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JOB SEQUENCING WITH FUZZY PROCESSING TIMES

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Abstract—In practice, processing times can be more accurately represented as intervals with the most probable completion time somewhere near the middle of the interval. A fuzzy number which is essentially a generalized interval can represent this processing time interval exactly and naturally. In this work, triangular and trapezoidal fuzzy numbers are used to represent those vague job processing times in job shop production systems. The job sequencing algorithms of Johnson and Ignall and Schrage are modified to accept fuzzy job processing times. Fuzzy makespans and fuzzy mean flow times are then calculated for greater decision-making information. Numerous examples are used to illustrate the approach.

INTRODUCTION

In the literature dealing with job sequencing problems, the processing time is assumed to be known exactly. However, in practical situations, this is seldom the case. In general, the processing time can only be estimated as within a certain interval. The particular characteristics of this processing time interval can be exactly represented by a fuzzy number. Thus, fuzzy set theory is ideally suited for solving job sequencing problems.

One of the problems in using fuzzy set theory to solve job sequencing problems is that fuzzy numbers only form partial order and thus comparison of fuzzy numbers to obtain a total or linear order can be a problem. However, several effective comparison methods have been developed recently [1–4]. By the use of these ranking methods, the job sequencing problems with fuzzy processing times can be solved effectively.

One of the advantages in using fuzzy numbers for job sequencing is that the processing time can be represented realistically and naturally there is no need to force the manager to give a precise single number. Furthermore, useful information is retained throughout the manipulation. This detailed information can be used by the decision maker for sensitivity analysis and other purposes.

Prade [5, 6], Dubois and Prade [7] and Dumitru and Luban [8] have applied fuzzy set theory to the n job, M machine sequencing problem. Dubois and Prade incorporate fuzzy processing times into the deterministic scheduling algorithm developed by Erschler *et al.* [9]. The number of machines per operation per job and the earliest start time and latest finish times for each job remain deterministic, however. Furthermore, only some threshold values were used to perform the comparisons. The use of threshold values for comparing fuzzy numbers does not exactly correspond to the use of the extension principle. Dumitru and Luban [8] extended their job sequencing problem which was formulated as a mathematical programming problem [10] into a mathematical programming problem with fuzzy membership functions and fuzzy constraints.

As can be seen, both of the above approaches are fairly limited in scope. In the present investigation, a very general approach based on the recent development in ranking fuzzy numbers is presented.

FUZZY NUMBERS

In this work, only triangular fuzzy numbers (TFN) and trapezoidal fuzzy numbers (TrFN) are used to represent the fuzzy processing times. Although general fuzzy numbers can be used to represent this processing time, the increase in computational effort by the use of a general fuzzy number is tremendous. Furthermore, since the fuzzy processing time is only an approximate estimate, it is difficult, if not impossible, to estimate a general fuzzy number representation of this processing time.

The TFN and TrFN will be represented by (a, b, c) and (a, b, c, d), respectively. The membership function is 1 or maximum at b for TFN and at b–c for TrFN. This membership function becomes zero at the two end points which are a and c for the TFN and a and d for the TrFN. These numbers are illustrated in Fig. 1, where $\mu(x)$ is the membership function and x is the processing time. It should be emphasized that a fuzzy number is essentially a generalized "interval" used frequently in engineering applications. In fact, by the use of α-cuts, fuzzy numbers can be represented by a sequence of intervals.

TFN can represent the estimated processing time naturally. For example, a manager may say that the processing time for Job A is generally b min. But, due to other factors which cannot be controlled, the processing time may be occasionally as slow as c min or as fast as a min. This result is naturally a TFN fuzzy number. TrFN allows more flexibility in this estimated processing time. Insteady of generally finishing Job A in b min, TrFN says that Job A is generally finished in b–c min.

The manipulation of fuzzy numbers and the definition of a general fuzzy number are discussed in detail in the literature [11]. For the present purpose, only the manipulations of TNF and TrFN are needed.

Only makespan (M) and mean flow time (MFT) are used as the performance criteria in this work. The fuzzy makespan can be expressed as

$$\tilde{\mathrm{M}} = \tilde{\max}\, \tilde{C}_i, \tag{1}$$

where the symbol "~" indicates fuzzy and $\tilde{C}_i$ is the fuzzy completion time of job i. The fuzzy mean flow time can be calculated as:

$$\mathrm{M\tilde{F}T} = \left[\overset{n}{\underset{i=1}{(+)}} (\tilde{q}_i (+) \tilde{p}_i) \right] \Big/ n \tag{2}$$

where $\tilde{q}_i$ is the fuzzy waiting time and $\tilde{p}_i$ is the fuzzy processing time for job i, n is the number of jobs to be processed, $(+)$ indicates fuzzy addition, and $(+)_{i=1}^{n}$ indicates fuzzy summation.

n JOBS, ONE WORKSTATION

The simplest job sequencing problem is to route all jobs through a single identical workstation. The optimal sequence is defined as the sequence which minimizes $\mathrm{M\tilde{F}T}$. It has been proven that the use of the shortest processing time (SPT) sequencing rule guarantees the optimal sequence.

When the processing times are fuzzy, all that must be done is a comparison of the fuzzy processing times. Here, the Lee–Li [3] method is used. The following examples illustrate the approach:

Job	Example 1 TFN processing time	Example 2 TrFN processing time
1	(3, 7, 9)	(3, 4, 8, 9)
2	(5, 6, 8)	(5, 6, 7, 8)
3	(7, 8, 9)	(7, 7.5, 8, 9)
4	(4, 5, 8)	(4, 5, 6, 8)

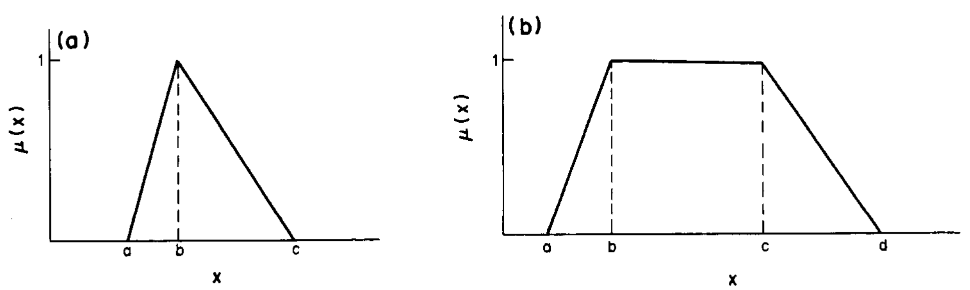

Fig. 1. (a) Triangular fuzzy number; (b) trapezoidal fuzzy number.

To use the Lee–Li method, the generalized mean value (GMVs) of the fuzzy numbers is first calculated. For a TFN with a uniform density the GMV is calculated as

$$m(\tilde{A}) = \tfrac{1}{3}(a + b + c), \tag{3}$$

where $\tilde{A}$ represents the fuzzy number $\tilde{A}$ and the GMV for a TrFN with a uniform density is

$$m(\tilde{A}) = \frac{\int_S x\mu_{\tilde{A}}(x)\,\mathrm{d}x}{\int_S \mu_{\tilde{A}}(x)\,\mathrm{d}x}, \tag{4}$$

where S represents the support of the fuzzy number and $\mu_{\tilde{A}}(x)$ represents the membership function value of x in the support of fuzzy number $\tilde{A}$.

The generalized mean values of the processing times of Examples 1 and 2 are:

Job	Example 1	Example 2
1	6.3	6.0
2	6.3	6.5
3	8.0	7.9
4	5.7	5.8

Note that jobs 1 and 2 with triangular processing times have the same GMVs. To "break the tie," the spread, $s(\tilde{A})$, is calculated for each fuzzy number and the one with the smaller spread is judged the smaller. The calculation of the spread of a triangular fuzzy number with a uniform density function is:

$$s(\tilde{A}) = \tfrac{1}{18}(a^2 + b^2 + c^2 - ab - ac - bc). \tag{5}$$

For jobs 1 and 2, Example 1, we have:

$$s(J1) = 1.56, \quad s(J2) = 0.39.$$

Therefore, the smallest to the largest fuzzy numbers are 4, 2, 1, 3, and the corresponding job sequence with triangular processing times is 4, 2, 1, 3.

If the spread of a trapezoidal fuzzy number with uniform density was of interest, it could be calculated by:

$$s(\tilde{A}) = \left[\frac{\int_S x^2\mu_{\tilde{A}}(x)\,\mathrm{d}x}{\int_S \mu_{\tilde{A}}(x)\,\mathrm{d}x} - [m(A)]^2\right]^{1/2}. \tag{6}$$

Note that the smallest to largest ranking of the jobs for the trapezoidal processing times yields a different sequence: 4, 1, 2, 3.

Performance criteria

The fuzzy makespans of the triangular and trapezoidal cases are simply the sum of all the fuzzy job processing times in the one workstation case and are

Example 1: $\tilde{\mathrm{M}} = (19, 26, 34)$.

Example 2: $\tilde{\mathrm{M}} = (19, 22.5, 29, 34)$.

The fuzzy mean flow times or $\mathrm{M\tilde{F}T}$s are calculated using equation (2). For the triangular and trapezoidal fuzzy processing time cases, the $\mathrm{M\tilde{F}T}$s are

Example 1: $\mathrm{M\tilde{F}T} = (11, 15, 20.75)$.

Example 2: $\mathrm{M\tilde{F}T} = (10.5, 12.9, 17.5, 21)$.

If the problem were solved in a deterministic fashion using the modes (or b values) of the triangular fuzzy processing times, the optimal job sequence would again be 4, 2, 1, 3 with a makespan of 26 and a MFT of 15. In the triangular processing time case, what is gained by using the fuzzy number representation is the range of values for $\tilde{\mathrm{M}}$ and $\mathrm{M\tilde{F}T}$ which are truly fuzzy numbers if the processing times are fuzzy. The decision-maker now knows the spread of the $\tilde{\mathrm{M}}$ and MFT, whereas it was assumed away and lost in the deterministic simplification. Furthermore, in the trapezoidal processing time case, there is no deterministic approximation—the fuzzy method must be used.

n JOBS, TWO WORKSTATIONS

In this situation, the optimal sequence is defined as the sequence which minimizes the makespan of the n jobs. Johnson's algorithm [12] has been used with great success to find the optimal sequence when the processing times are deterministic. A fuzzified version of this algorithm is used to solve two examples. The first example is represented by triangular fuzzy numbers:

Job	W1	W2
1	(3, 4, 6)	(8, 11, 12)
2	(3, 7, 10)	(6, 7, 8)
3	(1, 3, 5)	(6, 10, 14)
4	(8, 12, 15)	(6, 8, 9)
5	(10, 11, 15)	(5, 10, 12)
6	(7, 9, 13)	(10, 13, 15)

The GMVs of Lee–Li's method for each fuzzy processing time, assuming uniform distributions, are calculated as:

Job	W1	W2
1	4.3	10.3
2	6.7	7.0
3	3.0	10.0
4	11.7	7.7
5	12.0	9.0
6	9.7	12.7

Using these GMVs, with Johnson's algorithm yields the job sequence of 3, 1, 2, 6, 5, 4.

Performance criteria

The M$\tilde{\text{F}}$T is a bit more complicated to calculate in the two workstation cases because the queue or waiting time between workstations 1 and 2 must be calculated for each job using fuzzy subtraction.

$$\tilde{q}_{i2} = \tilde{C}_{k2}(-)\tilde{C}_{i1}, \tag{7}$$

where $(-)$ represents fuzzy subtraction. $\tilde{C}_{k2}$ is the fuzzy completion time of job k at workstation 2, $\tilde{C}_{i1}$ is the fuzzy conpletion time of job i at workstation 1, job k proceeds job i in the sequence, and all jobs pass through workstation 1 first, Note that

$$\tilde{C}_{i1} = \tilde{q}_{il}(+)\tilde{p}_{i1}. \tag{8}$$

The fuzzy queue time before workstations 1 or q_{i1}, is simple the fuzzy finish time of the previous job or

$$\tilde{q}_{i1} = \tilde{C}_{k1}. \tag{9}$$

The fuzzy completion time at workstation 2, $\tilde{C}_{i2}$, for each job is the sum of both waiting times and both processing times

$$\tilde{C}_{i2} = \overset{2}{\underset{j=1}{(+)}}\,[\tilde{q}_{ij}(+)\tilde{p}_{ij}] \tag{10}$$

the M$\tilde{\text{F}}$T is then

$$\text{M}\tilde{\text{F}}\text{T} = \left[\overset{n}{\underset{i=1}{(+)}}\,\tilde{C}_{i2}\right]\Big/ n. \tag{11}$$

Table 1 lists all the waiting, processing and completion times for each of the six jobs in the example. Since a negative waiting is unrealistic, the negative portion of this fuzzy number is deleted. However, this leaves a non-triangular fuzzy number. This will cause subsequent waiting time fuzzy numbers to degenerate into many-pieced membership functions, and make subsequent calculations

Table 1. Fuzzy parameters of the two workstation Example 1

Job	$\tilde{q}_{i1}$	$\tilde{p}_{i1}$	$\tilde{C}_{i1}$	$\tilde{q}_{i2}$	$\tilde{p}_{i2}$	$\tilde{C}_{i2}$
3	0	(1, 3, 5)	(1, 3, 5)	0	(6, 10, 14)	(7, 13, 19)
1	(1, 3, 5)	(3, 4, 6)	(4, 7, 11)	(0, 6, 15)	(8, 11, 12)	(12, 24, 38)
2	(1, 3, 5)	(3, 7, 10)	(7, 14, 21)	(0, 10, 31)	(6, 7, 8)	(13, 31, 60)
6	(7, 14, 21)	(7, 9, 13)	(14, 23, 34)	(0, 8, 46)	(10, 13, 15)	(24, 44, 95)
5	(14, 23, 34)	(10, 11, 15)	(24, 34, 49)	(0, 10, 71)	(5, 10, 12)	(29, 54, 132)
4	(24, 34, 49)	(8, 12, 15)	(32, 46, 64)	(0, 8, 100)	(6, 8, 9)	(38, 62, 173)

unwieldly. Therefore, the non-triangular fuzzy waiting time is modified to a triangular fuzzy waiting time as shown in Table 1.

The MF̃T can now be determined as:

$$\mathrm{M\tilde{F}T} = \left[\overset{n}{\underset{i=1}{(\tilde{+})}} \tilde{C}_{i2} \right] \Big/ n = (20.5, 38, 86.2).$$

The fuzzy makespan is the mãx of all the fuzzy completion times. In this example it is simply the fuzzy completion time of job 4 or (38, 62, 173).

The deterministic solution of this example problem using the means (or b values) of the fuzzy processing times yields the sequence: 3, 1, 2, 6, 5, 4 with an M of 62 and an MFT of 38. Note that this M and MFT exactly corresponds to the modes of $\tilde{\mathrm{M}}$ and MF̃T. What the fuzzy formulation contributes is how the spreads of the processing times affect the spread of the solution outputs. Note how wide the $\tilde{\mathrm{M}}$ spread is. The manager can now understand how long the makespan can potentially be, and plan accordingly.

Trapezoidal example

The second six job, two workstation example has trapezoidal fuzzy processing times, as listed below:

Job	W1	W2
1	(3, 4, 5, 6)	(8, 9, 10.12)
2	(3, 4, 7, 10)	(6, 7, 7.5, 8)
3	(1, 2, 3, 5)	(6, 8, 10, 14)
4	(8, 12, 13, 15)	(6, 7, 8, 9)
5	(10, 11, 14, 15)	(5, 9, 10, 12)
6	(7, 9, 10, 13)	(10, 12, 14, 15)

The corresponding GMVs, assuming uniform distributions, are:

Job	W1	W2
1	4.5	9.80
2	6.0	7.1
3	2.8	9.6
4	11.9	7.5
5	12.5	9.0
6	9.81	12.7

Using these generalized mean values in Johnson's algorithm yields the job sequence of 3, 1, 2, 6, 5, 4. Note that this is the same sequence found as in the triangular case, but this is coincidental. Using equations (7)–(9), the fuzzy waiting, processing and completion times for each of the jobs in the sequence 3, 1, 2, 6, 5, 4 are calculated and listed in Table 2. Negative portions of the fuzzy numbers are deleted, as before. The fuzzy makespan of this sequence is:

$$\tilde{\mathrm{M}} = \underset{i}{\tilde{\max}} (\tilde{C}_{i2}) = (38, 49, 94.5, 173).$$

Table 2. Fuzzy parameters of the two workstation Example 2

Job	$\tilde{q}_{i1}$	$\tilde{p}_{i1}$	$\tilde{C}_{i1}$	$\tilde{q}_{i2}$	$\tilde{p}_{i2}$	$\tilde{C}_{i2}$
3	0	(1, 2, 3, 5)	(1, 2, 3, 5)	0	(6, 8, 10, 14)	(7, 10, 13, 19)
1	(1, 2, 3, 5)	(3, 4, 5, 6)	(4, 6, 8, 11)	(0, 2, 7, 15)	(8, 9, 10, 12)	(12, 17, 25, 38)
2	(4, 6, 8, 11)	(3, 4, 7, 10)	(7, 10, 15, 21)	(0, 2, 15, 31)	(6, 7, 7.5, 8)	(13, 19, 37.5, 60)
6	(7, 10, 15, 21)	(7, 9, 10, 13)	(14, 19, 25, 34)	(0, 0, 18.5, 46)	(10, 12, 14, 15)	(24, 31, 57.5, 95)
5	(14, 19, 25, 34)	(10, 11, 14, 15)	(24, 30, 34, 49)	(0, 0, 27.5, 71)	(5, 9, 10, 12)	(29, 39, 76.5, 132)
4	(24, 30, 39, 49)	(8, 12, 13, 15)	(32, 42, 52, 62)	(0, 0, 34.5, 100)	(6, 7, 8, 9)	(38, 49, 94.5, 173)

The $\tilde{\text{MFT}}$ can now be determined by averaging the workstation 2 completion times:

$$\tilde{\text{MFT}} = \left[\overset{n}{\underset{i=1}{(+)}} \tilde{C}_{i2} \right] \Big/ 6 = (20.5, 26.8, 50.7, 86.2).$$

In the trapezoidal case, unlike the triangular case, a deterministic approximation cannot be used, so the analyst must use the fuzzified method, and the fuzzy makespan and $\tilde{\text{MFT}}$ can be calculated for further decision making.

n JOBS, THREE WORKSTATIONS

Ignall and Schrage's branch and bound algorithm [13] for a general three workstation flow shop problem will be used. The problem is represented using a tree structure where each node is a partial sequence. To determine the best partial sequence node from which to branch, the lower bounds (LB) of the makespans of all the partial sequence nodes re calculated, and the node with the lowest lower bound of the makespan is then selected. The process is continued until the sequence with the least lower bound is found. To fuzzify this algorithm by using fuzzy processing times, all of the lower bounds must be expressed as fuzzy numbers, and any comparison must use a fuzzy number comparison method. The Lee–Li method will be used.

To calculate the fuzzy lower bound on the fuzzy makespan of all schedules beginning with the sequence S_r, use

$$\tilde{\text{LB}}(S_r) = \max \begin{cases} \tilde{\text{TW}}1(S_r)(+)\underset{s_r}{(+)}\tilde{p}_{i1}(+)\underset{s_r}{\min}\,(\tilde{p}_{i2}(+)\tilde{p}_{i3}), \\ \tilde{\text{TW}}2(S_r)(+)\underset{s_r}{(+)}\tilde{p}_{i2}(+)\underset{s_r}{\min}\,(\tilde{p}_{i3}), \\ \tilde{\text{TW}}3(S_r)(+)\underset{s_r}{(+)}\tilde{p}_{i3}, \end{cases} \tag{12}$$

where $\tilde{p}_{ij}$ is the fuzzy processing time of job i at workstation j; s_r is the set of $(n - r)$ jobs not yet assigned; and $\tilde{\text{TW}}1(S_r)$, $\tilde{\text{TW}}2(S_r)$ and $\tilde{\text{TW}}3(S_r)$ are the fuzzy times at which workstations 1, 2 and 3, respectively, complete processing of the last job in the sequence S_r. It is assumed all the jobs pass through workstation 1 first and workstation 3 last. After finding the fuzzy lower bounds for the nodes, branch from the node with the lowest fuzzy lower bound. Create a new node for every job not yet scheduled. The fuzzy lower bounds of these new nodes are calculated using equation (12). The process continues until all the jobs have been scheduled into the sequence.

An illustrative example with triangular processing times is the following four job, three workstation scheduling problem:

Job	W1	W2	W3
1	(12, 14, 15)	(5, 6, 8)	(10, 15, 16)
2	(3, 8, 9)	(10, 11, 12)	(1, 4, 5)
3	(9, 10, 15)	(12, 13, 15)	(16, 17, 20)
4	(14, 16, 19)	(12, 15, 18)	(4, 5, 6)

The first level of fuzzy lower bounds using equation (12) are calculated as

$$\tilde{\text{LB}}(1) = \max \begin{cases} (12, 14, 15)(+)(26, 34, 43)(+)\min[(11, 15, 17),\ (28, 30, 35),\ (16, 20, 24)] \\ (17, 20, 23)(+)(34, 39, 45)(+)\min[(1, 4, 5),\ (16, 17, 20),\ (4, 5, 6)] \\ (27, 35, 39)(+)(21, 26, 31). \end{cases}$$

Using the Lee–Li method for finding the discrete minimum, the generalized mean values are calculated, and the fuzzy number with the smallest generalized mean value is deemed smallest. Continuing,

$$\tilde{\text{LB}}(1) = \max[(49, 63, 75),\ (52, 63, 73),\ (48, 61, 70)] = (52, 63, 73),$$

because

$$m[(49, 63, 75)] = 62.3, \; m[(52, 63, 73)] = 62.7, \; m[(48, 61, 70)] = 59.7,$$

similarly,

$$\mathrm{L}\tilde{\mathrm{B}}(2) = \max[(53, 68, 82), \; (46, 58, 68), \; (44, 60, 68)] = (53, 68, 82),$$

$$\mathrm{L}\tilde{\mathrm{B}}(3) = \max[(49, 63, 75), \; (49, 59, 73), \; (52, 64, 77)] = (52, 64, 77),$$

$$\mathrm{L}\tilde{\mathrm{B}}(4) = \max[(49, 63, 75), \; (54, 65, 77), \; (57, 72, 84)] = (57, 72, 84).$$

Now, to determine which node to branch from, the minimum of $\mathrm{L}\tilde{\mathrm{B}}(1)$, $\mathrm{L}\tilde{\mathrm{B}}(2)$, $\mathrm{L}\tilde{\mathrm{B}}(3)$ and $\mathrm{L}\tilde{\mathrm{B}}(4)$ must be chosen. Calculating the generalized mean values of each of these fuzzy numbers yields:

$$m[\mathrm{L}\tilde{\mathrm{B}}(1)] = \tfrac{1}{3}(52 + 63 + 73) = 62.7, m[\mathrm{L}\tilde{\mathrm{B}}(2)] = \tfrac{1}{3}(53 + 68 + 82) = 67.7,$$

$$m[\mathrm{L}\tilde{\mathrm{B}}(3)] = \tfrac{1}{3}(52 + 64 + 77) = 64.3, \; m[\mathrm{L}\tilde{\mathrm{B}}(4)] = \tfrac{1}{3}(57 + 72 + 84) = 71.0$$

Therefore sequences starting with job 1, since it has the lowest generalized mean value, will be investigated further. New $\mathrm{T}\tilde{\mathrm{W}}1(S_r)$, $\mathrm{T}\tilde{\mathrm{W}}2(S_r)$ and $\mathrm{T}\tilde{\mathrm{W}}3(S_r)$ must be calculated:

	S_r		
	(1, 2)	(1, 3)	(1, 4)
$\mathrm{T}\tilde{\mathrm{W}}1(S_r)$	(15, 22, 24)	(21, 24, 30)	(26, 30, 34)
$\mathrm{T}\tilde{\mathrm{W}}2(S_r)$	(25, 33, 36)	(33, 37, 45)	(38, 45, 52)
$\mathrm{T}\tilde{\mathrm{W}}3(S_r)$	(28, 39, 44)	(49, 54, 65)	(42, 50, 58)

The fuzzy lower bounds for these nodes are

$$\mathrm{L}\tilde{\mathrm{B}}(1, 2) = \max[(54, 68, 82), (53, 66, 75), (58, 61, 80)] = (54, 68, 82),$$

$$\mathrm{L}\tilde{\mathrm{B}}(1, 3) = \max[(49, 63, 75), \; (56, 67, 80), \; (54, 63, 76)] = (56, 67, 80),$$

$$\mathrm{L}\tilde{\mathrm{B}}(1, 4) = \max[(49, 63, 75), \; (61, 73, 84), \; (59, 71, 83)] = (61, 73, 84).$$

When comparing all the nodes at this stage, note that node 3 now has the lowest lower bound at (52, 64, 77). Therefore, all sequences starting with 3 must be investigated, and so on, through the algorithm.

The entire tree network is illustrated in Fig. 2. The GMVs supporting this figure are listed in Table 3. The sequence which yields the lowest lower bound for the entire makespan is 3, 4, 1, 2 with a fuzzy lower bound for the fuzzy makespan of (51, 66, 81). When the problem is solved deterministically with the means of the fuzzy numbers as the deterministic inputs, the results are identical: optimal sequence of 3, 4, 1, 2 with a deterministic makespan of 66 units. However, when

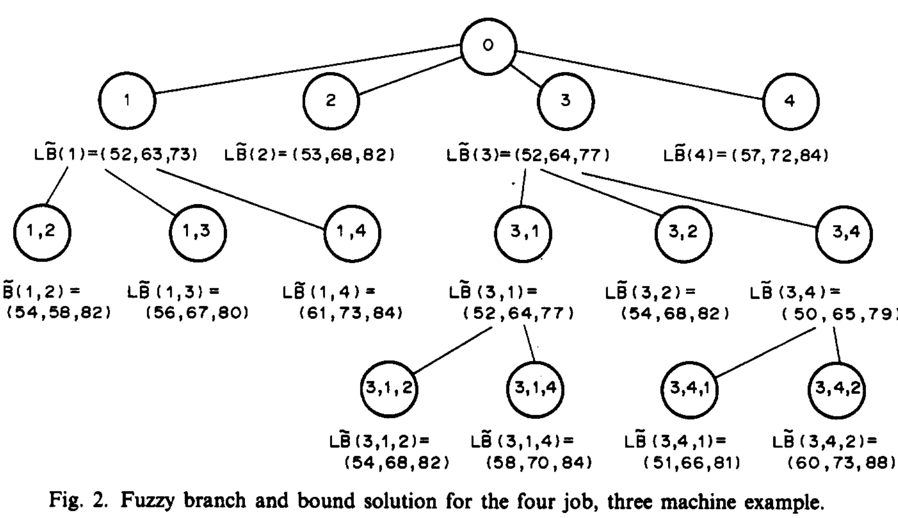

Fig. 2. Fuzzy branch and bound solution for the four job, three machine example.

Table 3. Corresponding GMVs illustrated in Fig. 2

	GMV
First level	
$L\tilde{B}(1)$	62.7†
$L\tilde{B}(2)$	67.7
$L\tilde{B}(3)$	64.3
$L\tilde{B}(4)$	71.0
Second level	
$L\tilde{B}(1,2)$	68.0
$L\tilde{B}(1,3)$	67.7
$L\tilde{B}(1,4)$	72.7
$L\tilde{B}(3,1)$	64.3†
$L\tilde{B}(3,2)$	68.0
$L\tilde{B}(3,4)$	64.7‡
Third level	
$L\tilde{B}(3,1,2)$	68.0
$L\tilde{B}(3,1,4)$	70.7
$L\tilde{B}(3,4,1)$	66.0
$L\tilde{B}(3,4,2)$	73.6
Solution = (3, 4, 1, 2)	

†Initially branched from this node.
‡Later branched from this node.

the times are fuzzy, the fuzzy makespan must be calculated as the maximum of the job completion times $\tilde{C}_{i3}$ or

$$\tilde{\mathrm{M}} = \underset{i}{\tilde{\max}}\,(\tilde{C}_{i3}), \tag{13}$$

where each $\tilde{C}_{i3}$, for the three workstation case, is calculated as

$$\tilde{C}_{i3} = \overset{3}{\underset{j=1}{(+)}}\,(\tilde{q}_{ij}(+)\tilde{p}_{ij}). \tag{14}$$

To calculate the fuzzy waiting time for workstation 3, we must use

$$\tilde{q}_{i3} = \tilde{C}_{k3}(-)\tilde{C}_{i2}, \tag{15}$$

where $\tilde{C}_{k3}$ is the fuzzy completion time of job k at workstation 3; $\tilde{C}_{i2}$ is the fuzzy completion time of job i at workstation 2; job k precedes job i and any negative portion of a fuzzy number is deleted. Table 4 lists all the fuzzy waiting, processing and completion times for each job by workstation.

Performance criteria

As can be seen in Fig. 3, the fuzzy makespan assuming the sequence of 3, 4, 1, 2, is

$$\mu_{\tilde{\mathrm{M}}}(x) = \begin{cases} \dfrac{x}{12} - \dfrac{50}{12}, & 50 \leqslant x \leqslant 52.4, \\[2ex] \dfrac{x}{17} - \dfrac{49}{17}, & 52.4 < x \leqslant 66, \\[2ex] \dfrac{x}{141} + \dfrac{207}{141}, & 66 < x \leqslant 207. \end{cases}$$

Note, this is non-triangular because $\tilde{\max}\,\tilde{C}_{3i}$ forms a non-triangular membership function.

If the $\mathrm{M\tilde{F}T}$ is of interest, it can be calculated as

$$\mathrm{M\tilde{F}T} = (43.8, 53.5, 118.5)$$

Table 4. Fuzzy parameters of the three workstation Example 1

Job	$\tilde{q}_{i1}$	$\tilde{p}_{i1}$	$\tilde{C}_{i1}$	$\tilde{q}_{i2}$	$\tilde{p}_{i2}$	$\tilde{C}_{i2}$	$\tilde{q}_{i3}$	$\tilde{p}_{i3}$	$\tilde{C}_{i3}$
3	0	(9, 10, 15)	(9, 10, 15)	0	(12, 13, 15)	(21, 23, 30)	0	(16, 17, 20)	(37, 40, 50)
4	(9, 10, 15)	(14, 16, 19)	(23, 26, 24)	(0, 0, 7)	(12, 15, 18)	(35, 41, 59)	(0, 0, 15)	(4, 5, 6)	(39, 46, 80)
1	(23, 26, 34)	(12, 14, 15)	(35, 40, 49)	(0, 1, 24)	(5, 6, 8)	(40, 47, 81)	(0, 0, 40)	(10, 15, 16)	(50, 62, 137)
2	(35, 40, 49)	(3, 8, 9)	(38, 48, 58)	(0, 0, 43)	(10, 11, 12)	(48, 59, 113)	(0, 3, 89)	(1, 4, 5)	(49, 66, 207)

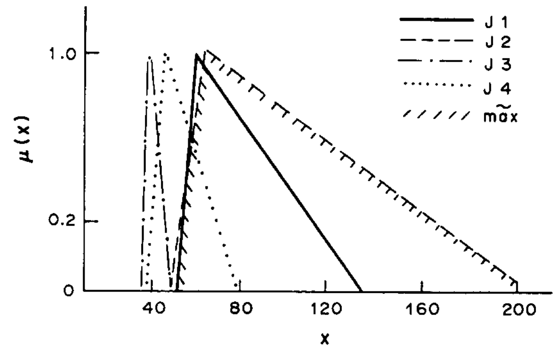

Fig. 3. Determination of the fuzzy makespan.

What the triangular decision maker gains by using the fuzzified procedure is an understanding of how fuzzy the makespan is, along with how fuzzy the job MFTs are, and can plan accordingly.

Trapezoidal example

An illustrative example with trapezoidal processing times is a variant of the previous example. The fuzzy times are listed below:

Job	W1	W2	W3
1	(12, 13, 14, 15)	(5, 6, 7, 8)	(10, 11, 15, 16)
2	(3, 4, 8, 9)	(10, 11, 11.5, 12)	(1, 2, 3, 5)
3	(9, 10, 11, 15)	(12, 13, 14, 15)	(16, 17, 18, 20)
4	(14, 16, 18, 19)	(12, 15, 16, 18)	(4, 4.5, 5, 6)

Figure 4 illustrates the entire tree network with node lower bounds for this problem. The generalized mean values for the nodes in Fig. 4 are listed in Table 5. The fuzzy lower bound for the fuzzy makespan is (51, 60, 70, 81) for the sequence 3, 4, 1, 2. The fuzzy makespan assuming the sequence of 3, 4, 1, 2 is

$$\mu_{\tilde{M}}(x) = \begin{cases} \dfrac{x}{12} - \dfrac{50}{12}, & 50 \leqslant x \leqslant 52.4, \\ \dfrac{x}{17} - \dfrac{49}{17}, & 52.4 < x \leqslant 66, \\ -\dfrac{x}{141} + \dfrac{207}{141}, & 66 < x \leqslant 207. \end{cases}$$

The value for $\mathrm{M\tilde{F}T}$ is

$$\mathrm{M\tilde{F}T} = (43.8, 53.5, 118.5).$$

Table 6 summarizes the fuzzy waiting, processing and completion times of the jobs in sequence 3, 4, 1, 2. The fuzzy makespan is then

$$\mu_{\tilde{M}}(x) = \underset{i}{\tilde{\max}}\, \tilde{C}_{i3} = (49, 56, 102.5, 207).$$

The $\mathrm{M\tilde{F}T}$ is found by averaging the workstation 3 completion times:

$$\mathrm{M\tilde{F}T} = \left[\overset{4}{\underset{i=1}{(+)}} \tilde{C}_{i3} \right] \Big/ 4 = (43.8, 49.4, 68.9, 118.5).$$

CONCLUSIONS

The job sequencing algorithms of Johnson and Ignall and Schrage were modified to accept two types of fuzzy job processing times: triangular and trapezoidal. The resultant job sequences were non-fuzzy, but the performance criteria of makespan and mean flow time were fuzzy using these

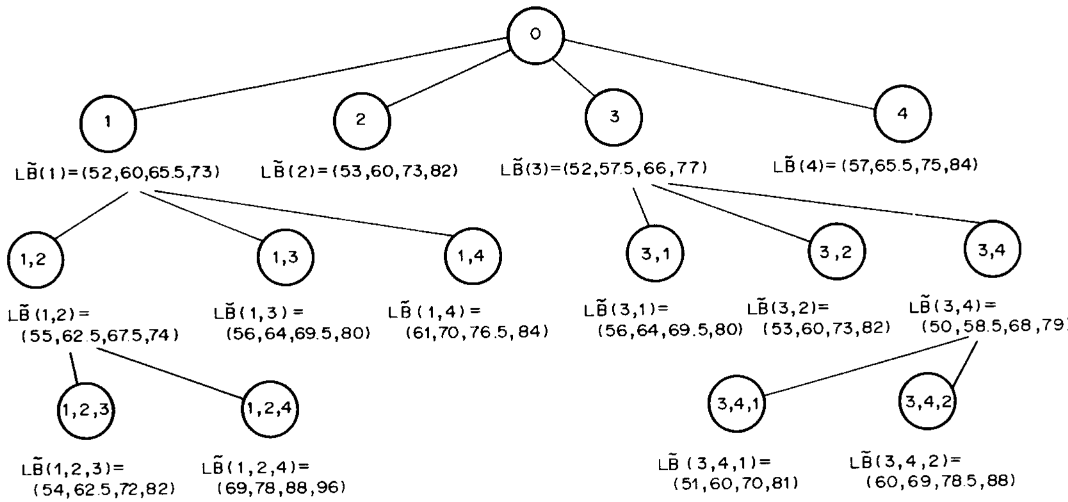

Fig. 4. Fuzzy branch and bound solution for the four job, three machine, Example 2.

Table 5. Corresponding GMVs illustrated in Fig. 4

	GMV
First level	
L$\tilde{B}$(1)	62.6†
L$\tilde{B}$(2)	67.1
L$\tilde{B}$(3)	63.4
L$\tilde{B}$(4)	70.4
L$\tilde{B}$(1, 2)	64.7‡
L$\tilde{B}$(1, 3)	67.5
L$\tilde{B}$(1, 4)	72.8
L$\tilde{B}$(3, 1)	67.5
L$\tilde{B}$(3, 2)	67.1
L$\tilde{B}$(3, 4)	64.0†
Third level	
L$\tilde{B}$(3, 4, 1)	65.5
L$\tilde{B}$(3, 4, 2)	73.9
L$\tilde{B}$(1, 2, 3)	82.7
L$\tilde{B}$(1, 2, 4)	82.7
Solution = (3, 4, 1, 2)	

†Initially branched from this node.
‡Later branched from this node.

Table 6. Fuzzy parameters of the three workstation Example 2

Job	$\tilde{q}_{i1}$	$\tilde{p}_{i1}$	$\tilde{C}_{i1}$	$\tilde{q}_{i2}$	$\tilde{p}_{i2}$	$\tilde{C}_{i2}$	$\tilde{q}_{i3}$	$\tilde{p}_{i3}$	$\tilde{C}_{i3}$
3	0	(9, 10, 11, 15)	(9, 10, 11, 15)	0	(12, 13, 14, 15)	(21523, 25, 30)	0	(16, 17, 18, 20)	(37, 40, 43, 50)
4	(9, 10, 11, 15)	(14, 16, 18, 19)	(23, 26, 29, 34)	(0, 0, 0, 7)	(12, 15, 16, 18)	(35, 41, 45, 59)	(0, 0, 2, 15)	(4, 4.5, 5, 6)	(39, 45.5, 52, 80)
1	(23, 26, 29, 34)	(12, 13, 14, 15)	(35, 39, 43, 49)	(0, 0, 6, 24)	(5, 6, 7, 8)	(40, 45, 56, 81)	(0, 0, 7, 40)	(10, 11, 15, 16)	(50, 56, 78, 137)
2	(35, 39, 43, 49)	(3, 4, 8, 9)	(38, 43, 51, 58)	(0, 0, 13, 43)	(10, 11, 11.5, 12)	(48, 54, 75.5, 113)	(0, 0, 24, 89)	(1, 2, 3, 5)	(49, 56, 102.5, 207)

modified algorithms. The fuzzy performance criteria could then be interpreted using possibility theory and/or fuzzy integrals.

In some cases, if the fuzzy processing times can be represented using triangular fuzzy numbers, the optimal sequence and means of the fuzzy makespan and fuzzy mean flow time can be determined by using a deterministic approximation. This approximation is made by using the modes (b values) of the triangular processing, times as deterministic with the non-modified (or original) sequencing algorithm. The problem with using this approximation is that the fuzzy makespan and mean flow time must be calculated anyway if further sensitivity analysis using possibility theory and/or fuzzy integrals is desired. If trapezoidal fuzzy numbers are used, then the fuzzified method must be used.

In summary, by keeping the fuzziness throughout the analysis procedure, the decision maker keeps intact information useful for subsequent sensitivity analysis of the performance criteria. If the input job processing times are truly fuzzy, then they should be modeled as fuzzy to obtain fuzzy results. Approximations can be used, but only in special cases, and without the comprehensive results obtained from the fuzzy procedure.

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