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Word representation of cords on a punctured plane ☆

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Dedicated to Professor Kunio Murasugi on his seventy second birthday

Abstract

In this paper a purely algebraic condition for a word in a free group to be representable by a simple curve on a punctured plane will be given.

As an application, an algorithm for simple closed curves on a punctured plane will be obtained. Our solution is different from any algorithm due to Reinhart [Ann. of Math. 75 (1962) 209], Zieschang [Math. Scand. 17 (1965) 17] or Chillingworth [Bull. London Math. Soc. 1 (1969) 310]. Although the study here will be confined to the case of a plane, similar arguments could be carried out on the 2-sphere. This research was motivated by monodromy problems appearing in Lefschetz fibrations and surface braids. See [Math. Proc. Cambridge Philos. Soc. 120 (1996) 237; Kamada, Braid and Knots Theory in Dimension Four, American Mathematical Society, 2002; Kamada and Matsumoto, in: Proceedings of the International Conference on Knot Theory "Knots in Hellas '98", World Scientific, 2000, p. 118; Kamada and Matsumoto, Enveloping monoidal quandles, Preprint, 2002; Matsumoto, in: S. Kojima et al. (Eds.), Proc. the 37th Taniguchi Sympos., World Scientific, 1996, p. 123].

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1. Introduction

Let n be a fixed integer $\geqslant 2$. Let $\mathbb{R}^2$ be the xy-plane, and let $P_n = \{p_1, \ldots, p_n\}$ be a set of n points on $\mathbb{R}^2$. To make our argument explicit, we will assume that for each $k = 1, \ldots, n$, the point p_k is given by the following coordinates:

$$p_k = (k, 0).$$

An (i, j)-*curve* on $(\mathbb{R}^2, P_n)$ is defined to be a continuous map

$$l : [0, 1] \to \left(\mathbb{R}^2 - P_n\right) \cup \{p_i, p_j\} \tag{1}$$

satisfying $l(0) = p_i, l(1) = p_j$, where $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Moreover, we assume that $l(t) = p_i$ if and only if $t = 0$ and that $l(t) = p_j$ if and only if $t = 1$.

If an (i, j)-curve l is simple (i.e., without self-intersections), it will be called an (i, j)-*cord*, or simply a *cord*. Two cords l and l' are *isotopic* if they are ambiently isotopic to each other by an isotopy of $\mathbb{R}^2$ which fixes P_n pointwise.

For each $k \in \{1, \ldots, n\}$, let L_k be the half-line defined as follows:

$$L_k = \left\{(k, y) \mid y \leqslant 0\right\}.$$

The half-line L_k is parallel to the y-axis and has terminal point p_k. An (i, j)-*curve* l is said to be *transverse* to $\bigcup_k L_k$ if in a neighborhood of each intersection point $p \in l([0, 1]) \cap \bigcup_k L_k$, the curve l is extended to a smooth curve whose velocity vectors are non-zero and transverse to $\bigcup_k L_k$. An (i, j)-curve which is transverse to $\bigcup_k L_k$ will be simply called a *transverse* (i, j)-curve. From the definition it follows that the intersection of a transverse (i, j)-curve l and $\bigcup_k L_k$ consists of a finite number of points.

Let F_n be a free group with preferred generators

$$x_1, x_2, \ldots, x_n. \tag{2}$$

Traversing a transverse (i, j)-curve l from $l(0)$ to $l(1)$ and reading the intersection points with $\bigcup_k L_k$ successively, we can associate with l a word $W(l)$ in F_n. (We will sometimes say that $W(l)$ is *represented* by l, or more simply, is the *reading* of l.) To be precise, in order to get $W(l)$, we start from $l(0) = p_i$ but do not count the starting point p_i in $W(l)$. Each time we meet an intersection point $p \in l \cap \bigcup_k L_k$ we read it as the generator x_k if at p the curve l crosses L_k in the positive direction with respect to the x-coordinate, and as the inverse x_k^{-1} if it crosses in the negative direction. Finally we arrive at the terminal point $l(1) = p_j$, but we do not count it to $W(l)$. Thus if an (i, j)-curve does not intersect $\bigcup_k L_k$ except at the end points p_i, p_j, we associate with it the empty word 1.

For example, the reading of a $(2, 6)$-cord shown in Fig. 1 is

$$W = x_1^{-1} x_3 x_4 x_5 x_4^{-1}.$$

Any prescribed word in F_n can be representable by an (i, j)-curve with self-intersections, but not necessarily by an (i, j)-cord. We are interested in the problem of characterizing those words in F_n that are representable by (i, j)-cords.

The following theorem is our main result, and gives a solution to this problem.

Theorem 1. *There exists an explicitly computable map*

$$R_{ij} : F_n \to F_n$$

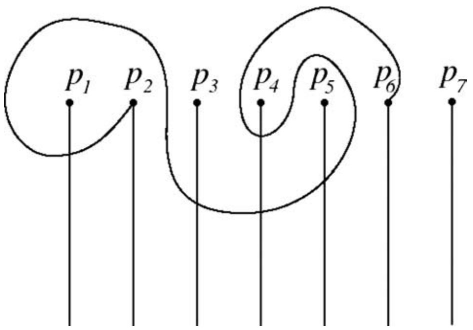

Fig. 1. A $(2,6)$-cord.

such that (i) R_{ij} *is a projection, namely* $R_{ij} \circ R_{ij} = R_{ij}$ *and* (ii) *a word* W *in* F_n *is representable by an* (i,j)*-cord if and only if*

$$R_{ij}(W) = W.$$

In other words, W *is representable by an* (i,j)*-cord if and only if* W *belongs to the image of* R_{ij}.

The map R_{ij} is a crossed anti-homomorphism twisted by an explicitly computable 'right representation'

$$D_{ij} : F_n \to \mathrm{Aut}(F_n).$$

The computations of R_{ij} and D_{ij} are purely algebraic, and even a computer could detect the representable words. See Section 6, particularly Theorem 36, Propositions 38, and 39.

In Section 7, we will apply Theorem 1 to obtain an algorithm to decide if a given word is representable by a simple closed curve on $\mathbb{R}^2 - P_n$. Our algorithm is considerably different from those of Reinhart [12], Zieschang [13] or Chillingworth [2]. See Theorem 41.

In the course of proving Theorem 1, we will have to study the relationship between the isotopy classes of cords and various cosets of the free group F_n. This will be discussed in Sections 2 and 3.

Theorem 1 will be proved in Sections 5 and 6. In fact, it is merely a statement putting together Lemmas 17, 18 and Theorem 36 proved in these sections.

In this paper, we will confine our investigation to a punctured plane $(\mathbb{R}^2, P_n)$ for simplicity, but it could be carried out similarly on the punctured sphere (S^2, P_n). We notice that if it is actually done, then in the special case where $n = 6$, we will have word representation of simple closed curves on a closed surface of genus 2 by taking a double branched covering of (S^2, P_6). In this sense, potentially, our work is related to the study of double torus knots by Hill [4], and Hill and Murasugi [5].

Finally, we remark that an independent treatment of $(2,3)$-cords on $(\mathbb{R}^2, P_3)$ (if said in our terminology) is found in Section 2 of Jin and Kim [6] in a different formulation.

2. Isotopy classes of (i, ∞)-cords

We take an auxiliary point p_∞ in $\mathbb{R}^2 - P_n$. To fix our idea, we assume that

$$p_\infty = (0, 1).$$

A *cord* on $(\mathbb{R}^2, P_n \cup \{p_\infty\})$ is defined just as in Section 1, and the meaning of an (i, ∞)-cord will be clear. The number $i \in \{1, \dots, n\}$ will be fixed throughout this section.

Let $\mathcal{A}_i$ denote the set of all (ambient) isotopy classes of (i, ∞)-cords on $(\mathbb{R}^2, P_n \cup \{p_\infty\})$. Then a map

$$f_i : F_n \to \mathcal{A}_i \tag{3}$$

is defined as follows.

First identify F_n with the fundamental group $\pi_1(\mathbb{R}^2 - P_n, p_\infty)$.

By Theorem 1.4 of Birman's book [1], there is an injective homomorphism j_* of the latter group to the pure braid group with the 'base' $P_n \cup \{p_\infty\}$, $P(\mathbb{R}^2, P_n \cup \{p_\infty\})$:

$$j_* : \pi_1\bigl(\mathbb{R}^2 - P_n, p_\infty\bigr) \to P\bigl(\mathbb{R}^2, P_n \cup \{p_\infty\}\bigr). \tag{4}$$

Given an element b of $P(\mathbb{R}^2, P_n \cup \{p_\infty\})$, there exists an isotopy $\{h_t\}_{0 \leqslant t \leqslant 1}$ of $\mathbb{R}^2$ onto itself such that $h_0 = \mathrm{id}$ and $(h_t(P_n \cup \{p_\infty\}), t)_{0 \leqslant t \leqslant 1}$ represents the braid b in $\mathbb{R}^2 \times [0, 1]$. (See [1].) Let

$$\mathcal{M}\bigl(\mathbb{R}^2, P_n \cup \{p_\infty\}\bigr)$$

denote the mapping class group of $(\mathbb{R}^2, P_n \cup \{p_\infty\})$ which fixes $P_n \cup \{p_\infty\}$ pointwise. By sending b to the final stage h_1 of the isotopy $\{h_t\}_{0 \leqslant t \leqslant 1}$, we have a natural map

$$d_* : P\bigl(\mathbb{R}^2, P_n \cup \{p_\infty\}\bigr) \to \mathcal{M}\bigl(\mathbb{R}^2, P_n \cup \{p_\infty\}\bigr). \tag{5}$$

Lemma 2. *The composite*

$$d_* \circ j_* : \pi_1\bigl(\mathbb{R}^2 - P_n, p_\infty\bigr) \to \mathcal{M}\bigl(\mathbb{R}^2, P_n \cup \{p_\infty\}\bigr)$$

is an injective homomorphism.

Proof. By Lemma 4.2.1 in [1], $\ker d_* \subset \mathrm{Center}(P(\mathbb{R}^2, P_n \cup \{p_\infty\})$. We are assuming $n \geqslant 2$, and the free group $\pi_1(\mathbb{R}^2 - P_n, p_\infty)$ is centerless. Since j_* is injective, this centerlessness implies

$$j_*\bigl(\pi_1\bigl(\mathbb{R}^2 - P_n, p_\infty\bigr)\bigr) \cap \ker d_* = \{1\}. \tag{6}$$

Now the injectivity $d_* \circ j_*$ follows from (6) and the injectivity of j_*. □

By Lemma 2, $F_n = \pi_1(\mathbb{R}^2 - P_n, p_\infty)$ is considered to be a subgroup of the mapping class group $\mathcal{M}(\mathbb{R}^2, P_n \cup \{p_\infty\})$, which turns out to be the subgroup of motions of p_∞ in $\mathbb{R}^2 - P_n$ (Birman [1, p. 10]).

Now we are in a position to define the map

$$f_i : F_n \to \mathcal{A}_i. \tag{7}$$

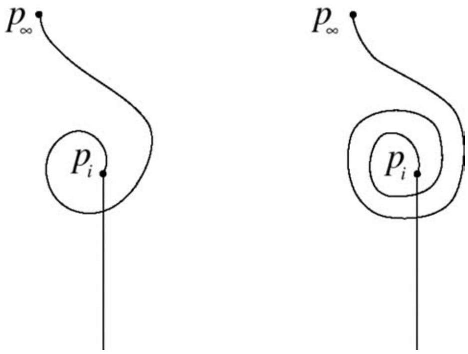

Fig. 2. $(l_{i\infty})x_i$ and $(l_{i\infty})x_i^2$.

Take a word V from F_n. By the above remark, we can regard V as an element of $\mathcal{M}(\mathbb{R}^2, P_n \cup \{p_\infty\})$. Let $l_{i\infty}$ be a special (i,∞)-cord which is a line segment joining p_i and p_∞:

$$l_{i\infty}(t) = (1-t)(i,0) + t(1,0), \quad 0 \leqslant t \leqslant 1.$$

For an (i,∞)-cord l we denote by $[l]$ its isotopy class $\in \mathcal{A}_i$. Then $f_i(V)$ is defined to be the isotopy class of the image of $l_{i\infty}$ under the action of the mapping class V:

$$f_i(V) := \big[(l_{i\infty})V\big]. \tag{8}$$

Here and in what follows, we will assume that $\mathcal{M}(\mathbb{R}^2, P_n \cup \{p_\infty\})$ acts on $(\mathbb{R}^2, P_n \cup \{p_\infty\})$ from the *right*.

Let C_k $(k \in \{1,\dots,n\})$ be a smooth simple closed curve on $\mathbb{R}^2 - P_n$ which starts and ends at p_∞, and crosses L_k only once, transversely in the positive direction. We also assume that $C_k \cap L_h = \emptyset$ if $k \neq h$. Then as an element of the mapping class group $\mathcal{M}(\mathbb{R}^2, P_n \cup \{p_\infty\})$, a generator x_k of F_n is the result of a motion whose support is within a sufficiently thin neighborhood of C_k and which moves the point p_∞ along the curve C_k. Similarly, x_k^{-1} is the result of a motion along C_k^{-1}, namely along the same curve C_k but in the opposite direction.

When $k = i$, the action of x_i has a special property. For example, see Fig. 2, where two (i,∞)-cords $(l_{i\infty})x_i$ and $(l_{i\infty})x_i^2$ are shown. Notice that these (i,∞)-cords are isotopic to $l_{i\infty}$ by isotopies which rotate a neighborhood of p_i round the point p_i.

More generally,

$$\big[(l_{i\infty})x_i^m\big] = [l_{i\infty}] \in \mathcal{A}_i, \quad \forall m \in \mathbb{Z}.$$

Since, for $V, W \in F_n$,

$$\big[(l_{i\infty})VW\big] = \big[(l_{i\infty})V\big]W, \tag{9}$$

we have

$$\big[(l_{i\infty})x_i^m W\big] = \big[(l_{i\infty})W\big].$$

Thus we have the following:

Lemma 3. *$f_i : F_n \to \mathcal{A}_i$ induces a map* (*denoted by f_i again*)

$$f_i : \langle x_i \rangle \backslash F_n \to \mathcal{A}_i,$$

where $\langle x_i \rangle \backslash F_n$ denotes the left cosets, in which $[V] = [W]$ if and only if $V = x_i^m W$ for some $m \in \mathbb{Z}$.

Next, we will define a homotopy set $\mathcal{H}_i$. We define an (i, ∞)-*curve* to be a continuous map (which may have self-intersections)

$$l : [0, 1] \to \left(\mathbb{R}^2 - P_n\right) \cup \{p_i\}$$

such that $l(t) = p_i$ if and only if $t = 0$ and such that $l(t) = p_\infty$ if $t = 1$. This definition of an (i, ∞)-curve differs slightly from that of an (i, j)-curve given in Section 1 in which $j \neq \infty$.

Two (i, ∞)-curves l and l' are said to be *i-homotopic* if there exists a homotopy

$$H : [0, 1] \times [0, 1] \to \left(\mathbb{R}^2 - P_n\right) \cup \{p_i\}$$

satisfying

(i) $H(0, t) = l(t)$ and $H(1, t) = l'(t)$, $\forall t \in [0, 1]$,
(ii) $H(s, t) = p_i$ if and only if $t = 0$, and
(iii) $H(s, 1) = p_\infty$, $\forall s \in [0, 1]$.

Notice the difference between the conditions (ii) and (iii); the “exit” of an i-homotopy is “closed” at p_i, while it is “open” at p_∞, which means that during the homotopy the interior of the curve is prohibited from going through p_i but is allowed through p_∞.

Let us define $\mathcal{H}_i$ to be the set of all i-homotopy classes of (i, ∞)-curves. Clearly we have a natural map

$$g_i : \mathcal{A}_i \to \mathcal{H}_i. \tag{10}$$

Lemma 4. *The map g_i is surjective.*

Proof. Let l be an (i, ∞)-curve. Deforming l via i-homotopy, if necessary, we may assume that l is smooth and has a finite number of transverse self-intersections. Then we can push out these self-intersections successively through the end point p_∞. See Fig. 3. The resulting (i, ∞)-curve l' is an (i, ∞)-cord and is i-homotopic to l. This proves the surjectivity of $g_i : \mathcal{A}_i \to \mathcal{H}_i$. □

Finally we will define a map

$$h_i : \mathcal{H}_i \to \langle x_i \rangle \backslash F_n. \tag{11}$$

Let $l : [0, 1] \to (\mathbb{R}^2 - P_n) \cup \{p_i\}$ be an (i, ∞)-curve. We can deform l by an i-homotopy to an (i, ∞)-curve l' which is transverse to $\bigcup_k L_k$. Let $W(l') \in F_n$ be the reading of l'. Then the map $h_i : \mathcal{H}_i \to \langle x_i \rangle \backslash F_n$ is defined to be the map sending the i-homotopy class of l to the coset of $W(l') \in \langle x_i \rangle \backslash F_n$.

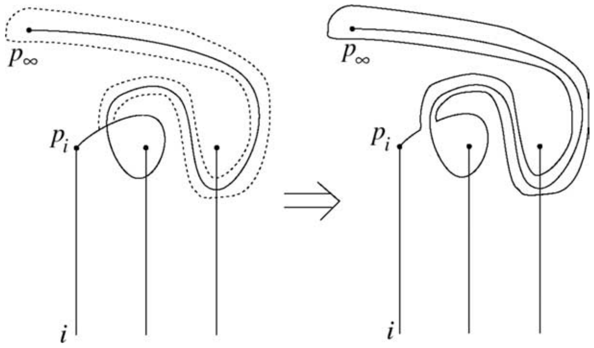

Fig. 3. Pushing out the self-intersections through p_∞.

Lemma 5. *The map*

$$h_i : \mathcal{H}_i \to \langle x_i \rangle \backslash F_n$$

is well-defined.

Proof. Suppose l and l' are transverse (i, ∞)-curves which are mutually i-homotopic. Then there exists an i-homotopy

$$H : [0, 1] \times [0, 1] \to (\mathbb{R}^2 - P_n) \cup \{p_i\}$$

satisfying (i), (ii), (iii) above.

From these properties, if $\varepsilon > 0$ is sufficiently small, it follows that

(a) the readings of restricted curves $l|[\varepsilon, 1]$ and $l'|[\varepsilon, 1]$ with respect to $\bigcup_k L_k$ are the same as $W(l)$ and $W(l')$, respectively, and
(b) the curve $H_\varepsilon(s) := H(s, \varepsilon)$, $0 \leqslant s \leqslant 1$, is contained in a small neighborhood N of p_i such that $N \cap P_n = \{p_i\}$. (The curve H_ε does not touch the point p_i.)

Perturbing a small part of H within N, if necessary, we may assume that the curve H_ε is transverse to $\bigcup_k L_k$. Then the reading of the curve H_ε will be x_i^m for some $m \in \mathbb{Z}$.

Now define a loop $L(\tau)$, $0 \leqslant \tau \leqslant 1$, on $\mathbb{R}^2 - P_n$ based at p_∞:

$$L(\tau) := \begin{cases} l(1 - 3\tau) & 0 \leqslant \tau \leqslant \frac{1}{3} - \frac{1}{3}\varepsilon, \\ H_\varepsilon\big((3\tau + \varepsilon - 1)/(1 + 2\varepsilon)\big) & \frac{1}{3} - \frac{1}{3}\varepsilon \leqslant \tau \leqslant \frac{2}{3} + \frac{1}{3}\varepsilon, \\ l'(3\tau - 2) & \frac{2}{3} + \frac{1}{3}\varepsilon \leqslant \tau \leqslant 1. \end{cases}$$

See Fig. 4.

It is obvious from (a) and (b) that the reading of the loop $L(\tau)$, $0 \leqslant \tau \leqslant 1$, is

$$W(l)^{-1} x_i^m W(l').$$

Since $H([0, 1] \times [\varepsilon, 1]) \subset \mathbb{R}^2 - P_n$, the loop $L(\tau)$ shrinks in $\mathbb{R}^2 - P_n$ to the base point p_∞. Therefore, in the group $F_n = \pi_1(\mathbb{R}^2 - P_n, p_\infty)$, we have

$$W(l)^{-1} x_i^m W(l') = 1,$$

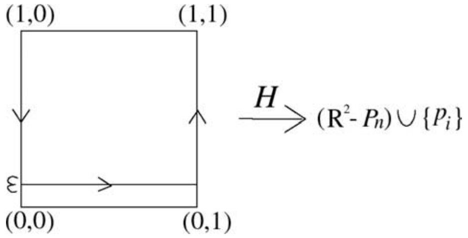

Fig. 4. Homotopy H and loop $L(\tau)$.

in other words,

$$\big[W(l)\big] = \big[W\big(l'\big)\big] \in \langle x_i \rangle \backslash F_n.$$

This proves Lemma 5. □

Lemma 6. *The map*

$$f_i : \langle x_i \rangle \backslash F_n \to \mathcal{A}_i$$

is surjective.

Proof. Let l be any (i, ∞)-cord from $\mathcal{A}_i$, which may be assumed to be smooth and transverse to $\bigcup_k L_k$. We will prove Lemma 6 by induction on the number N of the intersection points between l and $\bigcup_k L_k$. If $N = 1$, l does not meet $\bigcup_k L_k$ except at the starting point p_i. It is easily seen that such a cord l is isotopic to the line segment $l_{i\infty}$. Thus in this case

$$[l] = [l_{i\infty}] = f_i(1)$$

and $[l]$ is in the image of f_i. See (8).

Suppose Lemma 6 has been proved if the intersection points are less than a given N. We will prove Lemma 6 when the number equals N. Let p be the intersection point between l and $\bigcup_k L_k$ that we meet *last* when traversing l from $l(0)$ to $l(1)$. Suppose the point p is on the half-line L_k. We first assume that at p the cord l crosses L_k in the positive direction.

Let C_k be the simple closed curve based at p_∞, introduced before Lemma 3. Then we may assume that C_k intersects L_k at the point p and that the part of C_k between p and p_∞ is the same as the part of l between p and p_∞. Then consider the motion whose support is within a thin neighborhood of C_k and which carries p_∞ round along C_k^{-1}. Apply this motion to l. Then p will be removed from the intersections, and l will be moved to an (i, ∞)-curve l' having fewer intersection points with $\bigcup_k L_k$ than l.

Note that in $\mathcal{A}_i$,

$$\big[l'\big] = \big[(l)x_k^{-1}\big].$$

By induction hypothesis, $[l']$ is in the image of f_i, and we can find a word $V \in F_n$ such that

$$\big[l'\big] = \big[(l_{i\infty})V\big].$$

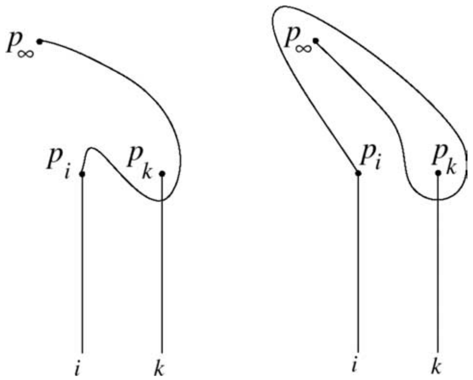

Fig. 5. $(l_{i\infty})x_k$ and $(l_{i\infty})x_k^{-1}$.

Thus

$$\big[(l)x_k^{-1}\big] = \big[(l_{i\infty})V\big].$$

In other words,

$$[l] = \big[(l_{i\infty})Vx_k\big] = f_i(Vx_k).$$

We have done in the case l crosses L_k at p in the positive direction. If it crosses in the negative direction, the argument is similar. This completes the proof of Lemma 6. □

Lemma 7. *The composite*

$$h_i \circ g_i \circ f_i : \langle x_i\rangle\backslash F_n \to \langle x_i\rangle\backslash F_n$$

is the identity.

Proof. We have only to prove that, for each $V \in F_n$, the reading of $(l_{i\infty})V$ is the same as V in $\langle x_i\rangle\backslash F_n$. Choose an arbitrary word V and fix it. By Lemma 5, the reading of an (i,∞)-cord does not change if we deform it by (i,∞)-isotopy, or more generally by i-homotopy. Thus we may assume that $(l_{i\infty})V$ is transverse to $\bigcup_k L_k$.

Write the word V in a *reduced form* of *length* N:

$$V = x_{\nu(1)}^{\epsilon(1)} x_{\nu(2)}^{\epsilon(2)} \cdots x_{\nu(N)}^{\epsilon(N)}.$$

That is to say, in this expression, $\epsilon(m) = \pm 1$, $\nu(m) \in \{1,2,\ldots,n\}$, $m = 1,2,\ldots,N$, and if $\nu(m) = \nu(m+1)$ for some m, then $\epsilon(m) \neq -\epsilon(m+1)$. If a word V has a reduced form of length N, this number N is called the *reduced length* of V. We will prove the lemma by induction on N.

First suppose $N = 1$, and draw a transverse curve $(l_{i\infty})x_k^\epsilon$. See Fig. 2 for the case $k = i$, and Fig. 5 for the case $k \neq i$. In the case $k = i$, we have seen that the reading of $(l_{i\infty})x_i^\epsilon$ is 1 as an element of $\langle x_i\rangle\backslash F_n$. (Lemma 3.) In the case $k \neq i$, by Fig. 5, we see that the reading of $(l_{i\infty})x_k^\epsilon$ is x_k^ϵ. Thus Lemma 7 is clear, if $N = 1$.

To proceed further, let us make a definition. For a transverse (i,∞)-cord l, its *honest reading* is defined to be the reading of the intersection points $l \cap \bigcup_k L_k$ without

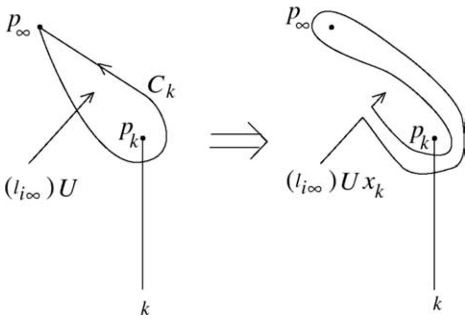

Fig. 6. C_k intersects $(l_{i\infty})U$.

canceling x_k and x_k^{-1} even if they appear successively in the course of traversing l. Thus a honest reading is not necessarily a reduced word.

Now suppose $N > 1$ and that Lemma 6 has been proved for smaller length. Suppose the reduced word V of length N is written as

$$V = Ux_k^\epsilon \quad (\epsilon = \pm 1),$$

where U is a reduced word of length $N - 1$ $(\geqslant 1)$. To draw the curve $(l_{i\infty})V$, we apply the mapping class x_k^ϵ to the curve $(l_{i\infty})U$. That is to say, we move $(l_{i\infty})U$ by the motion of p_∞ round along the curve C_k^ϵ. If C_k does not intersect $(l_{i\infty})U$ except at p_∞, then the reading of $(l_{i\infty})Ux_k^\epsilon$ is easily seen to be Ux_k^ϵ. But some complication appears if C_k intersects $(l_{i\infty})U$ at other points than the base point p_∞.

To see this, suppose $\epsilon = +1$, and suppose C_k intersects a part of $(l_{i\infty})U$ once as indicated by Fig. 6, left. Then by performing the motion of p_∞ along C_k, we have an (i, ∞)-cord $(l_{i\infty})Ux_k$.

Let us compare the honest readings of the cords before and after this motion. By Fig. 6, right, we see that the honest reading of $(l_{i\infty})Ux_k$ is obtained from that of $(l_{i\infty})U$ by multiplying x_k from the right and inserting a canceling pair $x_k x_k^{-1}$ somewhere in the honest reading of $(l_{i\infty})U$. By induction hypothesis, the reading of $(l_{i\infty})U$ is equal to U in $\langle x_i\rangle\backslash F_n$. Thus from the above observation, the reading of $(l_{i\infty})Ux_k$ is equal to Ux_k in $\langle x_i\rangle\backslash F_n$.

The argument is the same if $\epsilon = -1$ and/or if C_k intersects $(l_{i\infty})U$ more than once.

This proves Lemma 7 for the word $V = Ux_k^\epsilon$ of reduced length N, completing the proof of Lemma 7. □

The following theorem is the main result of Section 2.

Theorem 8. *The three maps*

$$f_i : \langle x_i\rangle\backslash F_n \to \mathcal{A}_i,$$
$$g_i : \mathcal{A}_i \to \mathcal{H}_i, \quad \textit{and}$$
$$h_i : \mathcal{H}_i \to \langle x_i\rangle\backslash F_n,$$

are bijective.

Proof. The theorem is obvious from Lemmas 4, 6 and 7. □

3. Isotopy classes of (i, j)-cords

Take and fix any $i, j \in \{1, 2, \ldots, n\}$ $(i \neq j)$ throughout this section. Let $\mathcal{A}_{ij}$ be the set of all isotopy classes of (i, j)-cords on $(\mathbb{R}^2, P_n)$. First we will parameterize $\mathcal{A}_{ij}$ by certain double cosets of F_n.

Theorem 9. *Let $N(x_j)$ be the normal subgroup of F_n generated by x_j. Then there exists a bijection*

$$\tilde{f}_{ij} : \langle x_i \rangle \backslash F_n / N(x_j) \to \mathcal{A}_{ij}.$$

Proof. Let Q_{n-1} denote the set of $n-1$ points defined by

$$Q_{n-1} = P_n - \{p_j\}. \tag{12}$$

Then obviously there is a homeomorphism

$$\bigl(\mathbb{R}^2, Q_{n-1} \cup \{p_\infty\}, p_\infty\bigr) \to \bigl(\mathbb{R}^2, P_n, p_j\bigr). \tag{13}$$

We will explicitly construct a homeomorphism (13).

Let $l_{j\infty}$ be the line segment joining p_j and p_∞ (caution: not $l_{i\infty}$). Consider a motion within a sufficiently small neighborhood of $l_{j\infty}$ which moves p_∞ to p_j along $l_{j\infty}$. Let φ_j (or simply φ, j being always understood) be the final stage of this motion. Then φ gives an explicit homeomorphism (13). Note that $\varphi(p_\infty) = p_j$ and φ fixes L_k $(k \neq j)$ pointwise.

It is easy to see that φ maps an (i, ∞)-cord on $(\mathbb{R}^2, Q_{n-1} \cup \{p_\infty\})$ to an (i, j)-cord on $(\mathbb{R}^2, P_n)$. By letting $\mathcal{A}_i(Q_{n-1})$ denote the set of isotopy classes of (i, ∞)-cords on $(\mathbb{R}^2, Q_{n-1} \cup \{p_\infty\})$, we have the bijection

$$\varphi_* : \mathcal{A}_i(Q_{n-1}) \to \mathcal{A}_{ij}. \tag{14}$$

By Theorem 8, the map

$$f_i : \langle x_i \rangle \backslash G_{n-1} \to \mathcal{A}_i(Q_{n-1}) \tag{15}$$

is a bijection, where G_{n-1} denotes the free group generated by $\{x_1, x_2, \ldots, x_n\} - \{x_j\}$. This group G_{n-1} is canonically isomorphic to $F_n / N(x_j)$. Thus we have a bijection (denoted by f_i again)

$$f_i : \langle x_i \rangle \backslash F_n / N(x_j) \to \mathcal{A}_i(Q_{n-1}). \tag{16}$$

Combining (14) and (16), we have the required bijection

$$\tilde{f}_{ij} := \varphi_* \circ f_i : \langle x_i \rangle \backslash F_n / N(x_j) \to \mathcal{A}_{ij}. \tag{17}$$

This completes the proof of Theorem 9. □

Remark 10. We can likewise prove that there exists a bijection

$$\tilde{f}'_{ij} : N(x_i) \backslash F_n / \langle x_j \rangle \to \mathcal{A}_{ij}$$

by exchanging the roles of $l_{i\infty}$ and $l_{j\infty}$ in the arguments.

We will give here a geometric interpretation of the bijection $\tilde{f}_{ij}$. For this, recall the simple closed curve C_k introduced before Lemma 3. Let C_k^j $(k \neq j)$ be the image of C_k under φ; $C_k^j := \varphi(C_k)$. Since $\varphi(p_\infty) = p_j$, C_k^j is a simple closed curve on $\mathbb{R}^2 - Q_{n-1}$ based at p_j and which intersects L_k transversely in a point. Also let l_{ij} be the image $\varphi(l_{i\infty})$. Then l_{ij} is an (i, j)-cord which does not intersect $\bigcup_k L_k$ except at the end points.

The homeomorphism φ induces a homomorphism between the mapping class groups:

$$\mathcal{M}\big(\mathbb{R}^2, Q_{n-1} \cup \{p_\infty\}\big) \to \mathcal{M}\big(\mathbb{R}^2, P_n\big). \tag{18}$$

Denoting the image of V under this homomorphism by V^φ, we see that x_k^φ $(k \neq j)$ acts on $(\mathbb{R}^2, P_n)$ as the result of the motion of p_j round along the simple closed curve C_k^j.

Proposition 11 (Geometric interpretation of $\tilde{f}_{ij}$). *Let V be a word ($\in F_n$) representing a coset $[V] \in \langle x_i \rangle \backslash F_n / N(x_j)$. We may assume that V does not contain x_j. Then $\tilde{f}_{ij}([V])$ is the isotopy class of the cord $(l_{ij})V^\varphi$.*

Proof. This is clear by the definition of f_i in Section 2 and the construction of $\tilde{f}_{ij}$ given above. □

There is another geometric interpretation of $\tilde{f}_{ij}$ which follows from Lemma 7 and Theorem 8. In fact, by Lemma 7 and Theorem 8, we have

$$f_i = g_i^{-1} \circ h_i^{-1} : \langle x_i \rangle \backslash F_n \to \mathcal{A}_i. \tag{19}$$

Thus,

$$\tilde{f}_{ij} = \varphi_* \circ f_i = \varphi_* \circ g_i^{-1} \circ h_i^{-1}. \tag{20}$$

This gives the second interpretation of $\tilde{f}_{ij}$:

Proposition 12 (Second geometric interpretation of $\tilde{f}_{ij}$). *Let V ($\in F_n$) be a word representing a coset $[V] \in \langle x_i \rangle \backslash F_n / N(x_j)$. (This time V may contain x_j.) Draw a smooth (i, j)-curve l, with self-intersections in general, which is transverse to $\bigcup_k L_k$ and whose reading is V. We assume that the self-intersections of l are transverse and finite in number. By homotopy, push out all the self-intersections of l through the terminal point p_j successively. Let l' be the resulting (i, j)-cord. Then $\tilde{f}_{ij}([V])$ is the isotopy class of the cord l'.*

The meaning of “homotopy” in Proposition 12 might be a little vague. Precisely speaking, it is the image of the i-homotopy in Section 2 under φ.

Proof of Proposition 12. From the proof of the surjectivity of $g_i : \mathcal{A}_i \to \mathcal{H}_i$ (Lemma 4), and the definition of $h_i : \mathcal{H}_i \to \langle x_i \rangle \backslash F_n$, Proposition 12 follows immediately. □

Fig. 7 illustrates Proposition 12, which shows how to obtain a $(1, 3)$-cord in the isotopy class $\tilde{f}_{13}([x_2^2 x_4])$, starting from a $(1, 3)$-curve whose reading is $x_2^2 x_4$.

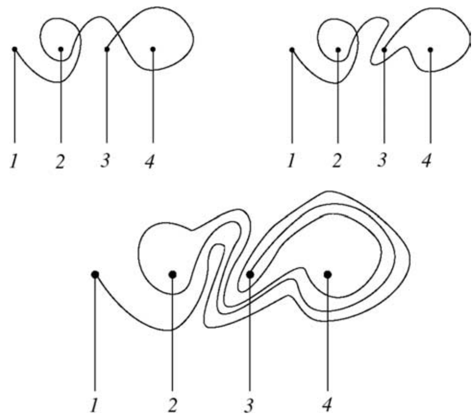

Fig. 7. (1, 3)-cord $\tilde{f}_{13}([x_2^2x_4])$.

In Fig. 7, the first $(1,3)$-curve reads as $x_2^2x_4$. Pushing out the intersection nearest to p_3, through p_3, we obtain the second curve. Its reading is $x_2^2x_3x_4$. Then pushing out the second intersection through p_3, we obtain a $(1,3)$-cord representing $\tilde{f}_{13}([x_2^2x_4])$. The reading of the $(1,3)$-cord is $x_2x_3x_4x_3x_4^{-1}x_3^{-1}x_2x_3x_4$. (Note that neglecting the generator x_3 in this final word, we recover the word $x_2^2x_4$.) In this way, the process of pushing out the intersections may be regarded as a process of successively rewriting the words. Thus the process will sometimes be referred to as the *rewriting process*.

Remark 13. By (20), it follows that

$$\tilde{f}_{ij}^{-1} = h_i \circ g_i \circ \varphi_*^{-1}.$$

Thus $\tilde{f}_{ij}^{-1}$ is explicitly described as follows: Let l be an (i,j)-cord on $(\mathbb{R}^2, P_n)$. Make it transverse to $\bigcup_k L_k$. Let $W(l)$ be the reading of l from $l(0) = p_i$ to $l(1) = p_j$. Then

$$\tilde{f}_{ij}^{-1}\bigl([l]\bigr) = \bigl[W(l)\bigr] \in \langle x_i\rangle\backslash F_n/N(x_j). \tag{21}$$

An implication of this equality is this: to determine the isotopy class of an (i,j)-cord l, we have only to know the reading of l modulo x_j.

4. Rewriting function R_{ij}

In this section, we will introduce a mapping $R_{ij} : F_n \to F_n$ which plays an important role in our investigation. We begin by defining the notion of (i,j)-homotopy of (i,j)-curves. Two (i,j)-curves l and l' are said to be (i,j)-*homotopic* if there exits a homotopy

$$H : [0,1] \times [0,1] \to \bigl(\mathbb{R}^2 - P_n\bigr) \cup \{p_i, p_j\}$$

satisfying

(i) $H(0,t) = l(t)$ and $H(1,t) = l'(t)$, $\forall t \in [0,1]$,

(ii) $H(s,t) = p_i$ if and only if $t = 0$, and
(iii) $H(s,t) = p_j$ if and only if $t = 1$.

The conditions (ii) and (iii) say that both the "exits" of an (i, j)-homotopy are "closed" at p_i and p_j. (Cf. Section 2.)

Let $\mathcal{H}_{ij}$ be the set of all (i, j)-homotopy classes of (i, j)-curves on $(\mathbb{R}^2, P_n)$.

Given an (i, j)-curve l, we make it transverse to $\bigcup_k L_k$. Let $W(l)$ be the reading of l from p_i to p_j.

Lemma 14. *The map*

$$\tilde{h}_{ij} : \mathcal{H}_{ij} \to \langle x_i \rangle \backslash F_n / \langle x_j \rangle$$

sending the (i, j)*-homotopy class of an* (i, j)*-curve* $[l]$ *to the double coset of its reading* $[W(l)]$ *is well-defined and is bijective.*

Caution: In this lemma, $\langle x_i \rangle \backslash F_n / \langle x_j \rangle$ is not $\langle x_i \rangle \backslash F_n / N(x_j)$; V and W belong to the same double coset in $\langle x_i \rangle \backslash F_n / \langle x_j \rangle$ if and only if $V = x_i^p W x_j^q$ for some $p, q \in \mathbb{Z}$.

Proof of Lemma 14. The well-definedness is proved by the (i, j)-homotopy version of the proof of Lemma 5.

The surjectivity of $\tilde{h}_{ij}$ is easy, because given a word V one can draw an (i, j)-curve whose reading is the word V if the curve is allowed to have self-intersections.

We will prove the injectivity. Let l be an (i, j)-curve. We may assume that it is smooth and transverse to $\bigcup_k L_k$. Observe that by giving rotations to l round p_i, we can multiply any power of x_i from the left of the reading $W(l)$ without changing the (i, j)-homotopy class of l. Similarly, we can multiply any power of x_j from the right of $W(l)$.

Now suppose that we are given (i, j)-curves l and l' and that their readings belong to the same double coset $\in \langle x_i \rangle \backslash F_n / \langle x_j \rangle$. By the above observation, adjusting the power of x_i from the left and that of x_j from the right, we may assume that the readings $W(l)$ and $W(l')$ are exactly the same: $W(l) = W(l') \in F_n$.

Also we may assume that the tangent vectors of l and l' at p_i (and at p_j) are the same, or more strongly, that there exists a small number $\varepsilon > 0$ such that as continuous maps $[0, 1] \to \mathbb{R}^2$, l and l' coincide if restricted to $[0, \varepsilon]$ and $[1 - \varepsilon, 1]$:

$$l|[0, \varepsilon] = l'|[0, \varepsilon], \qquad l|[1 - \varepsilon, 1] = l'|[1 - \varepsilon, 1].$$

Consider a loop L which starts at $l(\varepsilon)$, traverses l, arrives at $l(1 - \varepsilon) = l'(1 - \varepsilon)$, and returns to $l'(\varepsilon) = l(\varepsilon)$ along l'^{-1}. The loop L is completely contained in the punctured plane $\mathbb{R}^2 - P_n$, and its reading is $W(l)W(l')^{-1} = 1$. Since $\pi_1(\mathbb{R}^2 - P_n, l(\varepsilon)) \cong F_n$, L shrinks to a point in $\mathbb{R}^2 - P_n$. Making use of this homotopy, one can construct an (i, j)-homotopy between l and l'. This proves the injectivity of $\tilde{h}_{ij}$. □

Since an (i, j)-cord is an (i, j)-curve, and isotopic (i, j)-cords are (i, j)-homotopic, there is a natural map

$$\tilde{g}_{ij} : \mathcal{A}_{ij} \to \mathcal{H}_{ij} \tag{22}$$

Lemma 15 (Homotopy implies isotopy). $\tilde{g}_{ij}$ *is injective.*

Proof. Suppose that (i,j)-cords l and l' are mutually (i,j)-homotopic. We will prove that they are isotopic. By Lemma 14, the readings $W(l)$ and $W(l')$ belong to the same double coset in $\langle x_i\rangle\backslash F_n/\langle x_j\rangle$, thus evidently to the same double coset in $\langle x_i\rangle\backslash F_n/N(x_j)$. Then by Theorem 9 and Remark 13, l and l' are isotopic. □

Composing the three maps $\tilde{f}_{ij}$, $\tilde{g}_{ij}$ and $\tilde{h}_{ij}$, we have an injection denoted by

$$r_{ij}:\langle x_i\rangle\backslash F_n/N(x_j)\to\langle x_i\rangle\backslash F_n/\langle x_j\rangle. \tag{23}$$

The value $r_{ij}([V])$ is the reading of the (i,j)-cord $\tilde{f}_{ij}([V])$.

The mapping r_{ij} is computed geometrically by the rewriting process (pushing out the intersections through p_j) as explained in Section 3. For example, by Fig. 7, the reading of the $(1,3)$-cord $\tilde{f}_{13}([x_2^2x_4])$ is $x_2x_3x_4x_3x_4^{-1}x_3^{-1}x_2x_3x_4$. Thus we have

$$r_{13}\big(\big[x_2^2x_4\big]\big)=\big[x_2x_3x_4x_3x_4^{-1}x_3^{-1}x_2x_3x_4\big]\in\langle x_i\rangle\backslash F_n/\langle x_j\rangle. \tag{24}$$

This map r_{ij} can be lifted to a map

$$R_{ij}:F_n\to F_n \tag{25}$$

as follows: Take a word $V\in F_n$, consider its double coset $[V]\in\langle x_i\rangle\backslash F_n/N(x_j)$ and map it to $r_{ij}([V])$. Being an element of $\langle x_i\rangle\backslash F_n/\langle x_j\rangle$, $r_{ij}([V])$ has ambiguities of the left factor x_i^p and the right factor x_j^q. Adjust the exponents p (or q) to get a word W in F_n so that the total exponents of x_i (or x_j) in V and in W are equal. Here the *total exponent* of x_i in V means the sum of the exponents of x_i appearing in the word V. Similarly for x_j.

Then we define

$$R_{ij}(V)=W. \tag{26}$$

For example, if we want to get the R_{13}-image of the word $x_2^2x_4$, in which the total exponent of x_1 (and x_3) is 0, we have to adjust the right-hand side of (24) so that the resulting word has also the total exponent 0 w.r.t. x_1 and x_3. Thus we have

$$R_{13}\big(x_2^2x_4\big)=x_2x_3x_4x_3x_4^{-1}x_3^{-1}x_2x_3x_4x_3^{-2}. \tag{27}$$

We would like to call the map R_{ij} the *rewriting function*.

Obviously the following diagram commutes:

$$\begin{array}{ccc} F_n & \xrightarrow{R_{ij}} & F_n \\ \downarrow & & \downarrow \\ \langle x_i\rangle\backslash F_n/N(x_j) & \xrightarrow{r_{ij}} & \langle x_i\rangle\backslash F_n/\langle x_j\rangle \end{array} \tag{28}$$

The vertical arrows are natural projections.

In Section 6, we will give a formula to compute the rewriting function R_{ij} purely algebraically.

5. Some properties of R_{ij}

In this section, we give important properties of the rewriting function R_{ij}.

Lemma 16. *Let V and W be words in F_n. Then*

$$R_{ij}(V) = W,$$

if and only if they satisfy the following conditions:

(i) $[V] = [W] \in \langle x_i \rangle \backslash F_n / N(x_j)$,
(ii) $E_i(V) = E_i(W)$, $E_j(V) = E_j(W)$, *where $E_k(U)$ denotes the total exponent of x_k in the word U, and*
(iii) *W is the reading of an (i, j)-cord.*

Proof. Suppose $R_{ij}(V) = W$. Since by definition $R_{ij}(V)$ is a lifted reading of the (i, j)-cord $\tilde{f}_{ij}([V])$, $R_{ij}(V)$ and V belong to the same double coset $\in \langle x_i \rangle \backslash F_n / N(x_j)$ by Remark 13. Thus (i) is satisfied. The conditions (ii) and (iii) are satisfied by the definition of R_{ij}. This proves the *only if*-part.

Conversely, suppose that V and W satisfy (i), (ii) and (iii). Let l be an (i, j)-cord such that $W = W(l)$. Such a cord l exists by condition (iii). Just as above, by Remark 13, $R_{ij}(V)$ and V belong to the same double coset $\in \langle x_i \rangle \backslash F_n / N(x_j)$. By condition (i), V and W belong to the same double coset. Thus $[R_{ij}(V)] = [W] \in \langle x_i \rangle \backslash F_n / N(x_j)$. By Theorem 9 and Remark 13 again, the (i, j)-cords $\tilde{f}_{ij}([V])$ and l are isotopic. By Lemma 14, the readings $R_{ij}(V)$ and W of these isotopic cords coincide modulo left factor x_i^p and right factor x_j^q. But by the definition of R_{ij} and condition (ii), we have $E_i(R_{ij}(V)) = E_i(V) = E_i(W)$ and $E_j(R_{ij}(V)) = E_j(V) = E_j(W)$. Thus $R_{ij}(V) = W$. The *if*-part is proved. □

Lemma 17. *A word V is the reading of an (i, j)-cord if and only if*

$$R_{ij}(V) = V. \tag{29}$$

Proof. Suppose V satisfies (29). Then $r_{ij}([V]) = [V]$, and by the definition of r_{ij}, $[V]$ is the reading of the (i, j)-cord $\tilde{f}_{ij}([V])$. By giving rotations to this cord round p_i and p_j, we may adjust that the actual reading of the cord is V. This proves the *if*-part.

Conversely, suppose V is the reading of an (i, j)-cord l, then applying Lemma 16, we have

$$R_{ij}(V) = V.$$

This proves the *only if*-part. □

Lemma 18. *R_{ij} is a projection, that is, it satisfies*

$$R_{ij} \circ R_{ij} = R_{ij}.$$

Proof. For any word W, $R_{ij}(W)$ is a lifted reading of the isotopy class of (i, j)-cords $\tilde{f}_{ij}([W])$. Thus applying Lemma 17 to the word $V = R_{ij}(W)$, we have $R_{ij}(R_{ij}(W)) = R_{ij}(W)$. $\square$

Lemma 19. *For any $m \in \mathbb{Z}$, we have*

$$\begin{cases} R_{ij}\left(V_1 x_j^m V_2\right) = R_{ij}(V_1 V_2) x_j^m, \\ R_{ij}\left(x_i^m V\right) = x_i^m R_{ij}(V). \end{cases}$$

Proof. Since the words $V_1 x_j^m V_2$ and $V_1 V_2$ belong to the same double coset $\in \langle x_i \rangle \backslash F_n / N(x_j)$, the commutative diagram (28) implies that the images $R_{ij}(V_1 x_j^m V_2)$ and $R_{ij}(V_1 V_2)$ differ only in the left x_i- and the right x_j-powers. However,

$$\begin{cases} E_i\left(V_1 x_j^m V_2\right) = E_i(V_1 V_2), \quad \text{and} \\ E_j\left(V_1 x_j^m V_2\right) = E_j(V_1 V_2) + m, \end{cases}$$

and we know that $R_{ij}(\cdot)$ preserves the total i- and j-exponents. Thus we have the first equality.

The second equality is proved similarly. $\square$

6. Algebraic formula for R_{ij}

In this section, we will give a formula to compute $R_{ij} : F_n \to F_n$ purely algebraically.

As we remarked just before Proposition 11, $F_n/N(x_j)$ acts on $(\mathbb{R}^2, P_n)$ from the right. More precisely, an element $x_k \in F_n (k \neq j)$, acts on $(\mathbb{R}^2, P_n)$ as x_k^φ, which is the mapping class of the motion of p_j along the curve $C_k^j = \varphi(C_k)$. Incidentally, we also consider the case $k = j$, where taking Proposition 11 into account, we define the action of x_j^φ to be the trivial action on $(\mathbb{R}^2, P_n)$. Then the action of $F_n/N(x_j)$ lifts to the action of F_n on $(\mathbb{R}^2, P_n)$. We call this action the *j-action* of F_n to distinguish it from the action of F_n on $(\mathbb{R}^2, P_n \cup \{p_\infty\})$ introduced in Section 2.

Via the j-action, F_n acts on the (i, j)-homotopy set $\mathcal{H}_{ij}$. We would like to describe this action algebraically.

Recall that l_{kj} is the (k, j)-cord which does not intersect $\bigcup_h L_h$ except at the end points p_k, p_j.

Lemma 20. *The action of x_k^φ $(k \neq j)$ is nothing but the "$360°$-twist" along l_{kj}, namely, the mapping class whose support is contained in a disk neighborhood of the cord l_{kj} and which rotates the cord through $360°$ counterclockwise.*

This lemma is easily seen by figures. (Cf. the proof of Lemma 4.1 of [9].)

The j-action of F_n is generated by x_k^φ, $k = 1, \ldots, n$. By Lemma 20, the action of x_k^φ $(k \neq j)$ is the "360°-twist" along the (k, j)-cord l_{kj}, and the action of x_j^φ is the identity. Thus if we assume that all the (k, j)-cord l_{kj} are contained in the domain $y < 1 - \varepsilon$ (or more safely in $y < \frac{1}{2}$) of the xy-plane, then we may assume that the j-action of F_n is

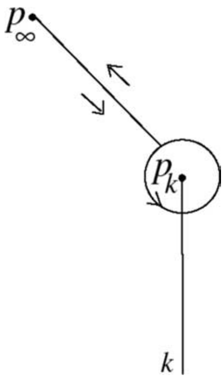

Fig. 8. The loop x_k.

trivial on the complementary region $y \geqslant 1 - \varepsilon$ (or $y \geqslant \frac{1}{2}$) in which $p_\infty = (0, 1)$ is. In the following arguments, we will always assume this *optimal condition* on the j-action of F_n.

Let us introduce some notations. Let α_i denote the line segment $l_{i\infty}$ considered to be an oriented simple curve from p_i to p_∞. Similarly let β_j denote the line segment $l_{j\infty}$ regarded as an oriented simple curve from p_∞ to p_j. Thus the composition of these curves $\alpha_i \cdot \beta_j$ is isotopic to the cord l_{ij}.

The generator x_k of $\pi_1(\mathbb{R}^2 - P_n, p_\infty)$ is represented by the simple loop C_k. However, in studying the effect of the j-action of F_n, we prefer to C_k the following loop as the representative of x_k, namely, the loop which starts at p_∞, going down along the line segment $l_{k\infty}$, arrives at a point near to p_k, then makes a small circle round p_k, and finally comes back to p_∞ along $l_{k\infty}$. (See Fig. 8.) We will also denote by x_k such a loop. Of course, the inverse x_k^{-1} is represented by the loop traversing x_k in the opposite direction.

Let V be a word ($\in F_n$), and define an (i, j)-curve $L_{ij}(V)$ as follows:

$$L_{ij}(V) := \alpha_i \cdot x_{k(1)}^{\epsilon(1)} \cdots x_{k(N)}^{\epsilon(N)} \cdot \beta_j, \tag{30}$$

where $V = x_{k(1)}^{\epsilon(1)} \cdots x_{k(N)}^{\epsilon(N)}$, $\epsilon(m) \in \{+1, -1\}$ and $k(m) \in \{1, 2, \ldots, n\}$ for $m = 1, \ldots, N$. Every (i, j)-homotopy class has $L_{ij}(V)$ as its representative for some V. Note that the reading of this (i, j)-curve is V, and that by Lemma 14 $L_{ij}(V)$ and $L_{ij}(W)$ belong to the same (i, j)-homotopy class if and only if $[V] = [W] \in \langle x_i \rangle \backslash F_n / \langle x_j \rangle$.

We are now in a position to study the effect of the j-action of F_n on $L_{ij}(V)$. For notational convenience, we will denote the (right) j-action W^φ of a word $W \in F_n$ on $(\mathbb{R}^2, P_n)$ by $(\cdot)T_W^j$, or understanding j being always fixed, simply by $(\cdot)T_W$.

Let us apply the j-action $(\cdot)T_W$ on the (i, j)-curve $L_{ij}(V)$ of (30). Then by the optimal condition on the j-action, we have

$$\begin{aligned} \big(L_{ij}(V)\big)T_W &= (\alpha_i)T_W \cdot \big(x_{k(1)}^{\epsilon(1)}\big)T_W \cdots \big(x_{k(N)}^{\epsilon(N)}\big)T_W \cdot (\beta_j)T_W \\ &= (\alpha_i)T_W \cdot (V)T_W \cdot (\beta_j)T_W. \end{aligned} \tag{31}$$

Thus we can study the action $(\cdot)T_W$ on α_i, V, and β_j, separately.

First consider $(\alpha_i)T_W$. This is an (i,∞)-cord, and its i-homotopy class is represented by an (i,∞)-curve of the form

$$\alpha_i \cdot W' \tag{32}$$

where W' is a certain product of the loops $x_k^\epsilon \in F_n$. By Theorem 8, the word W' is well-defined up to left multiplication of x_i^p, that is, only its coset $[W'] \in \langle x_i\rangle\backslash F_n$ is well-defined. To fix this ambiguity, we impose the condition that the total exponent of x_i in W' should be 0:

$$E_i\big(W'\big) = 0. \tag{33}$$

Then the ambiguity is removed and W' is well-defined as an element of F_n. Let us denote this W' by $A_i(W)$. The following equalities are considered to be the definition of $A_i(W) \in F_n$:

$$\begin{cases} (\alpha_i)T_W = \alpha_i \cdot A_i(W), \\ E_i\big(A_i(W)\big) = 0. \end{cases} \tag{34}$$

Similarly, the (∞, j)-cord $(\beta_j)T_W$ is j-homotopic to the (∞, j)-curve $W'' \cdot \beta_j$, and $W'' \in F_n$ is proved to be well-defined up to right multiplication of x_j^q. Imposing the condition $E_j(W'') = 0$, we can eliminate this ambiguity. Denoting the well-defined W'' by $B_j(W)$, we have the following equalities, which are considered to be the definition of $B_j(W) \in F_n$:

$$\begin{cases} (\beta_j)T_W = B_j(W) \cdot \beta_j, \\ E_j\big(B_j(W)\big) = 0. \end{cases} \tag{35}$$

The part $(V)T_W$ is easily understood, because T_W acts on $F_n(= \pi_1(\mathbb{R}^2 - P_n, p_\infty))$ as a group automorphism. Thus denoting this automorphism by $H_{j,W} : F_n \to F_n$, we have

$$(V)T_W = H_{j,W}(V). \tag{36}$$

In these notations, (31) is rewritten as follows:

$$\big(L_{ij}(V)\big)T_W = \alpha_i \cdot A_i(W) \cdot H_{j,W}(V) \cdot B_j(W) \cdot \beta_j. \tag{37}$$

To further analyze these mappings $A_i, B_j, H_{j,W} : F_n \to F_n$, we check the simplest cases.

Lemma 21. *If $i < j$, then*

$$A_i(x_k) = \begin{cases} x_j x_k x_j^{-1} x_k^{-1} & k < i, \\ x_i x_j^{-1} x_i^{-1} & k = i, \\ 1 & k > i, \end{cases} \tag{38}$$

$$A_i\big(x_k^{-1}\big) = \begin{cases} x_k^{-1} x_j^{-1} x_k x_j & k < i, \\ x_j & k = i, \\ 1 & k > i. \end{cases} \tag{39}$$

If $i > j$, then

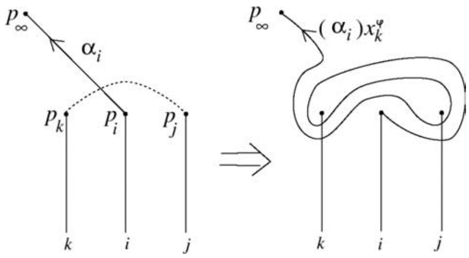

Fig. 9. The cords $(\alpha_i)x_k^\varphi = \alpha_i \cdot A_i(x_k),\ k < i < j$.

$$A_i(x_k) = \begin{cases} 1 & k < i, \\ x_j^{-1} & k = i, \\ x_k x_j x_k^{-1} x_j^{-1} & k > i, \end{cases} \tag{40}$$

$$A_i\left(x_k^{-1}\right) = \begin{cases} 1 & k < i, \\ x_i^{-1} x_j x_i & k = i, \\ x_j^{-1} x_k^{-1} x_j x_k & k > i. \end{cases} \tag{41}$$

Proof. To prove (38), we apply the j-action x_k^φ to the (i, ∞)-cord α_i. By Lemma 20, the resulting cords are as shown in Fig. 9. We can easily prove (38) by reading the intersections of the cords and $\bigcup_h L_h$. Notice that in the right hand side of the second equality of (38), we multiply x_i artificially to meet the requirement (34) on the total exponent of x_i. Other equalities (39), (40) and (41) are proved similarly. □

By the same method, we can prove the following lemma (x_j^{-1} in the third equality of (42) and x_j in the first of (43) are "artificially" multiplied to meet the condition (35)).

Lemma 22.

$$B_j(x_k) = \begin{cases} x_k & k < j, \\ 1 & k = j, \\ x_j x_k x_j^{-1} & k > j, \end{cases} \tag{42}$$

$$B_j\left(x_k^{-1}\right) = \begin{cases} x_j^{-1} x_k^{-1} x_j & k < j, \\ 1 & k = j, \\ x_k^{-1} & k > j. \end{cases} \tag{43}$$

Proof. Fig. 10 shows, in the case $k < j$, how β_j changes when it is acted on by x_k^φ. This proves the first equality of (42). Other cases are obtained by similar figures. □

Lemma 23. *Let ϵ denote $+1$ or -1.*
If $k < j$, then

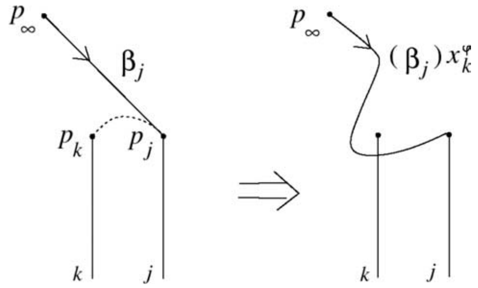

Fig. 10. The cord $(\beta_j)x_k^\varphi = B_j(x_k)\cdot\beta_j,\ k<j$.

$$H_{j,x_k}\big(x_l^\epsilon\big)=\begin{cases} x_l^\epsilon & l<k \text{ or } l>j,\\ x_kx_jx_k^\epsilon x_j^{-1}x_k^{-1} & l=k,\\ x_kx_jx_k^{-1}x_j^{-1}x_l^\epsilon x_jx_kx_j^{-1}x_k^{-1} & k<l<j,\\ x_kx_j^\epsilon x_k^{-1} & l=j,\end{cases} \tag{44}$$

$$H_{j,x_k^{-1}}\big(x_l^\epsilon\big)=\begin{cases} x_l^\epsilon & l<k \text{ or } l>j,\\ x_j^{-1}x_k^\epsilon x_j & l=k,\\ x_j^{-1}x_k^{-1}x_jx_kx_l^\epsilon x_k^{-1}x_j^{-1}x_kx_j & k<l<j,\\ x_j^{-1}x_k^{-1}x_j^\epsilon x_kx_j & l=j.\end{cases} \tag{45}$$

If $k>j$, then

$$H_{j,x_k}\big(x_l^\epsilon\big)=\begin{cases} x_l^\epsilon & l<j \text{ or } l>k,\\ x_jx_kx_j^\epsilon x_k^{-1}x_j^{-1} & l=j,\\ x_jx_kx_j^{-1}x_k^{-1}x_l^\epsilon x_kx_jx_k^{-1}x_j^{-1} & j<l<k,\\ x_jx_k^\epsilon x_j^{-1} & l=k,\end{cases} \tag{46}$$

$$H_{j,x_k^{-1}}\big(x_l^\epsilon\big)=\begin{cases} x_l^\epsilon & l<j \text{ or } l>k,\\ x_k^{-1}x_j^\epsilon x_k & l=j,\\ x_k^{-1}x_j^{-1}x_kx_jx_l^\epsilon x_j^{-1}x_k^{-1}x_jx_k & j<l<k,\\ x_k^{-1}x_j^{-1}x_k^\epsilon x_jx_k & l=k.\end{cases} \tag{47}$$

If $k=j$, then

$$H_{j,x_j}(x_l)=H_{j,x_j^{-1}}(x_l)=x_l. \tag{48}$$

Proof. Fig. 11 shows, in the case $k<l<j$, how the loop x_l^ϵ changes when it is acted on by x_k^φ. The third equality of (44) follows from this figure. Other cases are proved similarly. □

Lemma 24. *We have*

$$\begin{cases} H_{j,W_1W_2}(V)=H_{j,W_2}\big(H_{j,W_1}(V)\big),\\ H_{j,W}(V_1V_2)=H_{j,W}(V_1)H_{j,W}(V_2)\end{cases} \tag{49}$$

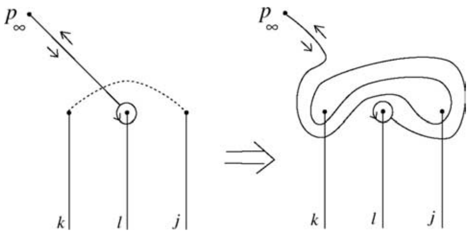

Fig. 11. The loop $(x_l^\epsilon)x_k^\varphi,\ k < l < j$.

and

$$\begin{cases} E_i\big(H_{j,W}(V)\big) = E_i(V), \\ E_j\big(H_{j,W}(V)\big) = E_j(V). \end{cases} \tag{50}$$

Proof. The equalities (49) follow from the definition of $H_{j,W}$ in (36) and the fact that T_W acts on F_n (from the right) as a group automorphism. By Lemma 23, we see that, for $W = x_k^{\pm 1}$, H_{j,x_k} and $H_{j,x_k^{-1}}$ preserve the total exponents $E_i(\cdot)$ and $E_j(\cdot)$. The general statement (50) follows from this special case and (49). □

We express (49) by saying that $H_j : F_n \to \mathrm{Aut}(F_n)$ $(W \mapsto H_{j,W}(\cdot))$ is a *right representation* of F_n to $\mathrm{Aut}(F_n)$.

Lemma 25.

$$A_i(W_1 W_2) = A_i(W_2) H_{j,W_2}\big(A_i(W_1)\big). \tag{51}$$

Proof. By (34) and (36),

$$\begin{aligned} \alpha_i \cdot A_i(W_1 W_2) &= (\alpha_i) T_{W_1 W_2} = \big((\alpha_i) T_{W_1}\big) T_{W_2} \\ &= \big(\alpha_i \cdot A_i(W_1)\big) T_{W_2} \\ &= (\alpha_i) T_{W_2} \cdot \big(A_i(W_1)\big) T_{W_2} \\ &= \alpha_i \cdot A_i(W_2) H_{j,W_2}\big(A_i(W_1)\big). \end{aligned}$$

On the other hand, by (50) and (34),

$$E_i\Big(A_i(W_2) H_{j,W_2}\big(A_i(W_1)\big)\Big) = E_i\big(A_i(W_2)\big) + E_i\big(A_i(W_1)\big) = 0.$$

Thus by the definition (34) of $A_i(\cdot)$, we have the lemma. □

We express (51) by saying that $A_i : F_n \to F_n$ is a *crossed anti-homomorphism* twisted by the right representation $H_j : F_n \to \mathrm{Aut}(F_n)$.

Lemma 26.

$$E_j\big(A_i(W)\big) = -E_i(W).$$

Proof. For $W = x_k^{\pm 1}$, this follows from Lemma 21. General cases are proved by induction on the word length of W and Lemmas 24 (50) and 25. □

Lemma 27.

$$B_j(W_1W_2) = H_{j,W_2}\big(B_j(W_1)\big)B_j(W_2). \tag{52}$$

Proof. By (35) and (36),

$$\begin{aligned} B_j(W_1W_2)\cdot\beta_j &= (\beta_j)T_{W_1W_2} = \big((\beta_j)T_{W_1}\big)T_{W_2} \\ &= \big(B_j(W_1)\cdot\beta_j\big)T_{W_2} \\ &= \big(B_j(W_1)\big)T_{W_2}\cdot(\beta_j)T_{W_2} \\ &= H_{j,W_2}\big(B_j(W_1)\big)B_j(W_2)\cdot\beta_j. \end{aligned}$$

On the other hand, by (50) and (35),

$$E_j\big(H_{j,W_2}\big(B_j(W_1)\big)B_j(W_2)\big) = E_j\big(B_j(W_1)\big) + E_j\big(B_j(W_2)\big) = 0.$$

Thus by the definition (35) of $B_j(\cdot)$, we have the lemma. □

We express (52) by saying that $B_j : F_n \to F_n$ is a *crossed homomorphism* twisted by the right representation $H_j : F_n \to \mathrm{Aut}(F_n)$.

Lemma 28.

$$E_i\big(B_j(W)\big) = E_i(W).$$

Proof. For $W = x_k^{\pm 1}$, this follows from Lemma 22. General cases are proved by induction on the word length of W and Lemmas 24 (50) and 27. □

Lemma 29. *For any words V and W, we have*

$$\big[H_{j,W}(V)\big] = [V] \in F_n/N(x_j). \tag{53}$$

Proof. For $W = x_k^{\pm 1}$, $V = x_l^{\pm 1}$, this holds by Lemma 23. General cases are proved using (49). □

Lemma 30. *For any word W, we have*

$$\big[A_i(W)\big] = 1 \in F_n/N(x_j). \tag{54}$$

Proof. For $W = x_k^{\pm 1}$, this holds by Lemma 21. For a general W, (54) is proved by induction on the word length of W, using Lemmas 25 and 29. □

Lemma 31. *For any word W, we have*

$$\big[B_j(W)\big] = [W] \in F_n/N(x_j). \tag{55}$$

Proof. For $W = x_k^{\pm 1}$, this holds by Lemma 22. For a general W, (55) is proved by induction on the word length of W, using Lemmas 27 and 29. □

The following lemma is rather technical, but it will be useful in calculating the rewriting function R_{ij}.

Lemma 32. *For any words V, W, and for any integer m, we have*

$$A_i(W)H_{j,W}\bigl(x_i^m V\bigr) = x_i^m A_i(W)H_{j,W}(V), \tag{56}$$

$$H_{j,W}\bigl(V x_j^m\bigr)B_j(W) = H_{j,W}(V)B_j(W)x_j^m. \tag{57}$$

Proof. By (34) and (36), we have the following equality (in the sense of i-homotopy) of (i,∞)-curves:

$$\begin{aligned}\alpha_i \cdot A_i(W) \cdot H_{j,W}\bigl(x_i^m V\bigr) &= (\alpha_i)T_W \cdot \bigl(x_i^m V\bigr)T_W \\ &= \bigl(\alpha_i \cdot x_i^m V\bigr)T_W.\end{aligned} \tag{58}$$

Rotating the (i,∞)-curve round p_i, we have

$$\alpha_i \cdot x_i^m V = \alpha_i \cdot V.$$

Substituting this into (58), we have

$$\begin{aligned}\alpha_i \cdot A_i(W) \cdot H_{j,W}\bigl(x_i^m V\bigr) &= (\alpha_i \cdot V)T_W \\ &= \alpha_i \cdot A_i(W) \cdot H_{j,W}(V).\end{aligned}$$

Thus by Theorem 8 the words $A_i(W)H_{j,W}(x_i^m V)$ and $A_i(W)H_{j,W}(V)$ coincide up to the left multiplication of x_i^p for some p.

By (34) and (50) ,

$$\begin{aligned}E_i\bigl(A_i(W)H_{j,W}\bigl(x_i^m V\bigr)\bigr) &= m + E_i(V) \\ &= E_i\bigl(x_i^m A_i(W)H_{j,W}(V)\bigr).\end{aligned}$$

Therefore, $A_i(W)H_{j,W}(x_i^m V) = x_i^m A_i(W)H_{j,W}(V)$. This proves (56).

The equality (57) is proved similarly using (∞, j)-curves. □

Now we are ready to study the rewriting function $R_{ij} : F_n \to F_n$.

Proposition 33. *For any word V, we have*

$$R_{ij}(V) = A_i(V)B_j(V)x_j^{\phi(V)}, \tag{59}$$

where $\phi(V)$ is defined by $\phi(V) = E_i(V) + E_j(V)$.

Proof. We will check the three conditions in Lemma 16 on the right-hand side of (59). First, condition (i). In fact, as equality in the quotient group $F_n/N(x_j)$, we have

$$\bigl[A_i(V)B_j(V)x_j^{\phi(V)}\bigr] = \bigl[A_i(V)\bigr]\bigl[B_j(V)\bigr] = [V].$$

Here we used Lemmas 30 and 31.

Next we will check condition (ii):

$$E_i\big(A_i(V)B_j(V)x_j^{\phi(V)}\big) = E_i(V) \quad \text{by (34) and Lemma 28}$$

and

$$\begin{aligned}&E_j\big(A_i(V)B_j(V)x_j^{\phi(V)}\big)\\ &\quad = -E_i(V) + \phi(V) = E_j(V) \quad \text{by (35) and Lemma 26.}\end{aligned}$$

Finally we will prove that $A_i(V)B_j(V)x_j^{\phi(V)}$ is the reading of an (i,j)-cord (condition (iii)). As a special case of (37) we have

$$\big(L_{ij}(1)\big)T_V = \alpha_i \cdot A_i(V) \cdot B_j(V) \cdot \beta_j.$$

The left-hand side $L_{ij}(1)T_V$ is nothing but $(l_{ij})V^{\varphi}$, and this is an (i,j)-cord. Therefore, $A_i(V)B_j(V)$ is the reading of an (i,j)-cord. Hence $A_i(V)B_j(V)x_j^{\phi(V)}$ is also.

Now the three conditions on the word $A_i(V)B_j(V)x_j^{\phi(V)}$ are verified. Thus by Lemma 16 we have the proposition. □

Remark 34. Recall that $R_{ij}(V)$ is a lifted reading of the (i,j)-cord $\tilde{f}_{ij}([V])$. By Proposition 11, $\tilde{f}_{ij}([V])$ is the isotopy class of $(l_{ij})V^{\varphi}$. In view of this, the result of Proposition 33 is quite natural.

We want an inductive formula to compute R_{ij}:

$$\begin{aligned}R_{ij}(V_1V_2) &= A_i(V_1V_2)B_j(V_1V_2)x_j^{\phi(V_1V_2)}\\ &= A_i(V_2)H_{j,V_2}\big(A_i(V_1)\big)H_{j,V_2}\big(B_j(V_1)\big)B_j(V_2)x_j^{\phi(V_1)+\phi(V_2)}\\ &= A_i(V_2)H_{j,V_2}\big(A_i(V_1)B_j(V_1)x_j^{\phi(V_1)}\big)B_j(V_2)x_j^{\phi(V_2)}\\ &= A_i(V_2)H_{j,V_2}\big(R_{ij}(V_1)\big)B_j(V_2)x_j^{\phi(V_2)}.\end{aligned} \tag{60}$$

Note that we applied Lemma 32 to get the third equality.

By (60), we have

$$\begin{aligned}R_{ij}(V_2)^{-1}R_{ij}(V_1V_2) &= x_j^{-\phi(V_2)}B_j(V_2)^{-1}A_i(V_2)^{-1}\\ &\qquad \times A_i(V_2)H_{j,V_2}\big(R_{ij}(V_1)\big)B_j(V_2)x_j^{\phi(V_2)}\\ &= x_j^{-\phi(V_2)}B_j(V_2)^{-1}H_{j,V_2}\big(R_{ij}(V_1)\big)B_j(V_2)x_j^{\phi(V_2)}.\end{aligned} \tag{61}$$

For any word W, define a mapping $D_{ij,W}: F_n \to F_n$ by setting

$$D_{ij,W}(V) = x_j^{-\phi(W)}B_j(W)^{-1}H_{j,W}(V)B_j(W)x_j^{\phi(W)}. \tag{62}$$

Then (61) is rewritten as

$$R_{ij}(V_1V_2) = R_{ij}(V_2)D_{ij,V_2}\big(R_{ij}(V_1)\big). \tag{63}$$

Proposition 35.

$$\begin{cases} D_{ij,W}(V_1V_2) = D_{ij,W}(V_1)D_{ij,W}(V_2), \\ D_{ij,W_1W_2}(V) = D_{ij,W_2}\big(D_{ij,W_1}(V)\big). \end{cases} \tag{64}$$

Proof. The first equality is easily seen by the definition (62) of $D_{ij,W}(\cdot)$. The second equality is proved as follows:

$$\begin{aligned} D_{ij,W_1W_2}(V) &= x_j^{-\phi(W_1W_2)} B_j(W_1W_2)^{-1} H_{j,W_1W_2}(V) B_j(W_1W_2) x_j^{\phi(W_1W_2)} \\ &= x_j^{-\phi(W_1)-\phi(W_2)} B_j(W_2)^{-1} H_{j,W_2}\big(B_j(W_1)\big)^{-1} H_{j,W_2}\big(H_{j,W_1}(V)\big) \\ &\quad \times H_{j,W_2}\big(B_j(W_1)\big) B_j(W_2) x_j^{\phi(W_1)+\phi(W_2)} \\ &= x_j^{-\phi(W_2)} B_j(W_2)^{-1} \\ &\quad \times H_{j,W_2}\big(x_j^{-\phi(W_1)} B_j(W_1)^{-1} H_{j,W_1}(V) B_j(W_1) x_j^{\phi(W_1)}\big) \\ &\quad \times B_j(W_2) x_j^{\phi(W_2)} \\ &= D_{ij,W_2}\big(D_{ij,W_1}(V)\big). \qquad \square \end{aligned}$$

This proposition claims that $D_{ij} : F_n \to \mathrm{Aut}(F_n)$ $(W \mapsto D_{ij,W}(\cdot))$ is a right representation of F_n to $\mathrm{Aut}(F_n)$.

Now we have arrived at our main theorem of this section:

Theorem 36. *The rewriting function $R_{ij} : F_n \to F_n$ is a crossed anti-homomorphism twisted by the right representation $D_{ij} : F_n \to \mathrm{Aut}(F_n)$;*

$$R_{ij}(V_1V_2) = R_{ij}(V_2)D_{ij,V_2}\big(R_{ij}(V_1)\big).$$

This theorem is obvious from (63) and Proposition 35.

Remark 37. For Theorem 36, R_{ij} very much resembles in algebraic nature the *q-inverse* $I : F_n \to F_n$ studied in [3].

Using Theorem 36 together with the following *initial formulae*, we can compute R_{ij} purely algebraically. To state the initial formulae, we introduce the notation

$$\mathrm{sgn}(i, j, k) \tag{65}$$

which takes the value $+1$ or -1 according as the permutation of (i, j, k) into its natural order is of even type or of odd type, where i, j, k are distinct integers. For example, if $i < j < k$, then $\mathrm{sgn}(i, j, k) = 1$, and if $j < i < k$, then $\mathrm{sgn}(i, j, k) = -1$.

Proposition 38 (Initial formula, I). *Let ϵ be $+1$ or -1. Then we have*

$$R_{ij}\big(x_k^\epsilon\big) = \begin{cases} x_k^\epsilon & k = i \text{ or } j, \\ x_k^\epsilon & \mathrm{sgn}(i, j, k) = -\epsilon, \\ x_j^\epsilon x_k^\epsilon x_j^{-\epsilon} & \mathrm{sgn}(i, j, k) = \epsilon. \end{cases} \tag{66}$$

Proof. By Proposition 33, we have

$$R_{ij}\left(x_k^\epsilon\right) = A_i\left(x_k^\epsilon\right)B_j\left(x_k^\epsilon\right)x_j^{\phi(x_k^\epsilon)}.$$

Suppose $\epsilon = 1$ and $i < j < k$. Then

$$\begin{cases} A_i(x_k) = 1 & \text{by Lemma 21 (38)},\\ B_j(x_k) = x_j x_k x_j^{-1} & \text{by Lemma 22 (42)},\\ \phi(x_k) = 0. \end{cases}$$

The formula holds in this case. Other cases are proved similarly. □

Proposition 39 (Initial formulae, II). *Let ϵ be $+1$ or -1.*

$$D_{ij,x_i^\epsilon}(x_s) = \begin{cases} x_i & s = i,\\ x_j & s = j,\\ x_i^{-\epsilon}x_j^{-\epsilon}x_s x_j^\epsilon x_i^\epsilon & \operatorname{sgn}(i,j,s) = -\epsilon,\\ x_j^{-\epsilon}x_i^{-\epsilon}x_s x_i^\epsilon x_j^\epsilon & \operatorname{sgn}(i,j,s) = \epsilon, \end{cases} \tag{67}$$

$$D_{ij,x_j^\epsilon}(x_s) = x_j^{-\epsilon}x_s x_j^\epsilon. \tag{68}$$

If i, j, k are distinct, then

$$D_{ij,x_k^\epsilon}(x_s) = \begin{cases} x_j & s = j,\\ x_j^\epsilon x_k x_j^{-\epsilon} & s = k,\\ x_k^{-\epsilon}x_s x_k^\epsilon & \operatorname{sgn}(j,k,s) = -\epsilon,\\ \left(x_j^\epsilon x_k^{-\epsilon}x_j^{-\epsilon}\right)x_s\left(x_j^\epsilon x_k^\epsilon x_j^{-\epsilon}\right) & \operatorname{sgn}(j,k,s) = \epsilon. \end{cases} \tag{69}$$

Proof. By the definition (62) of $D_{ij}(\cdot)$, we have

$$D_{ij,x_k^\epsilon}(x_s) = x_j^{-\phi(x_k^\epsilon)}B_j\left(x_k^\epsilon\right)^{-1}H_{j,x_k^\epsilon}(x_s)B_j\left(x_k^\epsilon\right)x_j^{\phi(x_k^\epsilon)}.$$

Suppose $\epsilon = +1$ and consider the case $i < j < k < s$. Then we have

$$\begin{cases} \phi(x_k) = 0,\\ B_j(x_k) = x_j x_k x_j^{-1} & \text{by Lemma 22 (42)},\\ H_{j,x_k}(x_s) = x_s & \text{by Lemma 23 (44)}. \end{cases}$$

Thus

$$D_{ij,x_k}(x_s) = x_j x_k^{-1}x_j^{-1}x_s x_j x_k x_j^{-1}.$$

This proves the fourth equality of (69). Other cases are proved similarly. □

For example, let us calculate $R_{ij}(x_s^2)$ under the assumption $i < s < j$. By (66),

$$R_{ij}(x_s) = x_s.$$

By (63) and (69),

$$
\begin{aligned}
R_{ij}\big(x_s^2\big) &= R_{ij}(x_s)D_{ij,x_s}\big(R_{ij}(x_s)\big)\\
&= x_s D_{ij,x_s}(x_s)\\
&= x_s x_j x_s x_j^{-1}.
\end{aligned}
$$

We obtain $R_{ij}(x_s^2) = x_s x_j x_s x_j^{-1}$ in the case $i < s < j$.

Another concrete example is this (use the above result with $i = 1$, $j = 3$, $s = 2$):

$$
\begin{aligned}
R_{13}\big(x_2^2 x_4\big) &= R_{13}(x_4)D_{13,x_4}\big(R_{13}\big(x_2^2\big)\big)\\
&= \big(x_3x_4x_3^{-1}\big)D_{13,x_4}\big(x_2x_3x_2x_3^{-1}\big)\\
&= \big(x_3x_4x_3^{-1}\big)\big(x_3x_4^{-1}x_3^{-1}\big)x_2\big(x_3x_4x_3^{-1}\big)x_3\big(x_3x_4^{-1}x_3^{-1}\big)x_2\big(x_3x_4x_3^{-1}\big)x_3^{-1}\\
&= x_2x_3x_4x_3x_4^{-1}x_3^{-1}x_2x_3x_4x_3^{-2}.
\end{aligned}
$$

The final result coincides with (27) which was obtained diagrammatically.

Remark 40. For any word W, define a mapping $\widetilde{D}_{ij,W} : F_n \to F_n$ by setting

$$\widetilde{D}_{ij,W}(V) = A_i(W)H_{j,W}(V)A_i(W)^{-1}. \tag{70}$$

Then $\widetilde{D}_{ij} : F_n \to \mathrm{Aut}(F_n)$ $(W \mapsto \widetilde{D}_{ij,W}(\cdot))$ is a right representation of F_n to $\mathrm{Aut}(F_n)$. By a similar argument to the proof of (63), we can prove

$$R_{ij}(V_1V_2) = \widetilde{D}_{ij,V_2}\big(R_{ij}(V_1)\big)R_{ij}(V_2). \tag{71}$$

Thus R_{ij} is not only a crossed anti-homomorphism twisted by D_{ij}, but also a *crossed homomorphism* twisted by $\widetilde{D}_{ij}$. We do not know which is the natural formulation, but the initial formulae for $\widetilde{D}_{ij}$ are a bit more complicated than those for D_{ij}.

7. Application to simple closed curves

Let C be an oriented simple closed curve on the punctured plane $\mathbb{R}^2 - P_n$. Deforming C by homotopy, we may assume that it is smooth and transverse to $\bigcup_k L_k$. By taking and fixing a *starting point* s_0 on C, the *reading* of (C, s_0) is well-defined as an element of F_n. We denote this reading by $W(C, s_0)$. (Note that the staring point should be taken from $C - \bigcup_k L_k$.) The reading $W(C, s_0)$ depends only on the homotopy class of C fixing s_0.

If we take different starting point s_1 on C, the reading $W(C, s_1)$ is a cyclic conjugation of $W(C, s_0)$. Here two word W and W' are *cyclically conjugate* to each other, if W is a product V_1V_2 in a certain way, and W' is written as $W' = V_2V_1$.

Let C^{-1} denote the same curve C but with the opposite orientation. Obviously, $W(C^{-1}, s_0) = W(C, s_0)^{-1}$.

Theorem 41. *A word $V \in F_n$ is the reading of a simple closed curve on the punctured plane $\mathbb{R}^2 - P_n$ if and only if V or V^{-1} is cyclically conjugate to a word V' which satisfies*

$$R_{0,n+1}(V') = V'. \tag{72}$$

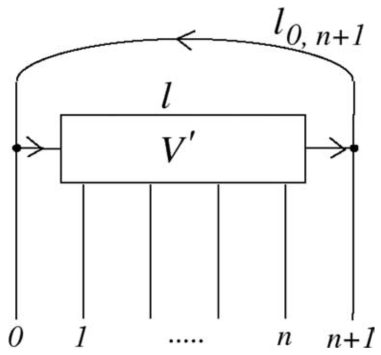

Fig. 12. Simple closed curve $C = l \cup l_{0,n+1}$.

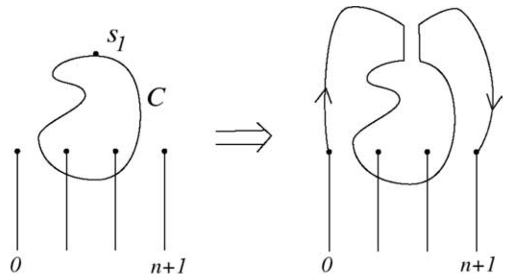

Fig. 13. Cutting open the loop C to a $(0, n+1)$-cord.

The free group F_n was generated by $\{x_1, \ldots, x_n\}$. If we define F_{n+2} to be the free group generated by $\{x_0, x_1, \ldots, x_n, x_{n+1}\}$, then F_n is naturally identified with a subgroup of F_{n+2}. In (72), the word $V' \in F_n$ is acted on by the rewriting function $R_{0,n+1} : F_{n+2} \to F_{n+2}$ under this natural identification.

Proof of Theorem 41. In the argument below, we may assume that the points p_0 and p_{n+1} are given by the coordinates

$$(-N, 0) \text{ and } (N, 0) \in \mathbb{R}^2$$

respectively with sufficiently large number $N > 0$.

Suppose that V or V^{-1} is cyclically conjugate to a word V' which satisfies (72). Then by Lemma 17, V' is the reading of a $(0, n+1)$-cord l on $(\mathbb{R}^2, P_n \cup \{p_0, p_{n+1}\})$. The reading of this $(0, n+1)$-cord (i.e., V') does not contain x_0 nor x_{n+1}. Thus the cord l does not intersect L_0 or L_{n+1} except at the end points. Then these points p_0 and p_{n+1} can be connected by a "large semi-circle"$l_{0,n+1}$ so that $C := l \cup l_{0,n+1}$ is a simple closed curve in $\mathbb{R}^2 - P_n$. (See Fig. 12.) The word V' is the reading of (C, p_0), and V is the reading of (C, s_0) or (C^{-1}, s_0) with some starting point s_0 on C. This proves the *if*-part.

Conversely, suppose V is the reading of an oriented simple closed curve with a starting point (C, s_0). Let s_1 be the highest point (or one of the highest points) of C with respect to the y-coordinate. Then we can "cut open" the loop C at this point s_1 to obtain a

$(0, n+1)$-cord l. (See Fig. 13.) The reading $W(l)$ coincides with the reading of (C, s_1) or of (C^{-1}, s_1), and is cyclically conjugate to V or V^{-1}. By Lemma 17, $W(l)$ satisfies

$$R_{0,n+1}(W(l)) = W(l).$$

This proves the *only if*-part, completing the proof of Theorem 41. □

This research was motivated by monodromy problems appearing in Lefschetz fibrations and surface braids. See [7–11].

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