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Semianalytical structural analysis based on combined application of finite element method and discrete-continual finite element method Part 3: Plate analysis

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Abstract

This paper is devoted to so-called semianalytical plate analysis, based on combined application of finite element method (FEM) [1,2] and discrete-continual finite element method (DCFEM) [3-11]. Kirchhoff model is under consideration. In accordance with the method of extended domain, the given domain is embordered by extended one. The field of application of DCFEM comprises structures with regular (constant or piecewise constant) physical and geometrical parameters in some dimension ("basic" dimension). DCFEM presupposes finite element mesh approximation for non-basic dimension of extended domain while in the basic dimension problem remains continual. Corresponding discrete and discrete-continual approximation models for subdomains and coupled multilevel approximation model for extended domain are under consideration. Brief information about software and verification sample are presented as well.

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Keywords: discrete-continual finite element method; finite element method; semianalytical strctural analysis; two-dimensional theory of elasticity

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1. Formulation of the problem and notation system

Let's consider problem of analysis of plate loaded by concentrated force with hinged ends (cross-sections) along basic dimension (Fig. 1). Some elements of notation system is presented at Fig. 1 as well.

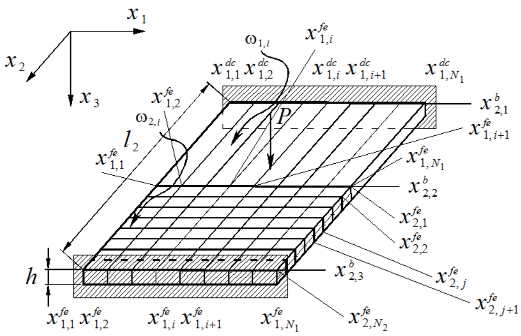

Fig. 1. Considering structure (thin plate).

Let's Ω be domain occupied by structure, $\Omega = \{ (x_1, x_2): \ 0 < x_1 < l_1, \ 0 < x_2 < l_2 \}$, where $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_k = \{ (x_1, x_2): \ 0 < x_1 < l_1, \ x^b_{2,k} < x_2 < x^b_{2,k+1} \}, \ k = 1, 2$; x_1, x_2 are coordinates (x_2 corresponds to basic dimension); $x^b_{2,1} = 0$, $x^b_{2,2} = l_{2,1}$, $x^b_{2,3} = l_{2,1} + l_{2,2} = l_2$ are coordinates of corresponding boundary points (cross-sections) along basic dimension; Ω_1 and Ω_2 are subdomains of Ω; ω_1 and ω_2 are extended subdomains, embordering subdomains $\Omega_1 \subset \omega_1$ and $\Omega_2 \subset \omega_2$; $\omega = \omega_1 \cup \omega_2$; $x^{dc}_{1,i}, \ i = 1, 2, ..., N^{dc}_1$ are coordinates (along x_1) of nodes (nodal lines) of discrete-continual finite elements, which are used for approximation of domain ω_1; $(N^{dc}_1 - 1)$ is the number of discrete-continual finite elements; $x^{fe}_{1,i}, \ i = 1, 2, ..., N^{fe}_1$ and $x^{fe}_{2,j}, \ j = 1, 2, ..., N^{fe}_2$ are coordinates (along x_1 and x_2) of nodes of finite elements, which are used for approximation of domain ω_2; $(N^{fe}_1 - 1)$ and $(N^{fe}_2 - 1)$ are numbers of finite elements along coordinates x_1 and x_2.

Two-index notation system is used for numbering of discrete-continual finite elements. Typical number of has the form (k, i), where k is the number of subdomain, i is the number of element (along x_1). Three-index system is used for numbering of finite elements. Typical number of has the form (k, i, j), where k is the number of subdomain, i and j are numbers of elements (along x_1 and x_2). Let's $N^{fe}_1 = N^{dc}_1 = N_1$ and $x^{fe}_{1,i} = x^{dc}_{1,i} = x_{1,i}, \ i = 1, 2, ..., N_1$.

2. Discrete-continual approximation model for subdomain

Discrete-continual approximation model is used for two-dimensional problems. It presupposes mesh approximation for non-basic dimension of extended domain (along x_1) while in the basic dimension (along x_2) problem remains continual. Thus extended subdomain ω_1 is divided into discrete-continual finite elements

$$\omega_1 = \bigcup_{i=1}^{N_1 - 1} \omega_{1,i} \ ; \quad \omega_{1,i} = \{ (x_1, x_2): \ x_{1,i} < x_1 < x_{1,i+1}, \ \ x^b_{2,1} < x_2 < x^b_{2,2} \} . \tag{1}$$

Flexural rigidity, Poisson's ratio and bedding value for discrete-continual finite element are defined by formulas:

$$\overline{D}_{1,i} = \theta_{1,i} D_1; \quad \overline{\nu}_{1,i} = \theta_{1,i} \nu_1; \quad \overline{c}_{1,i} = \theta_{1,i} c_1 \ ; \tag{2}$$

$$\theta_{1,i} = \begin{cases} 1, & \omega_{1,i} \subset \Omega_1; \\ 0, & \omega_{1,i} \not\subset \Omega_1; \end{cases} \quad D_k = \tilde{E}_k h_k^3 /[12(1-\nu_k^2)]; \tag{3}$$

where $\theta_{1,i}$ is the characteristic function of element $\omega_{1,i}$; h_k is thickness of plate; $\tilde{E}_k$ is the modulus of elasticity of material of plate. Let's w_1 be deflection of plate at subdomain ω_1.

Basic nodal unknown functions are the following functions:

$$y_1^{(1)} = y_1^{(1)}(x_1, x_2) = w_1(x_1, x_2); \quad y_i^{(1)} = y_i^{(1)}(x_1, x_2) = \partial_2^{i-1} w_1(x_1, x_2), \quad i = 2, 3, 4; \tag{4}$$

$$z_j^{(1)}(x_1, x_2) = \partial_1 y_j^{(1)}(x_1, x_2), \quad j = 1, 2, 3, 4. \tag{5}$$

Thus, $y_j^{(1)}(x_1, x_2)$, $j = 1, 2, 3, 4$ and $z_j^{(1)}(x_1, x_2)$, $j = 1, 2, 3, 4$ are basic nodal unknown functions (superscript hereinafter corresponds to the number of considered subdomain i.e. ω_1). Thus for node $(1, i)$ we have the following unknown functions: $y_j^{(1,i)}(x_2)$, $j = 1, 2, 3, 4$ and $z_j^{(1,i)}(x_2)$, $j = 1, 2, 3, 4$.

Polynomial (cubic) approximation along x_1 is used for $y_j^{(1,i)}(x_2)$, $j = 1, 2, 3, 4$ within discrete-continual finite element. Approximation formulas for $z_j^{(1)}$, $j = 1, 2, 3, 4$ can be obtained after derivation in accordance with (5).

DCFEM is reduced at some stage to the solution of systems of $8N_1$ first-order ordinary differential equations:

$$\bar{Y}_1(x_2) = A_1 \bar{Y}_1(x_2) + \bar{\tilde{R}}_1(x_2), \tag{6}$$

where $\bar{Y}_1(x_2)$ is global vector of nodal unknown functions (subscript corresponds to the number of subdomain ω_1),

$$\bar{Y}_1 = \bar{Y}_1(x_2) = [\,(\bar{y}_{1,1})^T \quad (\bar{y}_{1,2})^T \quad (\bar{y}_{1,3})^T \quad (\bar{y}_{1,4})^T\,]^T; \tag{7}$$

$$\bar{y}_{1,j} = \bar{y}_{1,j}(x_2) = [\,(\bar{y}_{n,j}^{(1,1)})^T \quad (\bar{y}_{n,j}^{(1,2)})^T \quad \ldots \quad (\bar{y}_{n,j}^{(1,N_1)})^T\,]^T, \quad j = 1, 2, 3, 4; \quad \bar{y}_{n,j}^{(1,i)} = [\, y_j^{(1,i)} \quad z_j^{(1,i)}\,]^T, \quad j = 1, 2, 3, 4; \tag{8}$$

A_1 is global matrix of coefficients of order $8N_1$; $\bar{\tilde{R}}_1(x_2)$ is the right-side vector of order $8N_1$.

Correct analytical solution of (6) is defined by formula

$$\bar{Y}_1(x_2) = E_1(x_2)\bar{C}_1 + \bar{S}_1(x_2), \tag{9}$$

where $\bar{C}_1$ is the vector of constants of order $8N_1$;

$$E_1(x_2) = \varepsilon_1(x_2 - x_{2,1}^b) - \varepsilon_1(x_2 - x_{2,2}^b); \quad \bar{S}_1(x_2) = \varepsilon_1(x_2) * \bar{\tilde{R}}_1(x_2); \tag{10}$$

$\varepsilon_1(x_2)$ is the fundamental matrix-function of system (6), which is constructed in the special form convenient for problems of structural mechanics [3]; $*$ is convolution notation.

3. Discrete (finite element) approximation model for subdomain

Discrete (finite element) approximation model for the considering two-dimensional problems presupposes finite element approximation along x_1 and x_2. Thus extended subdomain ω_2 is divided into finite elements

$$\omega_2 = \bigcup_{i=1}^{N_1-1} \bigcup_{j=1}^{N_2-1} \omega_{2,i,j}; \quad \omega_{2,i,j} = \{\,(x_1, x_2):\; x_{1,i} < x_1 < x_{1,i+1}, \;\; x_{2,j}^{fe} < x_2 < x_{2,j+1}^{fe}\,\}. \tag{11}$$

Flexural rigidity, Poisson's ratio and bedding value for finite element are defined by formulas:

$$\overline{D}_{2,i,j} = \theta_{2,i,j} D_2; \quad \overline{\nu}_{2,i,j} = \theta_{2,i,j}\nu_2; \quad \overline{c}_{2,i,j} = \theta_{2,i,j} c_2; \tag{12}$$

$$\theta_{2,i,j} = \begin{cases} 1, & \omega_{2,i,j} \subset \Omega_2; \\ 0, & \omega_{2,i,j} \not\subset \Omega_2; \end{cases} \tag{13}$$

where $\theta_{2,i,j}$ is the characteristic function of element $\omega_{2,i,j}$,

Basic nodal unknowns are nodal values of function of deflection of plate and corresponding derivatives with respect to x_1 and x_2 (deflection angles), i.e. the following functions

$$w_2(x_1,x_2) = y_1^{(2)}(x_1,x_2); \quad \theta_{2,1}(x_1,x_2) = \partial_2 w_2(x_1,x_2) = y_2^{(2)}(x_1,x_2); \quad \theta_{2,2}(x_1,x_2) = -\partial_1 w_2(x_1,x_2) = -z_1^{(2)}(x_1,x_2). \tag{14}$$

Thus for node $(2,i,j)$ we have the following unknown functions: $y_1^{(2,i,j)}$, $y_2^{(2,i,j)}$ and $-z_1^{(2,i,j)}$.

Formula for approximation of deflection $w_2(x_1,x_2)$ within discrete-continual finite element $\omega_{2,i,j}$ has the form:

$$\begin{aligned} w_2(x_1,x_2) = \alpha_1^{(2,i,j)} &+ \alpha_2^{(2,i,j)} x_1 + \alpha_3^{(2,i,j)} x_2 + \alpha_4^{(2,i,j)} x_1^2 + \alpha_5^{(2,i,j)} x_1 x_2 + \alpha_6^{(2,i,j)} x_2^2 + \\ &+ \alpha_7^{(2,i,j)} x_1^3 + \alpha_8^{(2,i,j)} x_1^2 x_2 + \alpha_9^{(2,i,j)} x_1 x_2^2 + \alpha_{10}^{(2,i,j)} x_2^3 + \alpha_{11}^{(2,i,j)} x_1^3 x_2 + \alpha_{12}^{(2,i,j)} x_1 x_2^3, \end{aligned} \tag{15}$$

where $\alpha_p^{(2,i,j)}$, $p = 1,2,...,12$ are polynomial coefficients.

In other words, we find it convenient to use polynomials as form functions, which are defined by 12 coefficients (the fourth-order polynomials with several zero coefficients can be used). It should be noted that formula (15) has certain advantages. In particular, deflection $w_2(x_1,x_2)$ along line $x_1 = const$ or line $x_2 = const$ is described by cubic polynomial. All of the external boundaries and boundaries between the elements consists precisely of such lines. Since the third-order polynomial is uniquely defined by four coefficients, displacement along the boundary are uniquely determined by nodal displacements and nodal deflection angles at the ends of this boundary. Function $w_2(x_1,x_2)$ is continuous along any boundary between elements because values of polynomials at the ends of the boundary are the same for the adjacent elements. Besides, it can be noted that the gradient of function $w_2(x_1,x_2)$ with respect to normal to any boundary is described by third-order polynomial along this boundary (for instance, function $\partial_1 w_2(x_1,x_2)$ along line $x_1 = const$). Since we have only two given values of deflection angles at these lines, the third-order polynomial is ambiguously determined and deflection angle may be discontinuous (i.e. continuity of the first-order derivatives at boundaries between several finite elements is not provided). Thus, we have so-called nonconforming form function and nonconforming finite elements [12-19].

We should introduce additional nodal basic unknown, i.e. nodal value of function (mixed derivative)

$$\tau^{(2)}(x_1,x_2) = \partial_1 \partial_2 w_2(x_1,x_2) = z_2^{(2)}(x_1,x_2) \tag{16}$$

in order to obtain conforming finite elements. Corresponding formula instead of (15) has the form

$$w_2(x_1,x_2) = \overline{w}_2(x_1,x_2) + \alpha_{13}^{(2,i,j)} x_1^3 x_2^2 + \alpha_{14}^{(2,i,j)} x_1^2 x_2^3 + \alpha_{15}^{(2,i,j)} x_1^2 x_2^2 + \alpha_{16}^{(2,i,j)} x_1^3 x_2^3, \tag{17}$$

where $\overline{w}_2(x_1,x_2)$ is defined by formula (15); $\alpha_p^{(2,i,j)}$, $p = 1,2,...,16$ are polynomial coefficients.

As known, FEM is reduced to the solution of systems of $4N_1N_2$ linear algebraic equations:

$$K_2 \overline{Y}_2 = \overline{R}_2, \tag{18}$$

where $\overline{U}_2$ is global vector of nodal unknowns (subscript corresponds to the number of subdomain ω_2),

$$\overline{Y}_2 = [\,(\overline{y}_n^{(2,1,1)})^T \quad (\overline{y}_n^{(2,2,1)})^T \quad \ldots \quad (\overline{y}_n^{(2,N_1,1)})^T \quad \ldots \quad (\overline{y}_n^{(2,1,2)})^T \quad (\overline{y}_n^{(2,2,2)})^T \quad \ldots \quad (\overline{y}_n^{(2,N_1,2)})^T \quad \ldots \quad \ldots \quad (\overline{y}_n^{(2,1,N_2)})^T \quad (\overline{y}_n^{(2,2,N_2)})^T \quad \ldots \quad (\overline{y}_n^{(2,N_1,N_2)})^T \;]^T ; \tag{19}$$

$$\overline{y}_n^{(2,i,j)} = [\, y_1^{(2,i,j)} \quad z_1^{(2,i,j)} \quad y_2^{(2,i,j)} \quad z_2^{(2,i,j)} \,]^T , \quad i = 1, 2, \ldots, N_1, \quad j = 1, 2, \ldots, N_2 ; \tag{20}$$

K_2 is global stiffness matrix of order $4N_1N_2$; $\overline{R}_2$ is global right-side vector of order $4N_1N_2$ (global load vector).

4. Multilevel approximation model for domain

System (18) can be rewritten for all nodes with indexes $1 < j < N_2$ (i.e. $x_{2,2}^b < x_2 < x_{2,3}^b$) in the following form (resolving system of $4N_1(N_2 - 2)$ linear algebraic equations):

$$\widetilde{K}_2 \overline{Y}_2 = \widetilde{\overline{R}}_2 , \tag{21}$$

where $\widetilde{K}_2$ is reduced global stiffness matrix of size $[4N_1(N_2 - 2)] \times [4N_1N_2]$; $\widetilde{\overline{R}}_2$ is reduced right-side vector of order $4N_1(N_2 - 2)$.

Boundary conditions at section $x_2 = x_{2,1}^b$ (hinged edge) has the form ($4N_1$ equations):

$$y_1^{(1,i)}(x_{2,1}^b + 0) = 0, \quad i = 1, 2, \ldots, N_1 ; \quad z_1^{(1,i)}(x_{2,k}^b + 0) = 0, \quad i = 1, 2, \ldots, N_1 ; \tag{22}$$

$$y_3^{(1,i)}(x_{2,k}^b + 0) = 0, \quad i = 1, 2, \ldots, N_1 ; \quad z_3^{(1,i)}(x_{2,k}^b + 0) = 0, \quad i = 1, 2, \ldots, N_1 . \tag{23}$$

Equations (22)-(23) can be rewritten in matrix form:

$$B_1^+ \overline{Y}_1(x_{2,1}^b + 0) = \overline{g}_1^+ , \tag{24}$$

where B_1^+ is matrix of boundary conditions of size $4N_1 \times 8N_1$, which can be constructed in accordance with algorithm presented at Table 1; $\overline{g}_1^+$ is the zero vector of order $4N_1$ (i.e. $\overline{g}_1^+ = 0$).

Table 1. Algorithm of construction of matrix B_1^+ (All other elements of matrix B_1^+ are equal to zero).

Numbers (indexes) of elements	Element value	Corresponding boundary condition
$(i, 2i - 1), \quad i = 1, 2, \ldots, N_1$	1	The first equation from (22)
$(N_1 + i, 2i), \quad i = 1, 2, \ldots, N_1$	1	The second equation from (22)
$(2N_1 + i, 4N_1 + 2i - 1), \quad i = 1, 2, \ldots, N_1$	1	The first equation from (23)
$(3N_1 + i, 4N_1 + 2i), \quad i = 1, 2, \ldots, N_1$	1	The second equation from (23)

After substitution of (9) into (24) it can be obtained that

$$B_1^+ E_1(x_{2,1}^b + 0)\overline{C}_1 = \overline{g}_1^+ - B_1^+ \overline{S}_1(x_{2,1}^b + 0) \quad \text{or} \quad Q_1 \overline{C}_1 = \overline{G}_1 , \tag{25}$$

where Q_1 is the matrix of size $4N_1 \times 8N_1$; $\overline{G}_1$ is the vector of order $4N_1$;

$$Q_1 = B_1^+ E_1(x_{2,1}^b + 0) ; \quad \overline{G}_1 = \overline{g}_1^+ - B_1^+ \overline{S}_1(x_{2,1}^b + 0) . \tag{26}$$

Boundary conditions at section $x_2 = x_{2,2}^b$ (perfect contact) has the form ($4N_1$ equations):

$$y_1^{(1,i)}(x_{2,2}^b - 0) = y_1^{(2,i,j)},\ \ i = 1, 2, ..., N_1,\ \ \ j = 1;\ \ \ z_1^{(1,i)}(x_{2,2}^b - 0) = z_1^{(2,i,j)},\ \ i = 1, 2, ..., N_1,\ \ \ j = 1; \tag{27}$$

$$y_2^{(1,i)}(x_{2,2}^b - 0) = y_2^{(2,i,j)},\ \ i = 1, 2, ..., N_1,\ \ \ j = 1;\ \ \ z_2^{(1,i)}(x_{2,2}^b - 0) = [\partial_1 y_2]^{(2,i,j)},\ \ i = 1, 2, ..., N_1,\ \ \ j = 1; \tag{28}$$

$$M_2^{(1,i)}(x_{2,2}^b - 0) = M_2^{(2,i,j)},\ \ i = 1, 2, ..., N_1,\ \ \ j = 1;\ \ \ [\partial_1 M_2]^{(1,i)}(x_{2,2}^b - 0) = [\partial_1 M_2]^{(2,i,j)},\ \ i = 1, 2, ..., N_1,\ \ \ j = 1; \tag{29}$$

$$V_2^{(1,i)}(x_{2,2}^b - 0) = V_2^{(2,i,j)},\ \ i = 1, 2, ..., N_1,\ \ \ j = 1;\ \ \ [\partial_1 V_2]^{(1,i)}(x_{2,2}^b - 0) = [\partial_1 V_2]^{(2,i,j)},\ \ i = 1, 2, ..., N_1,\ \ \ j = 1; \tag{30}$$

where $M_2^{(1,i)}(x_2)$, $V_2^{(1,i)}(x_2)$ and $[\partial_1 M_2]^{(1,i)}(x_2)$, $[\partial_1 V_2]^{(1,i)}(x_2)$ are nodal functions (after corresponding averaging) of bending moment M_2 , adjusted shear force V_2 and corresponding derivatives with respect to x_1 ($\partial_1 M_2$, $\partial_1 V_2$) for discrete-continual finite element $(1, i)$; $M_2^{(2,i,j)}$, $V_2^{(2,i,j)}$ and $[\partial_1 M_2]^{(2,i,j)}$, $[\partial_1 V_2]^{(2,i,j)}$ are nodal bending moment M_2 , adjusted shear force V_2 and corresponding derivatives with respect to x_1 ($\partial_1 M_2$, $\partial_1 V_2$) for finite element $(2, i, j)$; $j = 1$.

Equations (27)-(30) can be rewritten in matrix form:

$$B_2^- \overline{Y}_1(x_{2,2}^b - 0) = B_2^+ \overline{Y}_2 , \tag{31}$$

where B_2^- is matrix of boundary conditions of size $8N_1 \times 8N_1$, which can be constructed in accordance with method of basis variations [3-11]; B_2^+ is matrix of boundary conditions of size $8N_1 \times 4N_1N_2$, which can be constructed in accordance with method of basis variations [3-11].

After substitution of (8) into (22) it can be obtained that

$$B_2^- \overline{Y}_1(x_{2,2}^b - 0)\overline{C}_1 - B_2^+ \overline{Y}_2 = -B_2^- \overline{S}_1(x_{2,2}^b - 0) \quad \text{or} \quad Q_{2,1}\overline{C}_1 + Q_{2,2}\overline{Y}_2 = \overline{G}_2 , \tag{32}$$

where $Q_{2,1}$ is the matrix of size $8N_1 \times 4N_1$; $Q_{2,2}$ is the matrix of size $8N_1 \times 4N_1N_2$; $\overline{G}_2$ is the vector of order $8N_1$,

$$Q_{2,1} = B_2^- E_1(x_{2,2}^b - 0);\quad Q_{2,2} = -B_2^+;\quad \overline{G}_2 = -B_2^- \overline{S}_1(x_{2,2}^b - 0) . \tag{33}$$

Boundary conditions at section $x_2 = x_{2,3}^b$ (hinged edge) has the form ($4N_1$ equations):

$$y_1^{(2,i,j)} = 0,\ \ i = 1, 2, ..., N_1,\ \ \ j = N_2;\ \ \ z_1^{(2,i,j)} = 0,\ \ i = 1, 2, ..., N_1,\ \ \ j = N_2; \tag{34}$$

$$[\partial_2^2 y_1]^{(2,i,j)} = 0,\ \ i = 1, 2, ..., N_1,\ \ \ j = N_2;\ \ \ [\partial_1 \partial_2^2 y_1]^{(2,i,j)} = 0,\ \ i = 1, 2, ..., N_1,\ \ \ j = N_2 . \tag{35}$$

Equations (34) and (35) can be rewritten in matrix form:

$$B_3^- \overline{Y}_2 = \overline{g}_3^- , \tag{36}$$

here B_3^- is matrix of boundary conditions of size $4N_1 \times 4N_1N_2$, which can be constructed in accordance with method of basis variations [3-11]; $\overline{g}_3^-$ is the zero vector of order $4N_1$ (i.e. $\overline{g}_3^- = 0$).

Thus, corresponding coupled system of $4N_1N_2 + 8N_1$ linear algebraic equations with $4N_1N_2 + 8N_1$ unknowns has the form:

$$\begin{bmatrix} Q_1 & 0 \\ Q_{2,1} & Q_{2,2} \\ 0 & \tilde{K}_2 \\ 0 & B_3^- \end{bmatrix} \begin{bmatrix} \overline{C}_1 \\ \overline{U}_2 \end{bmatrix} = \begin{bmatrix} \overline{G}_1 \\ \overline{G}_2 \\ \tilde{\overline{R}}_2 \\ \overline{g}_3^- \end{bmatrix}. \tag{37}$$

It should be noted that boundary conditions (36) can be taken into account automatically within construction of global stiffness matrix and global right-side vector corresponding to subdomain ω_2. Then we get (instead of (28)):

$$\begin{bmatrix} Q_1 & 0 \\ Q_{2,1} & Q_{2,2} \\ 0 & \approx{K}_2 \end{bmatrix} \begin{bmatrix} \overline{C}_1 \\ \overline{U}_2 \end{bmatrix} = \begin{bmatrix} \overline{G}_1 \\ \overline{G}_2 \\ \approx{\overline{R}}_2 \end{bmatrix}, \tag{38}$$

where $\overset{\approx}{K}_2$ is corresponding reduced global stiffness matrix of size $[4N_1(N_2-1)]\times[4N_1N_2]$; $\overset{\approx}{\overline{R}}_2$ is corresponding reduced global right-side vector of order $4N_1(N_2-1)$.

Bending moments, torque moments and shear forces are computed according to well-known formulas after solving of system (38).

5. Software and verification samples

We should stress that all methods and algorithms considered in this paper have been realized in software. The main purpose of Analysis system CSASA2DPL (DCFEM + FEM) is semianalytical plate analysis (Kirchhoff model), based on combined application of FEM and DCFEM. Programming environment is Microsoft Visual Studio 2013 Community and Intel Parallel Studio 2015XE with Intel MKL Library [20-22]. Software is designed for Microsoft Windows 8.1/10.

Corresponding verification samples (ANSYS Mechanical 15.0 [6,7] was used for verification purposes) proved that DCFEM is more effective in the most critical, vital, potentially dangerous areas of structure in terms of fracture (areas of the so-called edge effects), where some components of solution are rapidly changing functions and their rate of change in many cases can't be adequately taken into account by the standard FEM [1].

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