, \ldots \rangle \subset F(X).$$

But this implies $\beta_x(\partial\alpha'(W)/\partial X_i) - \beta_x(\partial\alpha(W)/\partial X_i) \in I_x$ for all $i = 1, \ldots, n$ (see for example Lemma 1.3 of [LM1]). Hence one has $\gamma_x(\beta_x(\partial\alpha'(W)/\partial X_i)) = \gamma_x(\beta_x(\partial\alpha(W)/\partial X_i))$. Setting $W = Y_j$, for $j = 1, \ldots, n$, proves (1).

(2) Consider now a map $\hat{\alpha}: F(X) \to F(Y)$ with $\beta_y \circ \hat{\alpha} = \beta_x$. For any $k = 1, \ldots, n$ one has $\beta_x(X_k) = \beta_y(\hat{\alpha}(X_k)) = \beta_x(\alpha(\hat{\alpha}(X_k)))$. As before one obtains $\gamma_x \beta_x(\partial X_k/\partial X_i) = \gamma_x \beta_x(\partial\alpha\hat{\alpha}(X_k)/\partial X_i)$ for all $i = 1, \ldots, n$. The chain rule for Fox derivatives gives

$$\partial\alpha\hat{\alpha}(X_k)/\partial X_i = \sum_{j=1}^{n} \alpha(\partial\hat{\alpha}(X_k)/\partial Y_j) \cdot \partial\alpha(Y_j)/\partial X_i,$$

and hence $\gamma_x \beta_x \alpha(\partial\hat{\alpha}(X_k)/\partial Y_j)_{k,j} \cdot \gamma_x \beta_x(\partial\alpha(Y_j)/\partial X_i)_{j,i} = \gamma_x \beta_x(\partial\alpha\hat{\alpha}(X_k)/\partial X_i)_{k,i} = \gamma_x \beta_x(\partial X_k/\partial X_i)_{k,i} = (\delta_{ki})_{k,i} = 1 \in GL(\mathbb{Z}G/I_x)$. The same argument shows that $\gamma_y \beta_y \hat{\alpha}(\partial\alpha(Y_j)/\partial X_i)_{j,i} \cdot \gamma_y \beta_y(\partial\hat{\alpha}(X_i)/\partial Y_h)_{i,h} = 1 \in GL(\mathbb{Z}G/I_y)$. But $\gamma\beta_y \hat{\alpha}(\partial\alpha(Y_j)/\partial X_i)_{j,i} = \gamma\beta_x(\partial\alpha(Y_j)/\partial X_i)_{j,i} = (\partial y/\partial x)$. So $(\partial y/\partial x)$ has both a right and a left inverse, which proves (2). ∎

Lemma 1.3 shows that the matrix $(\partial y/\partial x)$ lies in $GL_n(\mathbb{Z}G/I)$ and hence in $GL(\mathbb{Z}G/I) = \varinjlim GL_n(\mathbb{Z}G/I)$. Let T denote the subgroup of *trivial units* of $GL(\mathbb{Z}G/I)$: that is, the subgroup generated by all matrices of the form

$$\begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \pm g & & \\ & & & & 1 & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{bmatrix} \qquad g \in G.$$

DEFINITION 1.4. (a) Assume $I \neq \mathbb{Z}G$. We define the *torsion invariant* $\mathcal{N}(G)$ as the following quotient of the first K-group of the ring $\mathbb{Z}G/I$:

$$\mathcal{N}(G) = K_1(\mathbb{Z}G/I)/T.$$

For any two minimal generating systems x, y of G define

$$\mathcal{N}(y, x) = [(\partial y/\partial x)] \in \mathcal{N}(G).$$

(b) If $I = \mathbb{Z}G$ call $\mathcal{N}(G)$ *degenerate* and define formally $\mathcal{N}(G) = \{0\}$. In particular one always has $\mathcal{N}(y, x) = 0$.

Proof of Theorem I. It is convenient to prove the statements listed in the theorem in the following order:

(ii) The claim is an immediate application of the chain rule property of the Fox derivatives.

(i) Any two generating systems of G which are Nielsen equivalent differ by a sequence of elementary Nielsen operations (see [MKS, Chap. 3]). But one can see directly that if x and y differ by an elementary Nielsen operation, then $(\partial y/\partial x)$ is a generalized elementary matrix, i.e., a matrix with at most one non-zero off diagonal element and elements of $G \cup -G$ in the diagonal. Thus (i) follows directly from (ii) and the definition of $\mathcal{N}(G)$.

(iv) Any surjective homomorphism $f: G \to H$ of groups with equal rank maps a minimal generating system x of G to a minimal generating system y of H. Thus f is induced by an isomorphism $f': F(X) \to F(Y)$ via the canonical epimorphisms $\beta_x: F(X) \twoheadrightarrow G$ and $\beta_y: F(Y) \twoheadrightarrow H$ as defined in Section 0. The kernel of β_x is mapped by f' into the kernel β_y, and hence (by Remark 1.2(1)) I_x is mapped by $f: \mathbb{Z}G \to \mathbb{Z}H$ into I_y. Since x was an arbitrary minimal generating system of G, the map f induces a well defined ring homomorphism $f'': \mathbb{Z}G/I \to \mathbb{Z}H/I$. By Lemma 1.3(1) the homomorhism f'' maps $(\partial x/\partial x')$ to $(\partial f(x)/\partial f(x'))$ for any two minimal generating systems x, x' of G. Claim (iv) then follows directly from the functoriality of K_1 and the definition of $\mathcal{N}(G)$.

(iii) The above argument in (ii) works as well under the weaker hypothesis that a non-surjective homomorphism $f: G \to H$ maps a minimal generating set x of G to a set $f(x)$ which becomes a minimal generating set of H when suitable elements of H are added. The canonical injections $G \subsetneq H_i$, $i = 1, 2, 3$, give precisely this situation, as can be seen from the Gruško–Neumann theorem and abelianization of the right factor of H_i. Thus $\mathbb{Z}G/I$ maps canonically to $\mathbb{Z}H_i/I$.

In order to derive a presentation for H_2 from $G = \langle x_1, ..., x_n \mid R_1, R_2, ... \rangle$, one adds the generators v_i together with defining relators $v_i v_j v_i^{-1} v_j^{-1}$, $i \neq j \in \{1, ..., m\}$. The Fox derivatives then give

$$\partial v_i v_j v_i^{-1} v_j^{-1}/\partial v_k = \begin{cases} 0 & \text{for} \quad k \neq i, j \\ 1 - v_j & \text{for} \quad k = i \\ v_i - 1 & \text{for} \quad k = j. \end{cases}$$

Hence we get $\mathbb{Z}H_2/I = \mathbb{Z}G/I$, and the subgroups T of trivial units coincide.

The canonical injection $G \to H_2$ induces an isomorphism $\mathcal{N}(G) \to \mathcal{N}(H_2)$ with $\mathcal{N}(y, x) \to \mathcal{N}(y \cup v, x \cup v)$.

For H_3 one adds generators w_i and relators w_i^2. Differentiating we get

$$\partial w_i^2/\partial w_j = \begin{cases} 0 & i \neq j \\ 1 + w_i & i = j. \end{cases}$$

As before we get $\mathbb{Z}H_2/I = \mathbb{Z}G/I$ with a coinciding set T of trivial units, and the canonical injection $G \to H_3$ induces an isomorphism $\mathcal{N}(G) \to \mathcal{N}(H_3)$ with $\mathcal{N}(y, x) \to \mathcal{N}(y \cup v, x \cup v)$.

For H_1 one adds new generators u_i but no new relators. However, in this case there is a retraction map from $\mathbb{Z}H_1/I$ onto $\mathbb{Z}G/I$ induced by the map $H_1 \to 1$, which maps the trivial units of H_1 to those of G. The functoriality of K_1 implies then that $\mathcal{N}(G)$ injects into $\mathcal{N}(H)$, mapping $\mathcal{N}(y, x)$ to $\mathcal{N}(y \cup u, x \cup u)$. ∎

The reader may find it interesting to compare statement (iii) of Theorem I to the following still unsolved problem:

Conjecture 1.5 (M. Dunwoody and B. Zimmermann). If x, y are Nielsen inequivalent generating systems of a group G then $x \cup z$ and $y \cup z$ are inequivalent generating systems of $G * F(z)$.

We conclude this section with a computation of $\mathcal{N}(G)$ and $\mathcal{N}(y, x)$ in the case of G a finite abelian group.

EXAMPLE 1.6. (1) Let G be the cyclic group of prime order with a presentation $G = \langle a \mid a^p \rangle$. Consider the two generating systems $x = \{a\}$, $y = \{a^q\}$, $1 \leqslant q < p/2$. One can easily check that $I_x = (1 + a + a^2 + \cdots + a^{p-1})$ and that $I = I_x$. We have $\mathbb{Z}G/I = \mathbb{Z}[a, a^{-1}]/(1 + a + a^2 + \cdots + a^{p-1}) = \mathbb{Z}[\xi]$, where ξ is the p-th root of unity. It is well known that $K_1(\mathbb{Z}[\xi])$ is the group of units of $\mathbb{Z}[\xi]$ (see [M1, p. 32]). Consequently we obtain

$$\mathcal{N}(G) = K_1(\mathbb{Z}[\xi])/\pm G = \mathbb{Z}[\xi]^*/\langle \xi \rangle.$$

Furthermore $\mathcal{N}(y, x) = [(\partial y/\partial x)] = 1 + \xi + \xi^2 + \cdots + \xi^{q-1} \in \mathbb{Z}[\xi]^*/\langle \xi \rangle$ and is non-trivial as $|1 + \xi + \xi^2 + \cdots + \xi^{q-1}| \neq 1$.

(2) Let $G = \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_n\mathbb{Z} = \langle x_1, ..., x_n \mid x_i^{q_i}, [x_i, x_j]\ (i, j = 1, ..., n) \rangle$ be a finite abelian group which cannot be generated by less than n elements. Furthermore assume that $d = \text{g.c.d.}(q_1, ..., q_n) \neq 1$. The Fox derivatives of the relators are $\partial x_i^{q_i}/\partial x_i = 1 + x_i + \cdots + x_i^{q_i - 1}$ and $\partial [x_i, x_j]/\partial x_i = 1 - x_j$, $\partial [x_i, x_j]/\partial x_j = x_i - 1$. Thus one obtains $\mathbb{Z}G/I = \mathbb{Z}/d\mathbb{Z}$ and hence $\mathcal{N}(G) = \mathbb{Z}/d\mathbb{Z}^*/\{\pm 1\}$ (see [Si, p. 140]).

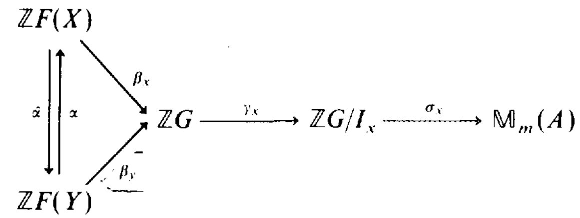
From the above computation and Theorem I(i) one recovers the known classification of Nielsen equivalence classes of minimal generating systems for finite abelian groups G (see [D]). (Note that by Theorem I (iii) and the Gruško-Neumann theorem this classification extends to the groups $G * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$. For d odd the previously known methods did not yield this extension.)

2. Evaluating $\mathcal{N}(G)$

The goal of this section is to present a method for computing non-zero elements $\mathcal{N}(y, x) \in \mathcal{N}(G)$ with relative ease (contrary to the situation in Whitehead groups). The idea is to evaluate $\mathcal{N}(G)$ via a representation of $\mathbb{Z}G/I$ in the ring of $(m \times m)$-matrices over a commutative ring with a unit element. Throughout this section A denotes such a ring. The first problem in finding representations of $\mathbb{Z}G/I$ is that according to Definition 1.1(b) the ideal $I \subset \mathbb{Z}G$ seems to require information about all Nielsen equivalence classes of minimal generating systems of G. The following lemma resolves this problem and explains the choice of A.

LEMMA 2.1. *Let x be a minimal generating system of G and A any commutative ring with $1 \in A$. Every ring homomorphism $\sigma_x : \mathbb{Z}G/I_x \to \mathbb{M}_m(A)$ with $\sigma_x(1) = 1$ maps $\gamma_x(I)$ to $0 \in \mathbb{M}_m(A)$ and hence induces a ring homomorphism $\sigma : \mathbb{Z}G/I \to \mathbb{M}_m(A)$.*

Proof. Let $G = \langle x \mid R \rangle = \langle y \mid S \rangle$ be two presentations for G, where x and y are both minimal. Consider any commutative diagram as described in the proof of Lemma 1.3 (2):



It suffices to show that $\sigma_x \gamma_x$ maps I_y to 0, where I_y is the two-sided ideal of $\mathbb{Z}G$ generated by $\{\beta_y(\partial S_h/\partial Y_j) \mid S_h \in S,\ Y_j \in Y\}$ (see Definition 1.1(a)). In the proof of Lemma 1.3 we have shown:

(1) For any $S_h \in S$ one has $\gamma_x \beta_x (\partial\alpha(S_h)/\partial X_i)_{h,i} = 0 \in \mathbb{M}_n(\mathbb{Z}G/I_x)$, as $\beta_x \alpha(S_h) = \beta_y(S_h) = 1$.

(2) $\gamma_x \beta_x \alpha(\partial\bar{\alpha}(X_k)/\partial Y_j)_{k,j} \cdot \gamma_x \beta_x (\partial\alpha(Y_j)/\partial X_i)_{j,i} = 1 \in GL(\mathbb{Z}G/I_x)$.

The map σ_x induces maps $\mathbb{M}_{k \times 1}((\mathbb{Z}G/I_x) \to \mathbb{M}_{mk \times m1}(A)$ for any k, $1 \in \mathbb{N}$

via "forgetting the brackets," which we also denote by σ_x. Since A is a commutative ring, the left inverse of a matrix in $\mathbb{M}_{mn}(A)$ is also a right inverse. Thus (1) and (2) above imply the following equations in $\mathbb{M}_{mn \times mn}(A)$ for any $S_h \in S$:

$$\begin{aligned}
&\sigma_x(\gamma_x\beta_x(\alpha(\partial S_h/\partial Y_j)_j)\\
&\quad = \sigma_x(\gamma_x\beta_x(\alpha(\partial S_h/\partial Y_j)_j)) \cdot \sigma_x(\gamma_x\beta_x\alpha(\partial\hat{\alpha}(X_k)/\partial Y_j)_{k,j} \cdot \gamma_x\beta_x(\partial\alpha(Y_j)/\partial X_i)_{j,i})\\
&\quad = \sigma_x(\gamma_x\beta_x\alpha(\partial S_h/\partial Y_j)_j)) \cdot \sigma_x(\gamma_x\beta_x(\partial\alpha(Y_j)/\partial X_i)_{j,i})\\
&\qquad \cdot \sigma_x(\gamma_x\beta_x\alpha(\partial\hat{\alpha}(X_k)/\partial Y_j)_{k,j})\\
&\quad = \sigma_x(\gamma_x\beta_x(\alpha(\partial S_h/\partial Y_j)_j \cdot (\partial\alpha(Y_j)/\partial X_i)_{j,i})) \cdot \sigma_x(\gamma_x\beta_x\alpha(\partial\hat{\alpha}(X_k)/\partial Y_j)_{k,j})\\
&\quad = \sigma_x(\gamma_x\beta_x(\partial\alpha(S_h)/\partial X_i)_i) \cdot \sigma_x(\gamma_x\beta_x\alpha(\partial\hat{\alpha}(X_k)/\partial Y_j)_{k,j})\\
&\quad = 0 \cdot \sigma_x(\gamma_x\beta_x\alpha(\partial\hat{\alpha}(X_k)/\partial Y_j)_{k,j}) = 0. \quad \blacksquare
\end{aligned}$$

All maps $\sigma\colon \mathbb{Z}G/I \to \mathbb{M}_m(A)$ induce by the functoriality of K_1 a map

$$K_1(\sigma)\colon K_1(\mathbb{Z}G/I) \to K_1(\mathbb{M}_m(A)) = K_1(A)$$

(where the last equation is induced by the "forgetting the brackets" map). On $K_1(A)$ we have the determinant map $\det\colon K_1(A) \to A^*$ into the multiplicative group of units A^* of A. Let τ_σ denote the composition map

$$\tau_\sigma\colon GL(\mathbb{Z}G/I) \xrightarrow[\text{definition}]{} K_1(\mathbb{Z}G/I) \xrightarrow[K_1(\sigma)]{} K_1(A) \xrightarrow[\det]{} A^*. \qquad (*)$$

We define the subgroup T_σ of A^* as the image $\tau_\sigma(T)$ of the set of trivial units $T \subset GL(\mathbb{Z}G/I)$ (see Definition 1.4). We now sum up with the following proposition:

PROPOSITION 2.2. *Let x be a minimal generating system of G and A a commutative ring with $1 \in A$. Any representation $\sigma\colon \mathbb{Z}G/I \to \mathbb{M}_m(A)$ (or equivalently $\sigma_x\colon \mathbb{Z}G/I_x \to \mathbb{M}_m(A)$) with $\sigma(1) = 1$ induces a homomorphism*

$$\mathcal{N}(\sigma)\colon \mathcal{N}(G) \to A^*/T_\sigma.$$

In particular, if y is a second minimal generating system of G and $\tau_\sigma[(\partial y/\partial x)]$ is not in T_σ, then y is not Nielsen equivalent to x. $\blacksquare$

For the purpose of a direct practical application we observe that for any representation $\sigma\colon \mathbb{Z}G/I \to \mathbb{M}_m(A)$ as above and any invertible matrix $M \in \mathbb{M}_n(\mathbb{Z}G/I)$ the map τ_σ associates to the matrix $\sigma(M) \in GL_{mn}(A) \subset GL(A)$ the value $\det(\sigma(M))$. In particular we see that for any generating system $x_1, \ldots, x_n$ of G the group T_σ is precisely the subgroup of A^* generated by all the values $\det(\sigma(\pm x_1)), \ldots, \det(\sigma(\pm x_n))$. Hence we obtain:

Proof of Theorem II. The statement of Theorem II is an immediate consequence of Lemma 2.1 and Proposition 2.2. ∎

We show next that all representations σ as in Proposition 2.2 can be derived from particular representations of $\mathbb{Z}G/I$ into "universal" rings A_m, $m \in \mathbb{N}$. From these representations one can compute all non-trivial values which can possibly be evaluated by any $(m \times m)$-representation σ as in Proposition 2.2.

Let $\langle x_1, ..., x_n \mid R_1, R_2, ... \rangle$ be a presentation for the rank n group G. We fix the size $m \in \mathbb{N}$ of the representations in question and consider the polynomial ring $\mathbb{Z}[a^k_{i,j}, a_k]$ on variables $a^k_{i,j}$ and a_k, where $1 \leqslant i, j \leqslant m$ and $1 \leqslant k \leqslant n$. The variables $a^k_{i,j}$ can be arranged as an $(m \times m)$-matrix $(a^k_{i,j})_{i,j}$, and in the quotient ring

$$\mathring{A} = \mathbb{Z}[a^k_{i,j}, a_k]/(a_k \cdot \det(a^k_{i,j})_{i,j} - 1)$$

the determinant of the matrix $(a^k_j)_{i,j}$ is invertible. Hence there is a homomorphism $\Gamma_m\colon F(X_1, ..., X_n) \to GL_m(\mathring{A})$ which maps X_k to the matrix $(a^k_{i,j})_{i,j}$ and which extends to a ring homomorphism $\Gamma_m\colon \mathbb{Z}F(X_1, ..., X_n) \to \mathbb{M}_m(\mathring{A})$. Let $J_m \subset \mathring{A}$ denote the two sided ideal generated by the entries of all $\Gamma_m(\partial R_h/\partial X_i)$. Define the *universal ring* A_m to be

$$A_m = \mathring{A}/J_m.$$

The fundamental formula for Fox derivatives (see [Fo] or [LS, p. 99]) is

$$R_h - 1 = \sum_{i=1}^{n} \partial R_h/\partial x_i \cdot (x_i - 1),$$

which implies that Γ_m induces a group homomorphism $\gamma_m\colon G \to GL_m(A_m)$. The canonical extension to $\gamma_m\colon \mathbb{Z}G \to \mathbb{M}_m(A_m)$ fits then into the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}F(X_1, ..., X_n) & \xrightarrow{\Gamma_m} & \mathbb{M}_m(\mathring{A}) \\ {\scriptstyle \beta_x}\downarrow & & \downarrow \\ \mathbb{Z}G & \xrightarrow{\gamma_m} & \mathbb{M}_m(A_m) \end{array}$$

By definition of A_m one has $\gamma_m(I_x) = 0$. Thus γ_m induces a representation

$$\sigma_m\colon \mathbb{Z}G/I_x \to \mathbb{M}_m(A_m).$$

For the rest of the paper we abbreviate the notation introduced above τ_{σ_m} to τ_m and T_{σ_m} to T_m.

LEMMA 2.3. *The ring $A_m(G)$ as well as the multiplicative subgroup $T_m \subset A_m(G)^*$ are independent of the presentation of G used in the construction.*

Proof. We first observe that for every commutative ring A with $1 \in A$ and any ring homomorphism $\sigma: \mathbb{Z}G/I_x \to \mathbb{M}_m(A)$ with $\sigma(1) = 1$, $\sigma(x_k) = (\sigma(x_k)_{i,j})$, there is a canonical map $\rho_\sigma: A_m \to A$ with the property that the induced map $\hat{\rho}_\sigma: \mathbb{M}_m(A_m) \to \mathbb{M}_m(A)$ satisfies $\sigma = \hat{\rho}_\sigma \circ \sigma_m$. Such a homomorphism $\rho_\sigma: A_m \to A$ is given by $a^k_{i,j} \to \sigma(x_k)_{i,j}$ and $a_k \to \det(\sigma(x_k))$. This follows from the definition of A_m and from the fact that σ is a ring homomorphism; i.e., the entries of the matrices $\sigma(x_k)$ have to satisfy all the relations given by the ideal J_m. The statement of the lemma now follows by symmetry: Let x' be a second generating system for G, with $T'_m \subset A'_m$, σ'_m, σ', and $\rho'_{\sigma'}$ denoting the corresponding objects as above. The argument above, applied to $\sigma = \sigma'_m$ and to $\sigma' = \sigma_m$, gives the equations $\sigma'_m = \hat{\rho}_{\sigma'_m} \circ \sigma_m$ and $\sigma_m = \hat{\rho}'_{\sigma_m} \circ \sigma'_m$. Hence we obtain

$$\sigma'_m = \hat{\rho}_{\sigma'_m} \circ \hat{\rho}'_{\sigma_m} \circ \sigma'_m \qquad \text{and} \qquad \sigma_m = \hat{\rho}'_{\sigma_m} \circ \hat{\rho}_{\sigma'_m} \circ \sigma_m.$$

These matrix equations imply equations on all entries. Thus the maps $\rho_{\sigma'_m}: A_m \to A'_m$ and $\rho'_{\sigma_m}: A'_m \to A_m$ are isomorphisms. As all the x'_1 are words in the x_k and conversely the isomorphism $\rho_{\sigma'_m}$ maps T_m to $T_{m'}$. ∎

The representation $\sigma_m: \mathbb{Z}G/I_x \to \mathbb{M}_m(A_m)$ satisfies the hypotheses of Proposition 2.2 and maps the group T of trivial units to the multiplicative subgroup $T_m \subset A^*_m$, which is generated by the images of all $\pm a_k$.

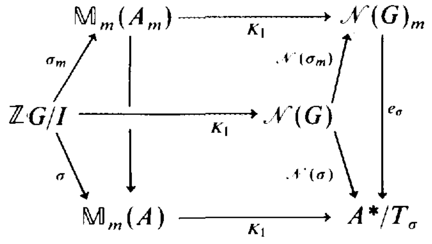
DEFINITION 2.4. Let $\mathcal{N}(G)_m$ be the image of $\mathcal{N}(G)$ under the evaluation map

$$\mathcal{N}(\sigma_m): \mathcal{N}(G) \to A^*_m / \langle \pm a_1, ..., \pm a_n \rangle.$$

The following proposition establishes the universality of $\mathcal{N}(G)_m$.

PROPOSITION 2.5. *Let A be any commutative ring with $1 \in A$, and let $\sigma: \mathbb{Z}G/I \to \mathbb{M}_m(A)$ be a ring homomorphism with $\sigma(1) = 1$. Then the induced map $\mathcal{N}(\sigma): \mathcal{N}(G) \to A^*/T_\sigma$ as given by Proposition 2.2 factors through $\mathcal{N}(\sigma_m): \mathcal{N}(G) \to \mathcal{N}(G)_m$, giving a map $e_\sigma: \mathcal{N}(G)_m \to A^*/T_\sigma$.*

Proof. As shown in the proof of Lemma 2.3 any $\sigma: \mathbb{Z}G/I \to \mathbb{M}_m(A)$ factors through $\sigma_m: \mathbb{Z}G/I \to \mathbb{M}_m(A_m)$. The proposition is equivalent to the existence of the lower left map in the right hand triangle of the commutative diagram below. But this is a straightforward consequence of the functoriality of K_1 and the existence of the maps in the left hand triangle, which was shown above:



▮

We finish this section by describing some particular evaluation methods, based on the invariant $\mathcal{N}(G)_m$, for exhibiting Nielsen inequivalent generating systems of G. The computational imput for these methods is

(1) a finite presentation $G = \langle x_1, ..., x_n \mid R_1, ..., R_s \rangle$, and

(2) a second generating system $y_1 = w_1(x_1, ..., x_n), ..., y_n = w_n(x_1, ..., x_n)$, given as words in the x_i.

From these data we derive polynomials in finitely many variables as follows: Consider the words $W_1 = w_1(X_1, ..., X_n), ..., W_n = w_n(X_1, ..., X_n)$ in the free group $F(X)$ and compute the Fox derivatives $\partial W_h/\partial X_k$ and $\partial R_q/\partial X_k$ in $\mathbb{Z}F(X)$ for all $1 \leqslant h,\ k \leqslant n$ and $1 \leqslant q \leqslant s$. Associate to every generator X_k the matrix $(a^k_{i,j})_{i,j}$ with entries in the introduced polynomial ring $\mathbb{Z}[a^k_{i,j}, a_k]$, $1 \leqslant i, j \leqslant m$ and $1 \leqslant k \leqslant n$ above. To X_k^{-1} associate the matrix $a_k \cdot \mathrm{Adj}((a^k_{i,j})_{i,j})$, where Adj denotes the adjoint. Although this does not define a ring homomorphism, we obtain in this way for every element of $\mathbb{Z}F(X)$ a well defined $(m \times m)$-matrix with entries in $\mathbb{Z}[a^k_{i,j}, a_k]$. Denote by $\omega(\partial W_h/\partial X_k)$ and $\omega(\partial R_q/\partial X_k)$ the matrices associated to $\partial W_h/\partial X_k$ and $\partial R_q/\partial X_k$, respectively. Thus we have derived from the data (1) and (2) in finitely many steps:

(3) Finitely many generators of an ideal $\hat{J}_m$ of $\mathbb{Z}[a^k_{i,j}, a_k]$, namely all entries of the matrices $\omega(\partial R_q/\partial X_k)$ together with the polynomials $a_k \cdot \det(a^k_{i,j})_{i,j} - 1$ (for $k = 1, ..., n$).

(4) The determinant $\delta_{y,x}$ of the $(mn \times mn)$-matrix over $\mathbb{Z}[a^k_{i,j}, a_k]$ which is given as $(n \times n)$-block matrix with $\omega(\partial W_h/\partial X_k)$ as (h, k)th block entry.

From the arguments above it is immediate that $\mathbb{Z}[a^k_{i,j}, a_k]/\hat{J}_m$ is precisely the ring A_m, and that $\delta_{y,x}$ modulo $\hat{J}_m$ is equal to $\tau_m([\partial y/\partial x])$. Hence $\mathcal{N}(y, x)$ is the coset of $\delta_{y,x}$ in A_m^* modulo $\langle \pm a_1, ..., \pm a_n \rangle$, and we obtain the following:

CRITERION 2.6. If, for any choice of exponents $s_k \in \mathbb{Z}$, none of the polynomials $\delta_{y,x} \pm a_1^{s_1} \cdots a_n^{s_n}$ is contained in the ideal $\hat{J}_m$, then x and y are not Nielsen equivalent.

Criterion 2.6 can be applied in practise, due to the existence of an algorithm (see [Tr]) which decides the following question: Given a polynomial ring $\mathbb{Z}[x_1, ..., x_n]$ and polynomials $p, p_1, ..., p_r \in \mathbb{Z}[x_1, ..., x_n]$, is p contained in the ideal $(p_1, ..., p_r) \subset \mathbb{Z}[x_1, ..., x_n]$?

This algorithm can be used in our context in the following way: Consider generators $p_1, ..., p_d$ of the ideal $J_m \subset \mathbb{Z}[a_{i,j}^k, a_k]$. For some large $t \in \mathbb{N}$ consider in addition the $2n$ polynomials $(\pm a_k)^t - 1$ for $k = 1, ..., n$, denoted by $p_{d+1}, ..., p_r$. Now apply the algorithm to $p = \delta_{y,x} \pm a_1^{s_1} \cdots a_n^{s_n}, p_1, ..., p_r$, for all $0 \leqslant s_k \leqslant t$ (i.e., finitely many times). If p is never contained in the ideal $(p_1, ..., p_r)$ then y is not Nielsen equivalent to x. If for some choice of the s_k the polynomial p is contained in $(p_1, ..., p_r)$. then repeat the test for larger $t \in \mathbb{N}$.

However there are groups for which the criterion can be applied in a "shorthand" version and one can distinguish Nielsen inequivalent generating systems with a simpler computation. For example:

(1) If one can find an ideal $J \subset \mathbb{Z}[a_{i,j}^k, a_k]$ which contains $\hat{J}_m$ such that $\mathbb{Z}[a_{i,j}^k, a_k]/\hat{J}_m$ is finite, then there are only finitely many elements of this quotient ring which are the images of any $a_1^{s_1} \cdots a_n^{s_n}$, $s_k \in \mathbb{Z}$. If the image of $\delta_{y,x}$ is not among them then y and x are Nielsen inequivalent.

For many examples even the rings $\mathbb{Z}/q\mathbb{Z}$ for a proper choice of $q \in \mathbb{N}$ yield (for $m = 1$!!) such inequivalent generating systems. This is shown in Example 2 of Section 4 (for G torsion free).

(2) Find a common solution for all generating polynomials of $\hat{J}_m$, i.e., a homomorphism $v: \mathbb{Z}[a_{i,j}^k, a_k]/\hat{J}_m \to \mathbb{C}$, with the additional property that the multiplicative subgroup U of $\mathbb{C}^*$ generated by $v(a_k)$ for $k = 1, ..., n$ is not dense in $\mathbb{C}$. Since U is finitely generated it is possible to decide whether $v(\delta_{y,x})$ is contained in the closure of U. If not, then y and x are Nielsen inequivalent. This method is illustrated in Example 1 of Section 4.

3. FREE PRODUCTS

In this section we prove an analogue of Stalling's direct sum theorem $\mathrm{Wh}(G_1 * G_2) = \mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2)$, for the functor $\mathcal{N}(\cdot)$ (see [St]). We first observe that in general the strict analogue is not true:

EXAMPLE 3.1. In Section 1 we computed

$$\mathcal{N}(\langle a \mid a^4 \rangle \oplus \langle b \mid b^4 \rangle) = (\mathbb{Z}/4\mathbb{Z})^*$$

and

$$\mathcal{N}(\langle c \mid c^6 \rangle \oplus \langle d \mid d^6 \rangle) = (\mathbb{Z}/6\mathbb{Z})^*$$

but

$$\mathcal{N}(\langle a \mid a^4 \rangle \oplus \langle b \mid b^4 \rangle \oplus \langle c \mid c^6 \rangle \oplus \langle d \mid d^6 \rangle) = (\mathbb{Z}/2\mathbb{Z})^*.$$

The situation can be even worse. For example, if $G_1 = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and $G_2 = \mathbb{Z}/q\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$ with relative prime $p, q \in \mathbb{N}$, then the torsion group $\mathcal{N}(G_1 * G_2)$ is degenerate, whereas both $\mathcal{N}(G_1) = (\mathbb{Z}/p\mathbb{Z})^*$ and $\mathcal{N}(G_2) = (\mathbb{Z}/q\mathbb{Z})^*$ are non-degenerate.

In order to avoid these phenomena we consider from now on group rings with field coefficients. We need to adapt the constructions of Section 2 to this situation:

DEFINITION 3.2. Let $\mathfrak{k}$ be a field. In the group ring $\mathfrak{k}[G]$ we define the ideal $I^{\mathfrak{k}}$ to be the $(\mathfrak{k} \otimes \iota)$-image of $\mathfrak{k} \otimes I$ in $\mathfrak{k} \otimes_{\mathbb{Z}} \mathbb{Z}G = \mathfrak{k}[G]$, where $\iota : I \subsetneq \mathbb{Z}G$ denotes the inclusion map. If $I^{\mathfrak{k}} \neq \mathfrak{k}[G]$ we define

$$\mathcal{N}(G; \mathfrak{k}) = K_1(\mathfrak{k}[G]/I^{\mathfrak{k}})/\mathfrak{k}^* \cdot G.$$

Otherwise we say $\mathcal{N}(G; \mathfrak{k})$ is "degenerate" and formally set $\mathcal{N}(G; \mathfrak{k}) = \{0\}$.

Let A be a commutative ring with 1, with a subring $\mathfrak{k} \cdot 1$ isomorphic to $\mathfrak{k}$. As in Proposition 2.2 every representation $\rho : \mathfrak{k}[G]/I^{\mathfrak{k}} \to \mathbb{M}_m(A)$ (with $\rho(\mathfrak{k}) = \mathfrak{k} \cdot 1$) induces a map $\mathcal{N}(G; \mathfrak{k}) \to A^*/T_\rho$. For technical reasons we quotient further to $A^*/\mathfrak{k}^* \cdot T_\rho$ and denote the composed map by $\mathcal{N}(\rho; \mathfrak{k})$.

DEFINITION 3.3. (1) Assume that there exists a representation ρ as above. Denote by $S\mathcal{N}(G; \mathfrak{k})$ the subgroup of all $\xi \in \mathcal{N}(G; \mathfrak{k})$ such that $\mathcal{N}(\rho; \mathfrak{k})(\xi) = 1$ for all ρ, A, and m as above. We define

$$\mathcal{N}(G; \mathfrak{k})^* = \mathcal{N}(G; \mathfrak{k})/S\mathcal{N}(G; \mathfrak{k}).$$

(2) Otherwise set $\mathcal{N}(G; \mathfrak{k})^* = 0$, and call $\mathcal{N}(G; \mathfrak{k})^*$ "degenerate."

Define a ring $A_m(G; \mathfrak{k})$ as the quotient ring of the polynomial ring $\mathfrak{k}[a^k_{i,j}, a_k]$ by the ideal $J_m(G; \mathfrak{k})$, which is defined analogously to J_m as in the discussion before Lemma 2.3. Denote by $\mathfrak{k}^* \cdot T_m$ the subgroup of $A_m(G; \mathfrak{k})^*$ generated by all elements $c \cdot a_k$, where $c \in \mathfrak{k}^*$. As in Section 2 there is a map $\sigma_m : \mathfrak{k}[G]/I^{\mathfrak{k}} \to \mathbb{M}_m(A_m(G; \mathfrak{k}))$, which induces a map $\mathcal{N}(\sigma_m) : \mathcal{N}(G; \mathfrak{k}) \to A^*(G; \mathfrak{k})/k^* \cdot T_m$. The image of $\mathcal{N}(\sigma_m)$ is denoted by $\mathcal{N}(G; \mathfrak{k})_m$; it has the analogous universality property as $\mathcal{N}(G)_m$ as stated in Proposition 2.5.

LEMMA 3.4. $\mathcal{N}(G;\mathfrak{k})^* = \operatorname{Im}(\prod_{m=1}^{\infty} \mathcal{N}(\sigma_m)) \subset \prod_{m=1}^{\infty} \mathcal{N}(G;\mathfrak{k})_m$.

Proof. From the definition of $S\mathcal{N}(G;\mathfrak{k})$ we obtain that all its elements are mapped to 0 by the product map $\prod \mathcal{N}(\sigma_m)$. Conversely, given an element ξ of $\mathcal{N}(G;\mathfrak{k})$ not contained in $S\mathcal{N}(G;\mathfrak{k})$, there are an $m \in \mathbb{Z}$, a commutative ring A, and a representation $\sigma: \mathfrak{k}[G]/I^{\mathfrak{k}} \to \mathbb{M}_m(A)$ such that $\mathcal{N}(\sigma)(\xi) \neq 0$. By the universality property of $\mathcal{N}(G;\mathfrak{k})_m$ this map factors through $\mathcal{N}(\sigma_m)$. Hence the image of ξ in $\prod \mathcal{N}(G;\mathfrak{k})_m$ is non-zero. This shows that the kernel of the product map $\prod \mathcal{N}(\sigma_m)$ is precisely $S\mathcal{N}(G;\mathfrak{k})$. ∎

THEOREM III. *Let G_1 and G_2 be groups and $\mathfrak{k}$ be a field, such that both $\mathcal{N}(G_1;\mathfrak{k})^*$ and $\mathcal{N}(G_2;\mathfrak{k})^*$ are non-degenerate. Then the natural embeddings $G_1 \subset G_1 * G_2$ and $G_2 \subset G_1 * G_2$ induce an isomorphism*

$$\mathcal{N}(G_1,\mathfrak{k})^* \oplus \mathcal{N}(G_2;\mathfrak{k})^* \xrightarrow{=} \mathcal{N}(G_1 * G_2;\mathfrak{k})^*.$$

Proof. By definition the Fox ideal $I^{\mathfrak{k}}(G_i) \subset \mathfrak{k}[G_i]$, $i = 1, 2$, is generated by the images of all Fox derivatives $\partial R/\partial x$, where x is an element of a minimal generating system X_i for G_i, and R is any element of $\ker(F(X_i) \to G_i)$. By Gruško's theorem any such generating system X_i is part of a minimal generating system for $G_1 * G_2$, and the above element R lies in the kernel of $\beta_{x_1 \cup x_2}: F(X_1, X_2) \to G_1 * G_2$ for some minimal generating systems X_1 of G_1, X_2 of G_2. Hence $\beta_{x_1 \cup x_2}(\partial R/\partial x) \in I^{\mathfrak{k}}(G_1 * G_2)$, and the embeddings $G_i \to G_1 * G_2$ induce ring homomorphisms

$$\Sigma^i: \mathfrak{k}[G_i]/I^{\mathfrak{k}}(G_i) \to \mathfrak{k}[G_1 * G_2]/I^{\mathfrak{k}}(G_1 * G_2) \qquad \text{for} \quad i = 1, 2.$$

The functoriality of $K_1(\cdot)$ then gives a map

$$\Sigma_1: K_1(\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1)) \oplus K_1(\mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2)) \to K_1(\mathfrak{k}[G_1 * G_2]/I^{\mathfrak{k}}(G_1 * G_2))$$

which maps $\mathfrak{k}^* \cdot G_1$ and $\mathfrak{k}^* \cdot G_2$ to $\mathfrak{k}^* \cdot (G_1 * G_2)$. Thus we obtain a natural map

$$\Sigma_2: \mathcal{N}(G_1;\mathfrak{k}) \oplus \mathcal{N}(G_2;\mathfrak{k}) \to \mathcal{N}(G_1 * G_2;\mathfrak{k}).$$

Assume that an element ξ in one of the $\mathcal{N}(G_i;\mathfrak{k})$ is mapped by Σ_2 to an element of $\mathcal{N}(G_1 * G_2;\mathfrak{k})$ which is not contained in $S\mathcal{N}(G_1 * G_2;\mathfrak{k})$. By definition this gives us a non-trivial representation of $\mathfrak{k}[G_1 * G_2]/I^{\mathfrak{k}}(G_1 * G_2)$ and an induced evaluation of $\mathcal{N}(G_1 * G_2;\mathfrak{k})$ with non-trivial value of ξ. By composition with Σ_1 we obtain a representation for $\mathfrak{k}[G_i]/I^{\mathfrak{k}}(G_i)$, which also gives a non-trivial value for ξ under the induced evaluation of $\mathcal{N}(G_i;\mathfrak{k})$. Hence ξ does not lie in $S\mathcal{N}(G_i;\mathfrak{k})$, and thus Σ_2 induces a map

$$\Sigma_3: \mathcal{N}(G_1;\mathfrak{k})^* \oplus \mathcal{N}(G_2;\mathfrak{k})^* \to \mathcal{N}(G_1 * G_2;\mathfrak{k})^*.$$

The above three maps Σ_i can be composed by definition to a commutative diagram:

$$\begin{array}{ccc}
\Sigma_1: K_1(\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1)) \oplus K_1(\mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2)) & \longrightarrow & K_1(\mathfrak{k}[G_1 * G_2]/I^{\mathfrak{k}}(G_1 * G_2)) \\
\downarrow & & \downarrow \\
\Sigma_2: \mathcal{N}(G_1; \mathfrak{k}) \oplus \mathcal{N}(G_2; \mathfrak{k}) & \longrightarrow & \mathcal{N}(G_1 * G_2; \mathfrak{k}) \\
\downarrow & & \downarrow \\
\Sigma_3: \mathcal{N}(G_1; \mathfrak{k})^* \oplus \mathcal{N}(G_2; \mathfrak{k})^* & \longrightarrow & \mathcal{N}(G_1 * G_2; \mathfrak{k})^*
\end{array}$$

CLAIM 1. *Σ_3 is surjective.*

Proof of Claim 1. Since all vertical maps in the above diagram are quotient maps it suffices to show that Σ_1 is surjective. This will be derived below as consequence of a Theorem of Casson (see [C]). We use his notation and definitions.

(1) We first show that the conditions of Theorems 1 and 2 of [C] are satisfied: It follows from Gruško's theorem and Remark 1.2(1) of Section 1 that $I^{\mathfrak{k}}(G_1 * G_2)$ is the ideal in $\mathfrak{k}[G_1 * G_2]$ generated by $I^{\mathfrak{k}}(G_1)$ and $I^{\mathfrak{k}}(G_2)$. Note that since $\mathfrak{k}[G_1]$ and $\mathfrak{k}[G_2]$ are vector spaces over $\mathfrak{k}$, the subrings $\mathfrak{k} \cdot 1 \subset \mathfrak{k}[G_i]$ are both pure (i.e., $\mathfrak{k}[G_i] = \mathfrak{k} \cdot 1 \oplus A$ as $\mathfrak{k}$-bimodule for some suitable A, see [C]). Hence we can form the amalgamated free product of rings $\mathfrak{k}[G_1] *_{\mathfrak{k}} \mathfrak{k}[G_2]$ and $\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1) *_{\mathfrak{k}} \mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2)$.

The proof of the following fact (**) is an elementary exercise, based on the universality property of the amalgamated free product of rings:

Given two $\mathfrak{k}$-algebras A, B with ideals $I \subset A$, $J \subset B$, then
$$A/I *_{\mathfrak{k}} B/J = (A *_{\mathfrak{k}} B)/(I, J). \qquad (**)$$

As an application of (**) we obtain

$$\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1) *_{\mathfrak{k}} \mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2) = \mathfrak{k}[G_1 * G_2]/I^{\mathfrak{k}}(G_1 * G_2).$$

(2) By the non-degeneracy assumption in Theorem III there are canonical injections $\alpha: \mathfrak{k} \to \mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1)$ and $\beta: \mathfrak{k} \to \mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2)$. For these injections Casson defines in [C] a group $K_1(\alpha, \beta)$ and homomorphisms $j: K_1(\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1)) \oplus K_1(\mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2)) \to K_1(\alpha, \beta)$ and $\tau: K_1(\alpha, \beta) \to K_1(\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1) *_{\mathfrak{k}} \mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2))$. It is immediate from his definitions (using the last equation) that $\tau \circ j$ differs from the above defined map Σ_1 only in the sign of the image of the second factor. Hence it suffices to show that $\tau \circ j$ is surjective.

(3) The map τ is surjective: This is proved in Theorem 3 of [C], under the hypotheses that first every $\mathfrak{k}$-bimodule is adirect limit of free $\mathfrak{k}$-bimodules and second that $\mathrm{Nil}(\mathfrak{k}) = 0$. But in the present situation $\mathfrak{k}$ is a

field and hence every $\mathfrak{k}$-bimodule is free (and hence the direct limit of free bimodules), and $\mathrm{Nil}(\mathfrak{k}) = K_1(\mathfrak{k}[t])/K_1(\mathfrak{k}) = 0$ by [M1, pp. 27–28].

(4) The map j is surjective: We apply Theorem 1 of [C], where j appears as part of an exact sequence:

$$K_1(\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1)) \oplus K_1(\mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2)) \xrightarrow{j} K_1(\alpha, \beta) \xrightarrow{\partial} K_0(\mathfrak{k})$$

$$\xrightarrow{i} K_0(\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1)) \oplus K_0(\mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2)).$$

The surjectivity of j is hence equivalent to the injectivity of the canonically induced map $i: K_0(\mathfrak{k}) \to K_0(\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1)) \oplus K_0(\mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2)$. The latter follows directly if one of the two rings $\mathfrak{k}[G_1]/I^{\mathfrak{k}}(G_1)$ or $\mathfrak{k}[G_2]/I^{\mathfrak{k}}(G_2)$ has the basis invariance property (see [M1, Chap. 1]). But the assumption that $\mathcal{N}(G_i; \mathfrak{k})^*$ is non-degenerate means that there exists a non-trivial ring representation of $\mathfrak{k}[G_i]/I^{\mathfrak{k}}(G_i)$ into a matrix ring over a commutative ring A. Such rings $\mathbb{M}_m(A)$ have the basis invariance property, since left invertible matrices over commutative rings are also right invertible. But for all ring homomorphisms $R \to S$ it follows that if S has the basis invariance property then so does R. This completes the proof of Claim 1.

CLAIM 2. *Σ_3 is injective.*

Proof of Claim 2. It suffices to show that for any non-zero element $\xi \in \mathcal{N}(G_i; \mathfrak{k})^*$ its image $\Sigma_3(\xi)$ is not contained in $\Sigma_3(\mathcal{N}(G_j; \mathfrak{k})^*) \subset \mathcal{N}(G_1 * G_2; \mathfrak{k})^*$, for $\{i, j\} = \{1, 2\}$. By Lemma 3.4 this follows if we can show:

For some $n \in \mathbb{N}$ the image of ξ in $\mathcal{N}(G_1 * G_2; \mathfrak{k})_n$ is not contained in the image of $\mathcal{N}(G_j; \mathfrak{k})^*$. (***)

From the universality property of the rings $A_m(G_i; \mathfrak{k})$ (see Lemma 3.4) it follows that for some $m \in \mathbb{N}$ the image of ξ in $\mathcal{N}(G_i; \mathfrak{k})_m$ is non-zero (and in particular $\mathcal{N}(G_i; \mathfrak{k})_m$ is non-degenerate). Similarily one obtains that for some $m' \in \mathbb{N}$ the group $\mathcal{N}(G_j; \mathfrak{k})_{m'}$ is non-degenerate.

Assumption 1. There exists a number $n \in \mathbb{N}$ such that

(1) both $\mathcal{N}(G_1; \mathfrak{k})_n$ and $\mathcal{N}(G_2; \mathfrak{k})_n$ are non-degenerate, and

(2) $\mathcal{N}(G_i; \mathfrak{k})_n$ maps surjectively and naturally onto $\mathcal{N}(G_i; \mathfrak{k})_m$. By naturalness here we mean that the surjection commutes with the maps $\mathcal{N}(G_i; \mathfrak{k}) \to \mathcal{N}(G_i; \mathfrak{k})^* \to \mathcal{N}(G_i, \mathfrak{k})_k$.

Assumption 2. Let $\mathfrak{k}$ be a field such that both $\mathcal{N}(G_1; \mathfrak{k})_n$ and $\mathcal{N}(G_2; \mathfrak{k})_n$ are non-degenerate. Then the natural embeddings $G_1 \subset G_1 * G_2$ and $G_2 \subset G_1 * G_2$ induce an injection

$$\Sigma^n: \mathcal{N}(G_1; \mathfrak{k})_n \oplus \mathcal{N}(G_2; \mathfrak{k})_n \hookrightarrow \mathcal{N}(G_1 * G_2; \mathfrak{k})_n.$$

From assumption 1 we obtain that $\Sigma^n(\xi) \in \mathcal{N}(G_i; \mathfrak{k})_n$ is non-trivial. From assumption 2 we obtain that $\Sigma^n(\xi)$ is not contained in $\Sigma^n(\mathcal{N}(G_j; \mathfrak{k})_n) \subset \mathcal{N}(G_1 * G_2; \mathfrak{k})_n$. By naturaliness the following square commutes:

$$\begin{array}{ccc} \mathcal{N}(G_j; \mathfrak{k})^* & \longrightarrow & \mathcal{N}(G_1 * G_2; \mathfrak{k})^* \\ \downarrow & & \downarrow \\ \mathcal{N}(G_j; \mathfrak{k})_n & \longrightarrow & \mathcal{N}(G_1 * G_2; \mathfrak{k})_n \end{array}$$

This gives the above fact (***). The assumptions 1 and 2 are proved below in Lemma 3.7 and Lemma 3.6 respectively. ∎

Before proving the two statements used in the last proof we show:

LEMMA 3.5. *Let $\mathfrak{k}$ be a field. Assume that both of the rings $A_m(G_1; \mathfrak{k})$ and $A_m(G_2; \mathfrak{k})$ are non-trivial. Then the injections $\alpha_i: G_i \hookrightarrow G_1 * G_2$ induce injective ring homomorphisms $\alpha_{\# i}: A_m(G_i; \mathfrak{k}) \to A_m(G_1 * G_2; \mathfrak{k})$, $i = 1, 2$. Furthermore one has*

$$\mathrm{im}(\alpha_{\# 1}) \cap \mathrm{im}(\alpha_{\# 2}) = \mathfrak{k} \cdot 1.$$

Proof. As $\mathfrak{k}$-vector spaces one has (non-canonical) direct sum decompositions $A_m(G_i, \mathfrak{k}) = \mathfrak{k} \cdot 1 \oplus A_i$, $i = 1, 2$. Hence $a_1 \to a_1 \otimes 1$, $a_2 \to 1 \otimes a_2$ for $a_i \in A_m(G_i; \mathfrak{k})$ defines maps $\varphi_i: A_m(G_i; \mathfrak{k}) \to A_m(G_1; \mathfrak{k}) \otimes_{\mathfrak{k}} A_m(G_2; \mathfrak{k}) = (\mathfrak{k} \cdot 1 \otimes_{\mathfrak{k}} \mathfrak{k} \cdot 1) \oplus (\mathfrak{k} \cdot 1 \otimes_{\mathfrak{k}} A_2) \oplus (A_1 \otimes_{\mathfrak{k}} \mathfrak{k} \cdot 1) \oplus (A_1 \otimes_{\mathfrak{k}} A_2)$ which are injective, and their images meet precisely in $(\mathfrak{k} \cdot 1 \otimes_{\mathfrak{k}} \mathfrak{k} \cdot 1)$. It suffices to show that there is a $\mathfrak{k}$-vector space isomorphism ψ from $A_m(G_1; \mathfrak{k}) \otimes_{\mathfrak{k}} A_m(G_2; \mathfrak{k})$ to $A_m(G_1 * G_2; \mathfrak{k})$ such that $\psi \cdot \varphi_i = \alpha_{\# i}$. This can be seen as follows:

We denote by P, P_1, and P_2 the polynomial rings $\mathfrak{k}[a_{ij}^k, a_k]$ for $G_1 * G_2$, G_1, and G_2, respectively, constructed via the presentations $G_1 = \langle X | R \rangle$, $G_2 = \langle Y | S \rangle$ and $G_1 * G_2 = \langle X, Y | R, S \rangle$ as introduced in the discussion before Lemma 3.4. We obtain a $\mathfrak{k}$-vector space isomorphism,

$$\eta: P_1 \otimes_{\mathfrak{k}} P_2 \to P,$$

if one just replaces the tensor product in every element of $P_1 \otimes_{\mathfrak{k}} P_2$ by the ring multiplication. But the rings $A_m(\cdot; \mathfrak{k})$ are defined as quotient of these polynomial rings modulo ideals $\hat{J}_m(\cdot)$, generated by the polynomials $\det(a_{i,j}^k) - a_k$ and the ideal $J_m(\cdot)$. It follows from the Gruško-Neumann Theorem that $\hat{J}_m(G_1 * G_2) = \ker(P \twoheadrightarrow A_m(G; \mathfrak{k}))$ is generated by the η-images of the ideals $\hat{J}_m(G_1) \otimes 1$ and $1 \otimes \hat{J}_m(G_2)$. Hence η induces a $\mathfrak{k}$-vector space isomorphism

$$A_m(G_1; \mathfrak{k}) \otimes_{\mathfrak{k}} A_m(G_2; \mathfrak{k}) \to A_m(G_1 * G_2; \mathfrak{k}). \quad ∎$$

The following proposition is used in the above proof for the injectivity of the map Σ_3, but is also relevant in itself for practical applications

PROPOSITION 3.6. *Let $\mathfrak{k}$ be a field such that both $\mathcal{N}(G_1;\mathfrak{k})_n$ and $\mathcal{N}(G_2;\mathfrak{k})_n$ are non-degenerate. Then the natural embeddings $G_1 \subset G_1 * G_2$ and $G_2 \subset G_1 * G_2$ induce an injection*

$$\Sigma^n: \mathcal{N}(G_1;\mathfrak{k})_n \oplus \mathcal{N}(G_2;\mathfrak{k})_n \hookrightarrow \mathcal{N}(G_1 * G_2;\mathfrak{k})_n.$$

Proof. By definition $\mathcal{N}(G_1 * G_2;\mathfrak{k})_n$ is the quotient of a subgroup of $A_n(G_1 * G_2;\mathfrak{k})^*$ modulo $\mathfrak{k}^* \cdot T_n(G_1 * G_2)$. By Lemma 3.5 the multiplicative group of units $A_n(G_1 * G_2;\mathfrak{k})^*$ contains isomorphic images of the groups $A_n(G_1;\mathfrak{k})^*$ and $A_n(G_2,\mathfrak{k})^*$ as subgroups which meet precisely in $\mathfrak{k}^* \cdot 1$. Hence one obtains an injection

$$A_n(G_1;\mathfrak{k})^*/\mathfrak{k}^* \oplus A_n(G_2;\mathfrak{k})^* \hookrightarrow A_n(G_1 * G_2;\mathfrak{k})^*/\mathfrak{k}^*.$$

It remains to show that the subgroups of $A_n(G_1 * G_2;\mathfrak{k})^*$ isomorphic to $A_n(G_i;\mathfrak{k})^*$ $(i=1,2)$ intersect $\mathfrak{k}^* \cdot T_n(G_1 * G_2)$ precisely in $\mathfrak{k}^* \cdot T_n(G_i)$. From the definition it follows directly that $\mathfrak{k}^* \cdot T_n(G_i)$ is contained in $A_n(G_1 * G_2;\mathfrak{k})^* \cap \mathfrak{k}^* \cdot T_n(G_1 * G_2)$. The converse inclusion follows from the fact that (1) $A_n(G_1;\mathfrak{k})^* \cap A_n(G_2;\mathfrak{k})^* = \mathfrak{k}^*$ and (2) $\mathfrak{k}^* \cdot T_n(G_1 * G_2)$ is an abelian image of $G_1 * G_2$ (since by definition it consists of the determinants of a representation of $G_1 * G_2$ in a matrix ring with abelian entries). ∎

LEMMA 3.7. *For any field $\mathfrak{k}$ and any two numbers m, $n \in \mathbb{N}$, such that m divides n and $\mathcal{N}(G;\mathfrak{k})_m$ is non-degenerate, there exists a surjection $\mathcal{N}(G,\mathfrak{k})_n \twoheadrightarrow \mathcal{N}(G;\mathfrak{k})_m$ which is natural in the following sense: It commutes with the quotient maps $\mathcal{N}(G;\mathfrak{k}) \twoheadrightarrow \mathcal{N}(G;\mathfrak{k})_n$, $\mathcal{N}(G;\mathfrak{k}) \twoheadrightarrow \mathcal{N}(G;\mathfrak{k})_m$.*

Note that, by Lemma 3.4, the surjection $\mathcal{N}(G;\mathfrak{k})_n \twoheadrightarrow \mathcal{N}(G;\mathfrak{k})_m$ also commutes with $\mathcal{N}(G;\mathfrak{k}) \twoheadrightarrow \mathcal{N}(G;\mathfrak{k})^* \twoheadrightarrow \mathcal{N}(G;\mathfrak{k})_n$, $\mathcal{N}(G;\mathfrak{k}) \twoheadrightarrow \mathcal{N}(G;\mathfrak{k})^* \twoheadrightarrow \mathcal{N}(G;\mathfrak{k})_m$.

Proof. It suffices to show the existence of a natural map $\mathcal{N}(G;\mathfrak{k})_n \to \mathcal{N}(G;\mathfrak{k})_m$, since by definition of naturalness such a map must be surjective. For any $(n \times n)$-matrix one can consider the sequence of n/m $(m \times m)$-block matrices along the diagonal. In each of the matrices $(a^k_{i,j})_{i,j}$ consider the elements $a^k_{i,j}$ which lie outside these diagonal blocks. Let A be the quotient of $A_n(G;\mathfrak{k})$ modulo the ideal generated by this subset.

For $h \in \{1, ..., n/m\}$ let A_h be the subring of A generated by all the variables $a^k_{i,j}$ which appear in the hth diagonal block of any of the matrices $(a^k_{i,j})_{i,j}$. For each of the rings A_h there is a "natural" isomorphism $i_h: A_m(G;\mathfrak{k}) \to A_h$, since the $a^k_{i,j}$ are subject to exactly the same relations as the corresponding generators of $A_m(G;\mathfrak{k})$. Here the precise meaning of "natural" is that the following diagram commutes, where the lower right horizontal map identifies an element $(M_1, ..., M_{n/m}) \in \bigoplus_{h=1,...,n/m} \mathbb{M}_m(A_h)$ with the corresponding block diagonal matrix in $\mathbb{M}_n(A)$:

$$\begin{array}{ccc}
\mathfrak{k}[G]/I^{\mathfrak{k}} & \xrightarrow{\sigma_n} & \mathbb{M}_n(A_n(G;\mathfrak{k})) \\
\big\downarrow{\scriptstyle \sigma_m} & & \big\downarrow \\
\mathbb{M}_m(A_m(G;\mathfrak{k})) & \xrightarrow{\oplus i_h} & \bigoplus \mathbb{M}_m(A_h) \to \mathbb{M}_n(A)
\end{array}$$

The subrings $A_1, ..., A_{n/m} \subset A$ meet precisely in $\mathfrak{k} \cdot 1$, by the same argument as in the proof of Lemma 3.5. As in the proof of Proposition 3.6 the maps i_h induce an injection of the direct sum of m/n copies of $\mathcal{N}(G, \mathfrak{k})_m$ into the quotient of A^* modulo the subgroup generated by all the "trivial units" $i_h(\mathfrak{k}^* \cdot T_m)$. Hence there is an isomorphism i from $\mathcal{N}(G; \mathfrak{k})_m$ to the diagonal subgroup D of this direct sum. Lemma 3.7 will be proved if we can show:

CLAIM. *The map $A_n(G; \mathfrak{k}) \to A$ induces a homomorphism j: $\mathcal{N}(G; \mathfrak{k})_n \to D$ which satisfies $j \circ \mathcal{N}(\sigma_n) = i \circ \mathcal{N}(\sigma_m)$.*

Proof of the Claim. Every element $\xi \in \mathcal{N}(G, \mathfrak{k})$ is represented by some square matrix $M \in \mathbb{M}_q(\mathfrak{k}[G]/I^{\mathfrak{k}})$, $q \in \mathbb{N}$. Applying the representation σ_n: $\mathfrak{k}[G]/I^{\mathfrak{k}} \to \mathbb{M}_n(A_n(G; \mathfrak{k}))$ to the entries of M gives a matrix $M' \in \mathbb{M}_q(\mathbb{M}_n(A_n(G; \mathfrak{k}))) = \mathbb{M}_{q \cdot n}(A_n(G; \mathfrak{k}))$, which by definition of σ_m has as determinant (a representative of) the image of ξ in $\mathcal{N}(G; \mathfrak{k})_n$.

If we apply the quotient map $A_n(G; \mathfrak{k}) \to A$ to the matrix M' it takes a form M'' which we describe schematically as follows (for the schematic representation we choose $q = 4$, $n/m = 3$):

$$\left[\begin{array}{cccc}
\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} \\
\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} \\
\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} \\
\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} & \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix}
\end{array}\right]$$

$$
= \begin{bmatrix}
A & & & A & & & A & & & A & & \\
& B & & & B & & & B & & & B & \\
& & C & & & C & & & C & & & C \\
A & & & A & & & A & & & A & & \\
& B & & & B & & & B & & & B & \\
& & C & & & C & & & C & & & C \\
A & & & A & & & A & & & A & & \\
& B & & & B & & & B & & & B & \\
& & C & & & C & & & C & & & C \\
A & & & A & & & A & & & A & & \\
& B & & & B & & & B & & & B & \\
& & C & & & C & & & C & & & C
\end{bmatrix}.
$$

By elementary row and column operations M'' is transformed into a diagonal block matrix:

$$
M''' = \begin{bmatrix}
A & A & A & A & & & & & & & & \\
A & A & A & A & & & & & & & & \\
A & A & A & A & & & & & & & & \\
A & A & A & A & & & & & & & & \\
& & & & B & B & B & B & & & & \\
& & & & B & B & B & B & & & & \\
& & & & B & B & B & B & & & & \\
& & & & B & B & B & B & & & & \\
& & & & & & & & C & C & C & C \\
& & & & & & & & C & C & C & C \\
& & & & & & & & C & C & C & C \\
& & & & & & & & C & C & C & C
\end{bmatrix}.
$$

For all $h = 1, ..., n/m$ the hth block of M''' has entries in the ring $A_h = i_h(A_m(G; \mathfrak{k}))$. By the commutativity of the above diagram each such block coincides with the image of M induced by the composed map $i_h \circ \sigma_m : \mathfrak{k}[G]/I^{\mathfrak{k}} \to A_m(G; \mathfrak{k}) \to A_h$. Since the determinant of M' is mapped by $A_n(G; \mathfrak{k}) \twoheadrightarrow A$ to the product of the n/m determinants of the $(mq \times mq)$-diagonal blocks of M''', this map induces a map from $\mathcal{N}(G, \mathfrak{k})_n$ into D which maps $\mathcal{N}(\sigma_n)(\xi)$ to $i \circ \mathcal{N}(\sigma_m)(\xi)$. ∎

4. Applications

In this section we give four examples of applications of the invariant $\mathcal{N}(G)$. These examples show that one can find non-zero elements of $\mathcal{N}(G)$ and in particular Nielsen inequivalent generating systems for many groups.

The first two examples are knot groups (which are of course torsion free). An application of $\mathcal{N}(G)$ to a much wider class of knot groups, including all 2-bridge knots/links and many Montesinos knots/links, is given in [LM2], where non-trivial values $\mathcal{N}(x, y)$ are used to determine inequivalent systems of unknotting tunnels. The third example concerns one relator quotients of free products of cyclics. A particular case of such groups are Fuchsian groups, for which $\mathcal{N}(G)$ computations gives the full classification of all Nielsen classes of minimal generating systems (see [LM1]). In our fourth example we present a group G and use the invariant $\mathcal{N}(G)$ to show that G admits an automorphism which does not lift to the free group specified in the presentation.

4.1. *Knot Groups*

Let $K \subset S^3$ be a knot and $X = \mathrm{cl}(S^3 - \mathrm{int}\, N(K))$ its exterior. The fundamental group $G = \pi_1(X)$ has a Wirtinger presentation of the form $G = \langle x_1, ..., x_n \mid r_1, ..., r_{n-1} \rangle$. The ring homomorphism $\varphi: \mathbb{Z}G \to \mathbb{Z}(G/[G, G]) = \mathbb{Z}[t, t^{-1}]$ given by the abelianization of G induces the map σ_1 (defined in Section 2) on the quotient ring $\mathbb{Z}G/I_x$. Let $\tilde{X}$ denote the infinite cyclic covering space of X. It is a well known fact that the matrix $(\varphi\beta_x(\partial r_k/\partial x_i))$ is a presentation matrix for $H_1(\tilde{X}) \oplus \mathbb{Z}[t, t^{-1}]$ as a $\mathbb{Z}[t, t^{-1}]$-module (see [G, Section 6]). We can get a presentation for $H_1(\tilde{X})$ by deleting any column from the matrix. Hence $\det(\varphi\beta_x(\partial r_k/\partial x_i))$ is the Alexander polynomial $\Delta_1^K(t)$. By Lemma 2.1 we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}G/I_x & \longrightarrow & \mathbb{Z}[t, t^{-1}]/(\varphi(I_x)) \\ & \searrow \quad \nearrow & \\ & \mathbb{Z}G/I & \end{array}$$

In the case of a 2-bridge knot $K(\alpha, \beta)$ (see [BZ]), where α, β are odd integers such that $-\alpha < \beta < \alpha$ and g.c.d. $(\alpha, \beta) = 1$, one obtains $G = \langle x_1, x_2 \mid r(\alpha, \beta) \rangle$ and $\varphi(I_x) = (\det(\varphi\beta_x(\partial r_k/\partial x_i))) = (\Delta_1^K(t))$ is the ideal generated by the first Alexander polynomial of the knot K. In particular, if $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are two generating systems for G with the property that $\det(\varphi\beta_x(\partial y_j/\partial x_i)) \neq \pm t^m \bmod(\Delta_1^K(t))$ for all $m \in \mathbb{Z}$, then $\mathcal{N}(y, x) \neq 0 \in \mathcal{N}(G)$.

In the case of a 3-bridge knot one has $G = \langle x_1, x_2, x_3 \mid r_1, r_2 \rangle$. It follows that $\varphi(I_x)$ is the second elementary ideal and is contained in the unique

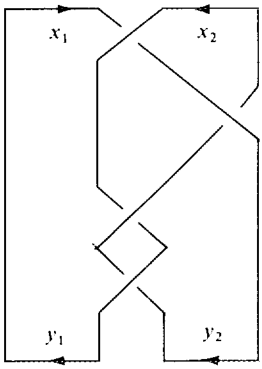


FIGURE 1

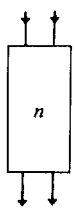
minimal principal ideal generated by the second Alexander polynomial. As before if $\det(\sigma_1 \gamma_x \beta_x(\partial y_j/\partial x_i)) \neq \pm t^m \operatorname{mod}(\Delta_2^K(t))$ for all $m \in \mathbb{Z}$, then $\mathcal{N}(y, x) \neq 0 \in \mathcal{N}(G)$. Similarly one can use the $(n-1)$th Alexander invariant to study $\mathcal{N}(G)$ for n-bridge knots.

EXAMPLE 4.1. Let K be the figure 8 knot, i.e., $K = K(5, 3)$. We choose generators $x = \{x_1, x_2\}$ and $y = \{y_1, y_2\}$ as indicated in Fig. 1. From the Wirtinger presentation we obtain the relations $x_1 = y_1$, and $y_2 = x_2^{-1} x_1 x_2$. It follows that

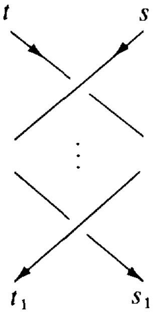
$$\left[\frac{\partial y}{\partial x}\right] = \begin{bmatrix} 1 & 0 \\ -x_2^{-1} & -x_2^{-1} + x_2^{-1} x_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ -t^{-1} & -t^{-1} + 1 \end{bmatrix}.$$

Furthermore $\det(\varphi \beta_x(\partial y_j/\partial x_i)) = t - 1$ up to units of $\mathbb{Z}[t, t^{-1}]$. The Alexander polynomial of K is $\Delta_1(t) = t^2 - 3t + 1$, and $\Delta_1(t) = 0$ implies that $t = (3 \pm \sqrt{5})/2$. Hence $|t|^{\pm m} = |(3 \pm \sqrt{5})/2|^{\pm m} \neq |(1 \pm \sqrt{5})/2| = |t - 1|$ for all $m \in \mathbb{Z}$. Therefore $\mathcal{N}(y, x)$ is a non-trivial unit. Similar arguments work for other 2-bridge knots.

EXAMPLE 4.2. Let $K \subset S^3$ be the oriented 3-bridge knot represented in Fig. 2 below. By



we mean n full right hand twists. Note that if



then $t_1 = (ts)^{-n} t(ts)^n$ and $s_1 = (ts)^{-n} s(ts)^n$. The projection in Fig. 2 gives us two presentations for the fundamental group G of the knot exterior,

$$G = \langle a, b, c \mid R_1, R_2, R_3 \rangle = \langle x, y, z \mid S_1, S_2, S_3 \rangle,$$

where the relators R_i, S_i come from the two possibilities of writing each of x, y, z as a product of a, b, c, and vice versa.

We get $x = (bc^{-1})^{-p} b(bc^{-1})^p$, $z = (ab^{-1})^{-r} b(ab^{-1})^r$, and $w_1 = (bc^{-1})^{-p} c(bc^{-1})^p$, $w_2 = (ca^{-1})^{-q} c(ca^{-1})^q$, $w_3 = (ca^{-1})^{-q} a(ca^{-1})^q$, and $w_4 = (ab^{-1})^{-r} a(ab^{-1})^r$. Hence $y = (w_1^{-1}w_2)^{-s} w_2(w_1^{-1}w_2)^s$, and $R_1 = x(w_1^{-1}w_2)^{-s} w_1^{-1}(w_1^{-1}w_2)^s$, $R_3 = z^{-1}(w_3^{-1}w_4)^{-u} w_4(w_3^{-1}w_4)^u$. The image

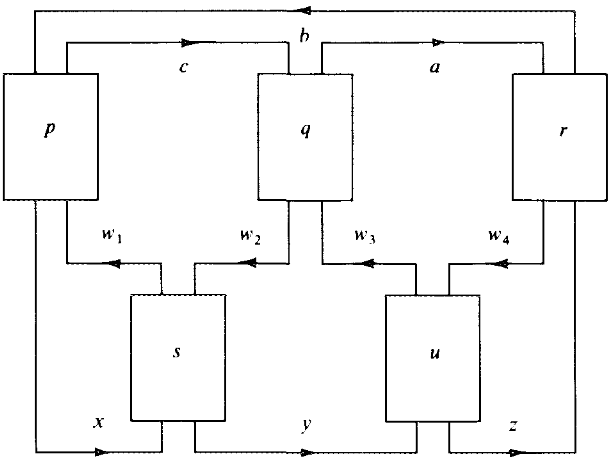


FIGURE 2

of the Fox ideal $I_{\{a,b,c\}}$ in $\mathbb{Z}[t, t^{-1}]$ is generated by the polynomials $\{qs(1-t)^2, ps(1-t)^2+t, qu(1-t)^2, ru(1-t)^2+t\}$. Hence we get a map

$$\varphi: \mathbb{Z}G/I_{\{a,b,c\}} \to \mathbb{Z}[t, t^{-1}]/(qs(1-t)^2, ps(1-t)^2+t, qu(1-t)^2, ru(1-t)^2+t).$$

PROPOSITION 4.3. *For all $p, q, r, s, u \in \mathbb{Z} - \{0\}$ which satisfy $q = 4kpr - 1$, $u = kp$, $s = kr$ for some $k \in \mathbb{Z}$, $k \geqslant 2$, the generating systems a, b, c and x, y, z as above are not Nielsen equivalent.*

Proof. Consider the image of the above map φ. We can further quotient (as in the discussion below Criterion 2.6) $\mathbb{Z}$ to $\mathbb{Z}/q\mathbb{Z}$ and t to -1 to get a map

$$\sigma: \mathbb{Z}G/I_{\{a,b,c\}} \to (\mathbb{Z}/q\mathbb{Z})/(4ps-1, 4ru-1) = \mathbb{Z}/q\mathbb{Z}.$$

The image of $(\partial\{x, y, z\}/\partial\{a, b, c\})$ under this map is the matrix

$$\sigma\gamma\beta(\partial\{x, y, z\}/\partial\{a, b, c\}) = \begin{bmatrix} 0 & 1-2p & 2p \\ 0 & 4ps & 1-4sp \\ -2r & 2r+1 & 0 \end{bmatrix}.$$

Taking the determinant we get $\tau_\sigma(\partial\{x, y, z\}/\partial\{a, b, c\}) = 4pr \neq \pm 1 \bmod q$. Hence $\mathcal{N}(\{x, y, z\}, \{a, b, c\})$ is a non-trivial unit, and the systems $\{x, y, z\}$, $\{a, b, c\}$ are not Nielsen equivalent by Theorem II. ▮

Remark 4.4. In fact, with no extra work the above examples give Nielsen inequivalent generating systems in groups of closed 3-manifolds: From $X = S^3 - \operatorname{int} N(K)$ we can obtain a closed manifold $K(m/n)$ by doing m/n-surgery, i.e., gluing a solid torus to the boundary of X along a rational slope m/n (see [R]). For, if $\{\mu, \lambda\}$ is a meridian longitude pair for K, we get a presentation for $\pi_1(K(m/n))$ by adding one relation of the form $\mu^m\lambda^n$ to G. If m is even then the images of the Fox derivatives of this relation is zero in $\mathbb{Z}/q\mathbb{Z}/(4ps-1, 4ru-1)$. This is because λ is in the second commutator subgroup of G, and μ is mapped to $t \in \mathbb{Z}[t, t^{-1}]$. Hence the image of the element $\mathcal{N}(\{x, y, z\}, \{a, b, c\}) \in \mathcal{N}(\pi_1(K(m/n)))$ is equal to the image of $\mathcal{N}(\{x, y, z\}, \{a, b, c\}) \in \mathcal{N}(G)$, and is therefore not a trivial unit. In other words, the generating systems $\{x, y, z\}$ and $\{a, b, c\}$ are not Nielsen equivalent in $\pi_1(K(m/n))$. This also implies that the ordered Heegaard splittings of the closed manifolds induced this way are not isotopic (see [LM2] for an extension of this result to regular Heegaard splittings).

Non-isotopic surfaces in 2-bridge knot complements which give non-isotopic ordered Heegaard splittings of the closed 3-manifolds $K(m/n)$ where obtained by a completely different method in [BM].

4.2. *One Relator Quotients of Free Products of Cyclics*

Let G be a group which is the one relator quotient of a free product of cyclic groups,

$$G = (\langle x_1 \mid x_1^{p_1} \rangle * \cdots * \langle x_n \mid x_n^{p_n} \rangle)/\langle w(x_1, ..., x_n)^m \rangle$$

with $w(x_1, ..., x_n) \neq x_i^k$, $p_i \geqslant 2$ for all i, k and with $m \geqslant 2$. For the simplicity of the computations we assume that the exponents p_i are pairwise relatively prime and odd.

PROPOSITION 4.5. *Two generating systems $u = \{x_1^{u_1}, ..., x_n^{u_n}\}$ and $v = \{x_1^{v_1}, ..., x_n^{v_n}\}$, with $0 \leqslant u_i, v_i < p_i$, are Nielsen equivalent if and only if $v_i = \pm u_i \bmod p_i$ for all $i = 1, ..., n$.*

Proof. The "if" direction of the proof is obvious. For the "only if" direction we prove that the image of $\mathcal{N}(v, u)$ in $\mathcal{N}(G)_2$ is not zero. For this purpose we use a representation technique described in [FHR]. There it is shown that the group G possesses a representation $\rho: \mathbb{Z}G \to PSL_2(\mathbb{C})$ which is faithful on each factor $\langle x_i \mid x_i^{p_i} \rangle$ and also on the subgroup of G generated by $w(x_1, ..., x_n)$. Using the same methods as in the proof given in [FHR] one can show that there is a representation $\hat{\rho}: \mathbb{Z}G \to SL_2(\mathbb{C})$ which is faithful on the same subgroups of G. The element $\mathcal{N}(v, u)$ is a product of cyclotomic units in some field $\mathbb{Q}(\xi)$, where ξ is a root of unity. The calculations performed in [LM1] for Fuchsian groups generalize straightforwardly to give the above claim. ∎

4.3. *Lifting Automorphism to Free Groups*

Let $F(a, b)$ be the free group with basis a, b and set $w = w(a, b) = ab^kab^{-k}a^{-1} \in F(a, b)$ for an arbitrary integer k. Consider the group

$$G = \langle a, b \mid w(a^{-1}, wb^{-1}w^{-1})\, wbw^{-1}w(a^{-1}, wb^{-1}w^{-1})^{-1} b^{-1} \rangle.$$

For any $k \in \mathbb{Z}$ the group G admits an involution $\varphi: G \to G$ defined by $\varphi(a) = a^{-1}$ and $\varphi(b) = wb^{-1}w^{-1}$.

PROPOSITION 4.6. *The involution φ does not lift to an automorphism of $F(a, b)$ for any $k \in \mathbb{Z} - \{0, 1, -1\}$; i.e., φ is non-tame.*[1]

Proof. It suffices to show that the generating system $\varphi(a)$, $\varphi(b)$ of G is not Nielsen equivalent to a, b. We consider the quotient map

[1] An interesting (somewhat disguised) application of $\mathcal{N}(G)$ has been given recently by Bryant *et al.* [BGLM], who showed that E. Stöhr's automorphism of $F_n/[F_n'', F_n]$ is non-tame.

$\sigma_1 : \mathbb{Z}G \to \mathbb{Z}G/[G, G]$. Using the fundamental formula for Fox derivatives one calculates that $\sigma_1(I_{\{a,b\}})$ is generated by

$$(1-b)((-a^{-1}-a^{-2}b^{-k}+a^{-2}) \\ +(2a^{-1}-a^{-1}b^{-k}+a^{-2}b^{-k}-a^{-2})(1+ab^k-a)).$$

Hence mapping $\sigma_1(a)$ and $\sigma_1(b^k)$ both to $(1+\sqrt{-3})/2$ gives a map $v: \mathbb{Z}G/[G, G] \to \mathbb{R}$ with $v(\sigma_1(I_{\{a,b\}}))=0$ and $v(\sigma_1(\pm g))$ a 6th root of unit y for all $g \in G$. The representation σ_1 maps $(\partial\{\varphi(a), \varphi(b)\}/\partial\{a, b\})$ to a lower triangular matrix with $-a^{-1}$ and $-ab^{-1}+a(a-1)(1-b^k)b^{-1}$ as diagonal entries. Hence we obtain

$$\det v\sigma_1(\partial\{\varphi(a), \varphi(b)\}/\partial\{a, b\}) = v\sigma_1(a^{-1}b^{-1}).$$

Choosing for $\sigma_1(b)$ an appropriate $|k|$th root of $(1+\sqrt{-3})/2$ gives for $v\sigma_1(a^{-1}b^{-1})$ a number which is not a 6th root of unity. Applying Theorem II gives the proposition. ∎

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