Finite sheeted covering maps over 2-dimensional connected, compact Abelian groups

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Abstract

Let \(G\) be a 2-dimensional connected, compact Abelian group and \(s\) be a positive integer. We prove that a classification of \(s\)-sheeted covering maps over \(G\) is reduced to a classification of \(s\)-index torsionfree supergroups of the Pontrjagin dual \(\hat{G}\). Using group theoretic results from earlier paper we demonstrate its consequences. We also prove that for a connected compact group \(Y\):

1. Every finite-sheeted covering map from a connected space over \(Y\) is equivalent to a covering homomorphism from a compact, connected group.
2. If two finite-sheeted covering homomorphisms over \(Y\) are equivalent, then they are equivalent as topological homomorphisms.

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1. Introduction and the main result

A procedure for obtaining finite-sheeted covering maps (with a connected total space) over connected paracompact spaces has been established [4] and it is based on shape-
theoretic techniques. In the paper the procedure was applied to the so-called solenoids, i.e. 1-dimensional, connected, compact Abelian groups, and we had a fairly simple conclusion. The present paper is devoted to an application of the procedure to 2-dimensional, connected, compact Abelian groups, i.e. the inverse limits of 2-dimensional tori. We call a 2-dimensional, connected, compact Abelian group toroidal group for short. As we see in the sequel, our investigation is reduced to the case of covering homomorphisms between toroidal groups. Then, it is again reduced to the investigation of torsionfree Abelian groups of rank two through the Pontrjagin duality. Starting from this point of view we have proved results of torsionfree Abelian groups of rank two [2]. Using these results we can classify finite-sheeted covers over each 2-dimensional toroidal group and investigate total spaces up to homeomorphism. Since we have also classified finite-index subgroups of torsionfree Abelian groups of rank two, we shall state their reflections, which correspond to covering homomorphisms from a toroidal group to other toroidal groups. Our main theorem is the following. Related definitions will be stated in the sequel.

**Theorem 1.1.** Every toroidal group $X$ is presented by $X = \lim\limits_{\leftarrow}(X_n, g_n: n < \omega)$ where $X_n$’s are copies of $\mathbb{T}^2$ and $g_n = \left(\begin{array}{cc} p_n & \alpha_n \\ 0 & t_n \end{array}\right) \in M_2(\mathbb{Z})$ such that $p_n, t_n > 0$ and $0 \leq \alpha_n < p$.

For a natural number $s$ let $F_s$ be the set of all positive integer $q$ satisfying $\text{GCD}(p_n, q) = \text{GCD}(t_n, q) = 1$ for almost all $n$ and that there exists $q_1$ such that $qq_1r = s$ and

(a) $\text{GCD}(p_n, q_1) = \text{GCD}(t_n, r) = 1$ for almost all $n$;
(b) if $q_1 > 1$, then $\text{GCD}(t_n, q_1) \neq 1$ for infinitely many $n$’s.

**Under the above presentation of $X$ the following hold:**

(1) For a natural number $s$, the number of non-equivalent $s$-sheeted covering maps over $X$ is $\sum_{q \in F_s} q$ and that of toroidal groups which admit mutually non-equivalent $s$-sheeted covering maps from $X$ is also $\sum_{q \in F_s} q$.

(2) Let $(\alpha_n: n < \omega)$ be semi-periodic and $p$ a positive integer. If $p_n = p, t_n = 1$ for almost all $n$ or if $p_n = t_n = p$ for almost all $n$, then total spaces of finite-sheeted covering maps over $X$ are homeomorphic to $X$ itself and all toroidal groups over which there are finite-sheeted covering maps from $X$ are also homeomorphic to $X$ itself.

(3) Let $p$ be a prime and $p_n = p, t_n = 1$ for every $n$ and $q$ be a natural number with $\text{GCD}(p, q) = 1$. If a $p$-adic integer $\sum_{n=0}^{\infty} \alpha_n p^n$ is not quadratic over $\mathbb{Q}$, then there are $(q + 1)$-many total spaces of $q^2$-sheeted covering maps over $X$ which are mutually non-homeomorphic and $(q + 1)$-many toroidal groups which admit $q^2$-sheeted covering maps from $X$ and which are mutually non-homeomorphic.

(4) Let $p$ be a prime and $t = 1$ and let $\alpha_n = 1$ if $n = 3^k$ for some $k$ and $\alpha_n = 0$ otherwise. Then, for each natural number $s$ total spaces of mutually non-equivalent $s$-sheeted covering maps are mutually non-homeomorphic. Moreover, toroidal groups which admit mutually non-equivalent $s$-sheeted covering maps from $X$ are mutually non-homeomorphic.

When we present a toroidal group as in Theorem 1.1, we can present a representative of each equivalence class of finite-sheeted covering maps by an inverse sequence of $(2 \times 2)$-
integer valued matrices. We shall state this in Appendix A. Theorem 1.1(3) generalizes the main result of [1] and (4) is what was previously announced in [1]. We state some basic notations. The circle group is denoted by $\mathbb{T}$ and identified with the quotient group $\mathbb{R}/\mathbb{Z}$. The identity of $\mathbb{T}$ and also $\mathbb{T}^2$ is denoted by 0. Covering maps $f : Y \to X$ and $g : Z \to X$ are equivalent, if there exists a homeomorphism $i : Y \to Z$ such that $f = g \circ i$. Covering maps $f : X \to Y$ and $g : X \to Y$ are equivalent, if there exists a homeomorphism $i : Y \to Z$ such that $g = i \circ f$. A sequence $(c_n : n < \omega)$ is said to be semi-periodic, if there exist $m < \omega$ and $1 \leq k < \omega$ such that $c_{n+k} = c_n$ for every $n \geq m$. We refer the reader to [6,5] for basic notions.

2. Covering maps over topological groups

In this section we consider covering maps over a topological group with connected total spaces. We state our result more generally than is necessary in the sequel. When we consider a topological group $X$ as a pointed space, we always choose the identity $e$ as a distinguished point. First let us recall a proof of

Lemma 2.1 [5, Theorem 79]. Let $Y$ be a path-connected, locally path-connected topological group and $f : (X,x_0) \to (Y,e)$ be a pointed covering map from a path-connected space $X$. Then there exists a multiplication on $X$ so that $X$ is a topological group and $f$ is a homomorphism.

The multiplication $\cdot$ in this lemma is defined as follows. Let $x$ and $x'$ be arbitrary points in $X$. Choose paths $p : [0,1] \to X$ and $p' : [0,1] \to X$ connecting $x_0$ to $x$ and $x_0$ to $x'$ respectively, i.e. $p(0) = x_0$, $p(1) = x$, $p'(0) = x_0$, $p'(1) = x'$. Define a path $r$ by: $r(s) = f(p(2s))$ for $0 \leq s \leq 1/2$ and $r(s) = f(x)f(p'(2s - 1))$ for $1/2 \leq s \leq 1$. We have a unique path $\tilde{r} : [0,1] \to X$ that covers $r$ and has the initial point $x_0$, i.e. $\tilde{r}(0) = x_0$ and $f \circ \tilde{r} = r$. Then $x \cdot x' = \tilde{r}(1)$.

Lemma 2.2. Let $Y$ be a path-connected, locally path-connected topological Abelian group and $f : (X,x_0) \to (Y,e)$ be a pointed covering map from a path-connected space $X$. Then the group $(X,\cdot,x_0)$ is Abelian.

Proof. Define a path $r' : [0,1] \to X$ and $\tilde{r}'$ similarly as $r$ and $\tilde{r}$ respectively, i.e. $r'(s) = f(p'(2s))$ for $0 \leq s \leq 1/2$ and $r'(s) = f(x')f(p(2s - 1))$ for $1/2 \leq s \leq 1$ and $\tilde{r}'(0) = x_0$ and $f \circ \tilde{r}' = r'$. Then $x \cdot x' = \tilde{r}(1)$ and $x' \cdot x = \tilde{r}'(1)$. Since $r(1) = f(x)f(x') = f(x')f(x) = r'(1)$, it suffices to show that $r$ and $r'$ are homotopic relative to $\{0,1\}$. Define $H : [0,1] \times [0,1] \to X$ by: $H(s,t) = f(p(2s))$ for $t \geq 2s$, $H(s,t) = f(p(2s - t))f(p'(2s - t))$ for $t \leq 2s$, $H(s,0) = f(x)f(p'(2s - 1))$ for $t \leq 2s - 1$. Then $H(s,0) = r(s), H(s,1) = r'(s), H(0,t) = x_0, H(1,t) = f(x)f(x')$ and $H$ is a homotopy between $r$ and $r'$ relative to $\{0,1\}$. $\square$

Lemma 2.3. Let $Y$ be a path-connected, locally path-connected topological group, let $X$ be a path-connected space and let $Z$ be a path-connected topological group. Let $f : (X,x_0) \to$
(Y, e) be a pointed covering map and let \( g : (Z, e) \to (Y, e) \), \( h : (Z, e) \to (X, x_0) \) be pointed maps such \( f \circ h = g \). If \( g \) is a homomorphism, then \( h : (Z, e) \to (X, \cdot, x_0) \) is a homomorphism.

**Proof.** We need to prove that \( h(z z') = h(z) \cdot h(z') \) for each \( z, z' \in Z \). Choose paths \( p \) in \( Z \) connecting \( e \) to \( z' \) and let \( q(t) = z p(t) \) for \( 0 \leq t \leq 1 \). Then \( q \) is a path connecting \( z \) and \( z z' \). Then \( h \circ p \) is a path in \( X \) connecting \( x_0 \) to \( h(z') \) and \( h \circ q \) is a path in \( X \) connecting \( h(z) \) to \( h(z z') \). Since \( f \circ h \circ q(t) = g(z p(t)) = g(z) g(p(t)) = ((f \circ h)(z)) ((f \circ h \circ p)(t)) \), \( ((f \circ h)(z)) ((f \circ h \circ p)(t)) = (f \circ h \circ q)(t) \), which implies \( h(z) \cdot h(z') = (h \circ q)(1) = h(z z') \). \( \square \)

**Lemma 2.4.** Let \( Y \) be a path-connected, locally path-connected topological group and let \( X \) be a path-connected topological group. If \( f : X \to Y \) is a covering homomorphism, then \( X \) is topologically isomorphic to \( (X, \cdot) \).

**Proof.** Consider \( f : (X, e) \to (Y, e) \) as a covering map and apply Lemma 2.3 for \( g = f \) and \( h = \text{id}_X \). \( \square \)

We omit straightforward proofs of the following two corollaries.

**Corollary 2.5.** Let \( Y \) be a path-connected, locally path-connected topological group and let \( X \) and \( Z \) be path-connected spaces. If \( f : (X, x_0) \to (Y, e) \) and \( g : (Z, z_0) \to (Y, e) \) are equivalent pointed covering maps via a homeomorphism \( \varphi : (X, x_0) \to (Z, z_0) \), i.e. \( g \circ \varphi = f \), then \( \varphi : (X, \cdot, x_0) \to (Z, \cdot, z_0) \) is an isomorphism of topological groups.

**Corollary 2.6.** Let \( X, Y, Z \) be path-connected, locally path-connected topological groups and \( f : X \to Y \) and \( g : Z \to Y \) be covering homomorphisms. If \( f \) and \( g \) are equivalent covering maps such that \( f \circ h = g \) with a pointed homeomorphism \( h : (Z, e) \to (X, e) \), then \( h \) is an isomorphism, i.e. \( f \) and \( g \) are equivalent as covering homomorphisms via a topological isomorphism \( h \).

**Lemma 2.7.** Let \( f : X \to \mathbb{T}^n \) be a finite sheeted covering map. Then there exists a covering homomorphism \( g : \mathbb{T}^n \to \mathbb{T}^n \) which is equivalent to \( f \).

**Proof.** We recall the correspondence between subgroups of a fundamental group and covering maps [6, Corollary 2.5.3]. Since the fundamental group of \( \mathbb{T}^n \) is the free Abelian group \( \mathbb{Z}^n \), the image of \( f_* : \pi_1(X) \to \pi_1(\mathbb{T}^n) \) is a subgroup of \( \mathbb{Z}^n \). Since \( f \) is finite-sheeted, the subgroup is also isomorphic to \( \mathbb{Z}^n \). By presenting \( f_* \) by an integer-valued matrix, we can see the existence of a covering homomorphism \( g : \mathbb{T}^n \to \mathbb{T}^n \) such that \( \text{Im}(g_*) = \text{Im}(f_*) \). (See also the remark to [2, Lemma 1.6] in Section 3.) Hence the lemma holds. \( \square \)

For the reader’s convenience we briefly review notions and claims that we need in the sequel. Let \( (X, x) = \lim \langle X_i, x_i \rangle \) and \( (Y, y) = \lim \langle Y_j, y_j \rangle \), where \( X_* = (\langle X_i, x_i \rangle, h_{ij} : i \leq j, i, j \in I) \) and \( Y_* = (\langle Y_j, y_j \rangle, g_{ij} : i \leq j, i, j \in I) \) be a directed set, and let \( f_i : (X_i, x_i) \to (Y_i, y_i) \) be
a pointed continuous map such that \( f_i, f_j, g_{ij} \) and \( h_{ij} \) form a pointed pull-back diagram for \( i \leq j \). When a pointed map \( f : (X, x) \to (Y, y) \) is the limit of \( f_i = (f_i : i \in I) \), we say that \((X_i, Y_i, f_i)\) is a pointed pull-back expansion of \( f \). Moreover, if all \( X_i \) and \( Y_i \) are ANR's, \((X_i, Y_i, f_i)\) is said to be a pointed ANR-pull-back expansion of \( f \). Omitting base points we get a definition of an \((ANR)\) pull-back expansion \((X, Y, f)\) of a map \( f : X \to Y \). Since we are interested in the limit, when we set \( X_i, Y_i, f_i \), we may use \( \{i \in I : i \geq i_0\} \) as the index set instead of \( I \), for arbitrary \( i_0 \in I \).

Having in mind that the total space \( X_i \) of a covering map \( f_i : X_i \to Y_i \) over an ANR \( Y_i \) is an ANR itself [3, Remark 7], and applying [4, Theorem 6] for compact connected space \( Y \) (see [4, Remark 1]) we get the following lemma.

**Lemma 2.8.** Let \( Y = \lim (Y_i, g_{ij} : i \leq j, i, j \in I) \), where each \( Y_i \) is a compact, connected ANR and let \( s \in \mathbb{N} \). A map \( f : (X, x) \to (Y, y) \) from a connected space \( X \) is a pointed \( s \)-sheeted covering map if and only if \( f \) admits a pointed ANR-pull-back expansion \((X_i, Y_i, f_i)\) such that \( Y_i = ((Y_i, y_i), g_{ij} : i \leq j, i \geq i_0) \) for some \( i_0 \in I \) and each map \( f_i : (X_i, x_i) \to (Y_i, y_i) \) is an \( s \)-sheeted pointed covering map from a compact, connected ANR \( X_i \).

The pointed ANR-pull-back expansion \((X_i, Y_i, f_i)\) from Lemma 2.8 is called a pointed development of the \( s \)-sheeted pointed covering map \( f : (X, x) \to (Y, y) \). Omitting base points we get a nonpointed version of Lemma 2.8 (see [4, Theorem 7]).

**Lemma 2.9.** Let \( Y = \lim (Y_i, g_{ij} : i \leq j, i, j \in I) \), where \( I \) is a directed set, each \( Y_i \) is a compact, connected group which is an ANR, and each bonding map \( g_{ij} \) is a homomorphism and let \( f : X \to Y \) be a finite-sheeted covering map from a connected space \( X \). Then there exists a multiplication defined on \( X \) such that \( X \) is a topological group and \( f \) is a homomorphism.

**Proof.** Let \( f : X \to Y \) be an \( s \)-sheeted covering map, where \( s \in \mathbb{N} \), let \( e_i \) be the identity element of \( Y_i \) and let \( e = (e_i : i \in I) \in Y \) be the identity element of \( Y \). As the identity element of \( X \) choose an arbitrary element \( x \in f^{-1}(e) \). According to Lemma 2.8 a pointed covering map \( f : (X, x) \to (Y, e) \) admits a pointed development \((X_i, Y_i, f_i)\). Since each pointed map \( f_i : (X_i, x_i) \to (Y_i, e_i) \) is an \( s \)-sheeted covering map from a connected ANR \( X_i \) we can apply Lemma 2.1. For each \( i \geq i_0 \) let \((X_i, \cdot, x_i)\) be a topological group according to Lemma 2.1. Then \( f_i : (X_i, \cdot, x_i) \to (Y_i, e_i) \) is an \( s \)-sheeted covering homomorphism. Since \( g_i \circ f_j : (X_j, \cdot, x_j) \to (Y_i, e_i) \) is a homomorphism and \( f_i \circ h_{ij} = g_i \circ f_j \), Lemma 2.3 implies that \( h_{ij} : (X_j, \cdot, x_j) \to (X_i, \cdot, x_i) \) is a homomorphism for each \( j \geq i \geq i_0 \). Now \( X \) is a topological group and \( f \) is a homomorphism. \( \square \)

Let \( G = (G_j, q_{ij} : i \leq j, i, j \in I) \) be a progroup and let \( H = (H_i, q_{ij} : i \leq j, i \geq i_0) \) be a subprogroup of \( G \). We say that the subprogroup \( H \subseteq G \) has index \( s \), provided there exists an index \( i_1 \geq i_0 \) such that for every \( i \geq i_1 \), the index \( [G_i : H_i] = s \) and a function \( r_{ij} : G_j / H_j \to G_i / H_i \), defined by \( r_{ij}(H_j g) = H_i q_{ij}(g) \), is a bijection, for \( j \geq i \geq i_1 \).

Subprogroups \( H = (H_i, q_{ij} : i \leq j, i \geq i_0) \) and \( H' = (H'_i, q_{ij} : i \leq j, i \geq i_0') \) of a progroup \( G = (G_i, q_{ij} : i \leq j, i, j \in I) \) are said to be conjugate subprogroups of \( G \).
provided there exist a $i^* \geq i_0, i'_0$ and a system of elements $g_i \in G_i$ ($i \geq i^*$) such that $H'_i = g_i^{-1}H_i g_i$ and $g_{ij}(g_j) \in H_i g_i$ for $j \geq i \geq i^*$.

Applying [4, Theorem 4] for a compact connected space $Y$, we get the following lemma.

**Lemma 2.10.** Let $Y = \lim(Y_i, g_{ij} : i \leq j, \ i, \ j \in I)$, where each $Y_i$ is a compact, connected ANR and let $s \in \mathbb{N}$. Then there is a bijection $F_s$ between the set of all pointed equivalence classes $[f]_s$ of $s$-sheeted pointed covering maps $f : (X, x) \to (Y, y)$ from a connected space $X$ and the set of all subgroups of index $s$ of the fundamental progroup $\pi_1(Y, y) = (\pi_1(Y_i, y_i), g_{ij} : i \leq j, \ i, \ j \in I)$.

The bijection $F_s$ is defined as follows. Let $f : (X, x) \to (Y, y)$ be a pointed $s$-sheeted covering map. Take a pointed development $(X_s, Y_s, f_s)$ of $f$ and put $H_s = (H_i, g_{ij} : i \leq j, \ i \geq i_0)$, where $H_i = f_i#(\pi_1(X_i, x_i))$. Then $F_s([f]_s) = H_s$ (see [4, Theorem 6]). The non-pointed version of Lemma 2.10 establishes a bijection $F$ between the set of all equivalence classes $[f]$ of $s$-sheeted covering mappings $f : X \to Y$ from a connected space $X$ and the set of all conjugacy classes of subgroups of index $s$ of the fundamental progroup $\pi_1(Y, y)$, where $y$ is an arbitrarily chosen point of $Y$. In this case we have $F([f]) = [F_s([f]_s)]$ [4, Theorems 5 and 7].

Let $Y = \lim(Y_i, g_{ij} : i \leq j, \ i, \ j \in I)$, where each $Y_i$ is a compact, connected group which is an ANR. Since the fundamental group of a path-connected topological group is Abelian, it follows that $\pi_1(Y, y)$ is an Abelian progroup and there is no difference between the pointed and nonpointed cases i.e. $F_s([f]_s) = F([f])$.

**Lemma 2.11.** Let $Y = \lim(Y_i, g_{ij} : i \leq j, \ i, \ j \in I)$, where each $Y_i$ is a compact, connected group which is an ANR and each bonding map $g_{ij}$ is a homomorphism. Let $f : X \to Y$ and $f' : X' \to Y$ be $s$-sheeted covering homomorphisms by pointed developments $(X_s, Y_s, f_s)$ and $(X'_s, Y_s, f'_s)$ respectively. If $f$ and $f'$ are equivalent covering maps, then there is a topological isomorphism $\varphi : X \to X'$ such that $f' \circ \varphi = f$, i.e. $f$ and $f'$ are equivalent as covering homomorphisms.

**Proof.** Let $F$ be the bijection between the set of all equivalence classes of $s$-sheeted covering maps over $Y$ and the set of all subgroups of index $s$ of $\pi_1(Y, e)$. Since $f$ and $f'$ are equivalent $s$-sheeted covering maps, it follows $F([f]) = F([f'])$ and therefore, there is an $i_1 \geq i_0, i_0'$ such that $(f_{i_0}(\pi_1(X_i, e)), g_{ij} : i_1 \leq i \leq i') = (f'_{i_0}(\pi_1(X'_i, e)), g_{ij} : i_1 \leq i \leq i')$. For each $i \geq i_1$ there is a pointed homeomorphism $\varphi_i : (X_i, e) \to (X'_i, e)$ such that $f_i = f'_i \circ \varphi_i$. Now, Corollaries 2.5 and 2.6 imply that $\varphi_i : X_i \to X'_i$ is a topological isomorphism. The isomorphisms $\varphi_i$ induce an isomorphism $\varphi : X \to X'$ such that $f' \circ \varphi = f$. □

The following lemma is well-known and follows from [5, Theorem 67]. Since the idea of its proof is also important in the proof of Theorem 2.13, we present a proof.

**Lemma 2.12** (Folklore). Every connected compact group $Y$ is the inverse limit of an inverse system of compact connected Lie groups $(Y_i, g_{ij} : i \leq j, \ i, \ j \in I)$ such that
Proof. Let \( U \) be a neighborhood base for the identity \( e \) of \( Y \). By [5, Theorem 67] for each \( U \in \mathcal{U} \) there exists a closed normal subgroup \( N_U \) of \( Y \) such that \( N_U \subseteq U \) and \( Y/N_U \) is a compact Lie group. Let \( I \) be the set of all non-empty finite subsets \( i \) of \( U \) and \( N_i = \bigcap_{U \in i} N_U \). Then \( I \) is a directed set under the set inclusion and \( Y/N_i \) is a Lie group by [5, 46(A)]. Since \( \mathcal{U} \) is a neighborhood base, \( Y \) is the inverse limit of \( (Y_i, g_{ij}: i \leq j, i, j \in I) \). \( \Box \)

**Theorem 2.13.** Let \( Y \) be a connected compact group. Then the following hold:

1. Every finite-sheeted covering map from a connected space over \( Y \) is equivalent to a covering homomorphism from a compact, connected group. Moreover, if \( Y \) is Abelian then the domain of the homomorphism is Abelian.

2. Let \( f : X \to Y \) and \( f' : X' \to Y \) be finite-sheeted covering homomorphisms over \( Y \). Then \( f \) and \( f' \) are equivalent if and only the two homomorphisms are equivalent as topological homomorphisms.

**Proof.** By Lemma 2.12 \( Y = \varprojlim (Y_i, g_{ij}: i \leq j, i \in I) \), where each \( Y_i \) is a compact, connected group which is an ANR and each bonding map \( g_{ij} \) is a homomorphism. Since a finite-sheeted covering map from a connected space is \( s \)-sheeted for some natural number \( s \), the domain of the map is compact. Now the first half of (1) follows from Lemma 2.9. The second half of (1) follows from Lemmas 2.2, 2.4 and 2.9.

To see the statement (2), let \( K = \text{Ker}(f) \) and \( Y \) be given as in Lemma 2.12. Then we have a neighborhood \( U_0 \) of \( e \in Y \) so that \( f^{-1}(U_0) \) is a disjoint sum of finitely many copies of \( U_0 \) and the restriction of \( f \) to each copy is a homeomorphism. Hence, there is a neighborhood \( V_0 \) of \( e \in X \) such that \( f|V_0 \) is a homeomorphism between \( V_0 \) and \( U_0 \). Since \( \mathcal{U} \) is a neighborhood base, we have \( U_1 \in \mathcal{U} \) so that \( uv \in V_0 \) and \( u^{-1} \in V_0 \) for each \( u, v \in f^{-1}(U_1) \cap V_0 \). Let \( f^{-1}(N_i) \cap V_0 = M_i \) for \( \{U_1\} \leq i \in I \). Since \( f^{-1}(N_i) \) is a subgroup of \( X \) and \( uv \in V_0 \) and \( u^{-1} \in V_0 \) for each \( u, v \in M_i \), \( M_i \) is a closed subgroup of \( X \) such that \( f|M_i \) is a topological isomorphism between \( M_i \) and \( N_i \). Since \( KM_i = f^{-1}(N_i) \), \( KM_i \) is a normal subgroup and \( K M_i \) is the disjoint sum of \( k M_i \)'s for \( k \in K \), where each \( k M_i \) is open in \( K M_i \). Let \( f_u(x) = x^{-1}ux \) for \( u \in M_i \). Then \( f_u(X) \) is connected and \( f_u(X) \cap M_i \neq \emptyset \). Hence \( f_u(X) \subseteq M_i \), which implies that \( M_i \) is a normal subgroup of \( X \).

For \( i, j \in I \) with \( \{U_1\} \leq i \leq j \), define \( h_{ij} : X/M_j \to X/M_i \) as \( h_{ij}(xM_j) = xM_i \) and \( f_i : X/M_i \to Y/N_i \) as \( f_i(xM_i) = f(x)N_i \) for \( x \in X \) respectively. Then we have the following pull-back diagram such that \( h_{ij}, f_i, f_j \) and \( g_{ij} \) are open surjective homomorphisms and \( \text{Ker}(f_i) \cong \text{Ker}(f_j) \cong K \):
This shows that \( f \) is a covering homomorphism by a pointed development \((X_\ast, Y_\ast, f_\ast)\)
where

\[
X_\ast = ((X/M_i, M_i), h_{ij}: \{U_1\} \leq i \leq j, i, j \in I) \quad \text{and} \quad Y_\ast = ((Y/N_i, N_i), g_{ij}: \{U_1\} \leq i \leq j, i, j \in I) \quad \text{and} \quad f_\ast = (f_i: i \in I).
\]

For \( f': X' \to Y \) we choose \( U'_0 \) and \( U'_1 \in \mathcal{U} \) in the same way as \( U_0 \) and \( U_1 \) for \( f \) and have a pointed development \( f' \) indexed by \( \{i \in I: \{U'_1\} \leq i\} \).

Let \( I_0 = \{i \in I: \{U_1, U'_1\} \leq i\} \).

\( \)Since the restriction of \((X_\ast, Y_\ast, f_\ast)\) to \( I_0 \) is also a development of \( f \) and the same thing holds for the development for \( f' \), the statement (2) follows from Lemmas 2.9 and 2.11. \( \Box \)

### 3. The Pontrjagin duality

The preceding results are related to finite-sheeted covering maps and homomorphisms. Here we explain materials about the Pontrjagin duality and refer the reader to [5] for the original source of basic knowledge.

For a topological group \( A \) let \( \widehat{A} \) be the Pontrjagin dual of \( A \), i.e. \( \widehat{A} \) is a topological group consisting of all the continuous homomorphisms from \( A \) to \( \mathbb{R}/\mathbb{Z} = \mathbb{T} \) and its topology is the compact open topology.

Let \( A, B \) be discrete Abelian groups and \( X, Y \) be compact Abelian groups. Then

(a) \( \widehat{A} \) is connected if and only if \( A \) is torsionfree and \( X \) is connected if and only if \( \widehat{X} \) is torsionfree;

(b) \( \widehat{A} \) is \( n \)-dimensional and connected if and only if \( A \) is torsionfree and rank \( n \) and \( X \) is connected and \( n \)-dimensional if and only if \( \widehat{X} \) is torsionfree and of rank \( n \).

(c) A homomorphism \( h: A \to B \) is injective if and only if a topological homomorphism \( \hat{h}: \widehat{B} \to \widehat{A} \) is surjective and a topological homomorphism \( g: X \to Y \) is surjective if and only if \( \hat{g}: \widehat{Y} \to \widehat{X} \) is injective.

We will use direct limits and inverse limits. Since the direct limit of compact Abelian groups in the category of topological groups and that in the category of compact Abelian groups are different, we should be careful in general. But we use only the direct limit of discrete Abelian groups and the inverse limit of compact Abelian groups and there are no such subtle situation. The inverse limit topology coincides with the topology as the Pontrjagin dual. Hence we have,

**Lemma 3.1.** *For the direct limit \( A = \lim\limits_{\to}(A_i, h_{ij}: i \leq j, i, j \in I) \) of discrete Abelian groups \( A_{ij} \), the Pontrjagin dual \( \widehat{A} \) is the inverse limit \( \lim\limits_{\leftarrow}(\widehat{A}_i, \widehat{h}_{ij}: i \leq j, i, j \in I) \).*

Let \( X \) and \( Y \) be connected compact Abelian groups and \( g: X \to Y \) be a surjective continuous homomorphism. Then \( \widehat{X} \) and \( \widehat{Y} \) are torsionfree Abelian groups and \( \hat{g}: \widehat{Y} \to \widehat{X} \).
is an injective homomorphism. Since \( X = \hat{X} \), we can define \( (X, H) \) to be an annihilator \( \{ x \in X: x(H) = \{0\} \} \) for a subset \( H \) of \( \hat{X} \). Then \( (\hat{X}/\operatorname{Im}(\hat{g}))^\sim = (X, \operatorname{Im}(\hat{g})) \).

**Lemma 3.2.** Let \( s \) be a positive integer. In the above correspondence, the cardinality of \( \operatorname{Ker}(g) \) is \( s \) if and only if \( \operatorname{Im}(\hat{g}) \) is an \( s \)-index subgroup of \( \hat{X} \).

**Proof.** Since \( \operatorname{Ker}(g) = (X, \operatorname{Im}(\hat{g})) = (\hat{X}/\operatorname{Im}(\hat{g}))^\sim \) and \( A \simeq \hat{A} \) for every finite Abelian group, we have \( \operatorname{Ker}(g) \simeq \hat{X}/\operatorname{Im}(\hat{g}) \), which implies the conclusion. \( \square \)

Since a finite subgroup of a topological group is discrete, for a surjective homomorphism \( g : X \to Y \) \( g \) is an \( s \)-sheeted covering homomorphism if and only if the cardinality of \( \operatorname{Ker}(g) \) is \( s \) by Lemma 3.2. Let \( X, X' \) be connected compact Abelian groups and \( g : X \to Y, g' : X' \to Y \) be a surjective continuous homomorphism. By the Pontrjagin duality the existence of a topological isomorphism \( i : X \to X' \) with \( g = g' \circ i \) is equivalent to that of an isomorphism \( j : \hat{X}' \to \hat{X} \) with \( \hat{g} = j \circ \hat{g}' \).

Corresponding to the equivalence between covering maps we introduced the equivalence between supergroups. In the scope of the Pontrjagin duality it is convenient to introduce the equivalence between embeddings of a group we define as follows. Let \( A \) be a discrete group and \( h : A \to B \) and \( h' : A \to B' \) be injective homomorphisms. We say \( h \) and \( h' \) are equivalent, if there exists an isomorphism \( i : B \to B' \) such that \( i \circ h = h' \). There is a related question to a finite-sheeted covering homomorphism concerning compact Abelian groups. For a compact Abelian group \( Y \) how can we classify \( s \)-sheeted covering homomorphisms \( g : Y \to X \)? The answer to this question is rather easy. It suffices to display all subgroups of \( Y \) whose cardinality are \( s \). Summing up the above we have the following: Let \( s \) be a positive integer. A homomorphism \( h : A \to B \) is injective and \( \operatorname{Im}(h) \) is an \( s \)-index subgroup if and only if a topological homomorphism \( \hat{h} : \hat{B} \to \hat{A} \) is an \( s \)-sheeted covering map and a topological homomorphism \( g : X \to Y \) is an \( s \)-sheeted covering map if and only if \( \hat{g} : \hat{Y} \to \hat{X} \) is injective and \( \operatorname{Im}(\hat{g}) \) is an \( s \)-index subgroup of \( \hat{X} \).

### 4. Proof of Theorem 1.1

According to results in Sections 2 and 3, it is sufficient to consider only the case of torsionfree Abelian groups of rank two as follows.

Let \( X \) be a toroidal group. We conclude the following:

1. A classification of \( s \)-sheeted covering maps over \( X \) from connected spaces is equivalent to that of torsionfree \( s \)-index supergroup of \( \hat{X} \);
2. A classification of \( s \)-sheeted covering homomorphisms from \( X \) to other topological groups corresponds to the class of \( s \)-index subgroups of \( \hat{X} \).

Moreover, since \( \hat{X} \) is isomorphic to the Čech cohomology group \( \check{H}^1(X) \) [7],

1. \( \hat{X} \) is isomorphic to \( \hat{Y} \) if and only if \( X \) is homeomorphic to \( Y \) for toroidal groups \( X \) and \( Y \).
In order to investigate s-index supergroups and subgroups of a torsionfree Abelian group $A$ of rank 2, we present $A$ by the direct limit of free Abelian groups of rank 2. This presentation corresponds to a presentation of the toroidal group $\hat{A}$. We review this here.

**Lemma 4.1** [2, Lemma 1.7]. Let $A$ be a torsionfree Abelian group of rank 2. Then there exist lower diagonal integer-valued matrices $f_n = \begin{bmatrix} p_n & 0 \\ \alpha_n & t_n \end{bmatrix}$ such that $p_n, t_n > 0$ and $0 \leq \alpha_n < p_n$, and the direct limit $\lim_{n<\omega} (A_n, f_n; n < \omega)$ is isomorphic to $A$ where $A_n$'s are copies of $\mathbb{Z} \oplus \mathbb{Z}$.

Here an element of $\mathbb{Z} \oplus \mathbb{Z}$ is presented by a $\mathbb{Z}$-valued column vector and a matrix acts from the left. An element of $T = \mathbb{R}/\mathbb{Z}$ is denoted by $x + \mathbb{Z}$. When we present an element of $T^2 = (\mathbb{R}/\mathbb{Z})$ by a $T$-valued column vector, the dual homomorphism of $\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]$ for $a, b, c, d \in \mathbb{Z}$ is denoted by $\left[ \begin{array}{c} a \\ b \\ c \\ d \end{array} \right]$ where $\left[ \begin{array}{cc} a & c \\ b & d \end{array} \right](\mathbb{Z} + \mathbb{Z}) = \left[ \begin{array}{c} ax + c + y + z \\ bx + d + y + z \end{array} \right]$. Therefore, for each toroidal group $X$ there exist upper diagonal integer-valued matrices $g_n = \begin{bmatrix} p_n & \alpha_n \\ 0 & t_n \end{bmatrix}$ such that $p_n, t_n > 0$ and $0 \leq \alpha_n < p_n$, and the inverse limit $\lim_{n<\omega} (G_n, g_n; n < \omega)$ is isomorphic to $X$ where $G_n$'s are copies of $T^2$.

Here we summarize the results in [2] about torsionfree Abelian groups of rank two.

**Lemma 4.2** [2, Theorems 3.5, 3.6, 6.9, 7.2, Lemma 3.8, Corollary 7.4]. Every torsionfree Abelian group $A$ of rank two is presented as $A = \lim_{n<\omega} (A_n, f_n; n < \omega)$ where $A_n$'s are copies of $\mathbb{Z} \oplus \mathbb{Z}$ and $f_n = \begin{bmatrix} p_n & 0 \\ \alpha_n & t_n \end{bmatrix} \in M_2(\mathbb{Z})$ such that $p_n, t_n > 0$ and $0 \leq \alpha_n < p_n$. For a natural number $s$ let $F_s$ be the set of all positive integer $q$ satisfying $\text{GCD}(p_n, q) = \text{GCD}(t_n, q) = 1$ for almost all $n$ and that there exists $q_1$ such that $qq_1r = s$ and

(a) $\text{GCD}(p_n, q_1) = \text{GCD}(t_n, r) = 1$ for almost all $n$;
(b) if $q_1 > 1$, then $\text{GCD}(t_n, q_1) \neq 1$ for infinitely many $n$'s.

Under the above presentation of $A$ the following hold:

1. For a natural number $s$, the number of non-equivalent $s$-index supergroups of $A$ is $\sum_{q \in F_s} q$ and that of $s$-index subgroups of $A$ is also $\sum_{q \in F_s} q$.
2. Let $(\alpha_n; n < \omega)$ be semi-periodic and $p$ a natural number. If $p_n = p, t_n = 1$ for almost all $n$ or if $p_n = t_n = p$ for almost all $n$, then all finite index supergroups and subgroups of $A$ are isomorphic to $A$ itself.
3. Let $p$ be a prime and $p_n = p, t_n = 1$ for each $n$ and $q$ be a natural number with $\text{GCD}(p, q) = 1$. If a $p$-adic integer $\sum_{n=0}^{\infty} \alpha_n p^n$ is not quadratic over $\mathbb{Q}$, then there are $(q + 1)$-many $q^2$-index supergroups of $A$ which are mutually non-isomorphic and $(q + 1)$-many $q^2$-index subgroups of $A$ which are non-isomorphic to each others.
4. Let $p$ be a prime and $p_n = p, t_n = 1$ for each $n$ and let $\alpha_n = 1$ if $n = 3^k$ for some $k$ and $\alpha_n = 0$ otherwise. Then, for each natural number $s$ non-equivalent $s$-index supergroups of $A$ are mutually non-isomorphic and distinct $s$-index subgroups of $A$ are mutually non-isomorphic.
Now, the statements in Theorem 1.1 correspond to those in Lemma 4.2 one by one and so we have Theorem 1.1.

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Appendix A

Here we present a representative of each equivalence class of finite-sheeted covering maps over a toroidal group by an inverse sequence of $(2 \times 2)$-integer valued matrices. We suppose that a toroidal group $X$ is presented as in Theorem 1.1. As its proof shows, a representative can be taken as the Pontrjagin dual of a torsionfree Abelian group of rank two which is presented as a direct limit of copies of a free Abelian group with $(2 \times 2)$-integer valued matrices as bonding homomorphisms.

We introduce two notions “super-admissible sequence” and “sub-admissible sequence” from [2, Section 2]. Assume $GCD(p_n,q) = GCD(t_n,r) = 1$ for sufficiently large $n$. A sequence $c_{qr}$ is super-admissible, if

- $c_{qr} : [n_0, \omega) \rightarrow \{0, 1, \ldots, q - 1\}$ for some $n_0 < \omega$;
- $p_n c_{qr}(n + 1) \equiv t_n c_{qr}(n) - r \alpha_n \pmod{q}$.

A sequence $c_{qr}$ is sub-admissible, if

- $c_{qr} : [n_0, \omega) \rightarrow \{0, 1, \ldots, r - 1\}$ for some $n_0 < \omega$;
- $p_n c_{qr}(n + 1) \equiv t_n c_{qr}(n) + q \alpha_n \pmod{r}$.

Two super-admissible sequences $c_{qr}$ and $c'_{qr'}$ are said to be equivalent, if $q = q'$, $r = r'$ and $c_{qr}(n) = c'_{qr'}(n)$ for sufficiently large $n$. For sub-admissible sequences the equivalence is defined in the same way.

Let $g_{qr} : [n_0, \omega) \rightarrow \mathbb{Z}$ by: $g_{qr}(n) = (p_n c_{qr}(n + 1) - t_n c_{qr}(n) + r \alpha_n)/q$ for a super-admissible sequence $c_{qr}$. Similarly let $e_{qr} : [n_0, \omega) \rightarrow \mathbb{Z}$ by: $e_{qr}(n) = (-p_n c_{qr}(n + 1) + t_n c_{qr}(n) + q \alpha_n)/r$ for a sub-admissible sequence $c_{qr}$.

By Theorem 2.13 a classification of finite-sheeted covering maps over a toroidal group $X$ is reduced to that of finite-sheeted covering homomorphisms over $X$ and then as shown in Section 3 it is reduced to that of finite-index supergroups of the Pontrjagin dual $\hat{X}$. By [2, Theorem 2.5] an $s$-index supergroup of $\hat{X}$ is again presented by a direct limit of $\mathbb{Z} \oplus \mathbb{Z}$’s whose bonding maps are $\left[\begin{array}{cc} p_n & 0 \\ 0 & t_n \end{array}\right]$ where $s = qr$. Therefore the total group corresponding to a super-admissible sequence $c_{qr}$ is presented as the inverse limit of $\mathbb{T} \times \mathbb{T}$ whose bonding maps are $\left[\begin{array}{cc} p_n & 0 \\ 0 & t_n \end{array}\right]$. Similarly a toroidal group which is an image of an $s$-sheeted continuous homomorphism from $X$ is presented as the inverse limit of $\mathbb{T} \times \mathbb{T}$ whose bonding maps are $\left[\begin{array}{cc} p_n & 0 \\ 0 & t_n \end{array}\right]$. 
Appendix B

Here, we give a different point of view to see [2, Lemma 1.6] using the Pontrjagin duality. Let $A_1, A_2, B_1, B_2$ be free Abelian groups of finite rank and there are injections $g_1 : A_1 \to A_2, \ g_2 : B_1 \to B_2, \ h_1 : A_1 \to B_1, \ h_2 : A_2 \to B_2$ such that $\text{Im}(g_2) + \text{Im}(h_2) = B_2$ and $\text{Im}(g_2) \cap \text{Im}(h_2) = \text{Im}(h_2 \circ g_1) = \text{Im}(g_2 \circ h_1)$.

\[
\begin{array}{c}
B_1 \xrightarrow{g_2} B_2 \\
\downarrow h_1 \quad \quad \downarrow h_2 \\
A_1 \xrightarrow{g_1} A_2
\end{array}
\]

We consider the Pontrjagin dual of this diagram. When we express $g_1, g_2, h_1, h_2$ by matrices in $M_n(\mathbb{Z})$, the dual homomorphisms are expressed by the transposed matrices, as we have explained in the rank 2 case. Then each injective homomorphism induces a surjective continuous homomorphism between tori. Representing homomorphism as a matrix in $M_n(\mathbb{Z})$ this fact corresponds to the fact that it is a regular matrix:

\[
\begin{array}{c}
\hat{B}_1 \xleftarrow{\hat{g}_1} \hat{B}_2 \\
\downarrow \hat{h}_1 \quad \quad \downarrow \hat{h}_2 \\
\hat{A}_1 \xleftarrow{\hat{g}_1} \hat{A}_2
\end{array}
\]

Now each of these continuous homomorphisms induces an injective homomorphism between fundamental groups which are free Abelian groups and the corresponding matrix is the same as that of the continuous homomorphism:

\[
\begin{array}{c}
\pi_1(\hat{B}_1) \xleftarrow{\hat{g}_1^*} \pi_1(\hat{B}_2) \\
\downarrow \hat{h}_1^* \quad \quad \downarrow \hat{h}_2^* \\
\pi_1(\hat{A}_1) \xleftarrow{\hat{g}_1^*} \pi_1(\hat{A}_2)
\end{array}
\]

Therefore this diagram is precisely the same as the diagram for $\mathbb{Z}$-duals in [2, Lemma 1.6]. There it is proved that $\text{Im}(g_1^*) + \text{Im}(h_1^*) = A_1^*$ and $\text{Im}(g_1^*) \cap \text{Im}(h_1^*) = \text{Im}(g_1^* \cdot h_2^*) = \text{Im}(h_1^* \cdot g_2^*)$:

\[
\begin{array}{c}
B_1^* \xleftarrow{g_2^*} B_2^* \\
\downarrow h_1^* \quad \quad \downarrow h_2^* \\
A_1^* \xleftarrow{g_1^*} A_2^*
\end{array}
\]

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