ELSEVIER

Discrete Mathematics 231 (2001) 453–457

DISCRETE MATHEMATICS

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The ratio of the longest cycle and longest path in semicomplete multipartite digraphs

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Received 14 July 1999; revised 8 March 2000; accepted 7 August 2000

Abstract

A digraph obtained by replacing each edge of a complete n-partite graph by an arc or a pair of mutually opposite arcs is called a semicomplete n-partite digraph. We call $\alpha(D) = \max_{1 \leqslant i \leqslant n}\{|V_i|\}$ the independence number of the semicomplete n-partite digraph D, where $V_1, V_2, \ldots, V_n$ are the partite sets of D. Let p and c, respectively, denote the number of vertices in a longest directed path and the number of vertices in a longest directed cycle of a digraph D. Recently, Gutin and Yeo proved that $c \geqslant (p+1)/2$ for every strongly connected semicomplete n-partite digraph D. In this paper we present for the special class of semicomplete n-partite digraphs D with connectivity $\kappa(D) = \alpha(D) - 1 \geqslant 1$ the better bound

$$c \geqslant \frac{\kappa(D)}{\kappa(D)+1}(p+1).$$

In addition, we present examples which show that this bound is best possible.

Keywords: Semicomplete multipartite digraphs; Paths; Cycles; Hamiltonian cycles

1. Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set of a digraph D is denoted by $V(D)$. An orientation of a complete n-partite graph is an *n-partite* or *multipartite tournament*. A digraph obtained by replacing each edge of a complete n-partite graph by an arc or a pair of mutually opposite arcs is called a *semicomplete n-partite digraph* or *semicomplete multipartite digraph*. If xy is an arc of a digraph D, then we say that x *dominates* y, and if X and Y are two disjoint subsets of $V(D)$ such that every vertex of X dominates every vertex of Y, then we say that

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PII: S0012-365X(00)00338-1

X dominates Y, denoted by $X \to Y$. The *out-neighborhood* $N_D^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N_D^-(x)$ is the set of vertices dominating x. For a vertex set X of D, we define $N_D^+(X) = \bigcup_{x \in X} N_D^+(X)$, and $D[X]$ as the subdigraph induced by X. The numbers $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ are called *outdegree* and *indegree* of x, respectively. Let x be a vertex and S be a vertex set of D. By $d_D^+(x,S)$ and $d_D^-(x,S)$ we denote the number of arcs from x to S and from S to x, respectively. All cycles and paths mentioned here are directed cycles and directed paths. A cycle of length m is called an *m-cycle*. A cycle (path) of a digraph D is *Hamiltonian* if it includes all the vertices of D. If a digraph contains a Hamiltonian cycle, then we speak of a *Hamiltonian* digraph. A digraph D is called *strongly connected* or *strong* if, for each pair of vertices u and v, there is a path in D from u to v. A digraph D with at least $k+1$ vertices is *k-connected* if for any set A of at most $k-1$ vertices, the subdigraph $D-A$ is strong. The *connectivity* of D, denoted by $k = \kappa(D)$, is then defined to be the largest value of k for which D is k-connected. A *1-factor* of a digraph D is a spanning subdigraph consisting of disjoint cycles. If $V_1, V_2, \ldots, V_n$ are the partite sets of a semicomplete n-partite digraph, then $\alpha(D) = \max_{1 \leqslant i \leqslant n}\{|V_i|\}$ is called the *independence number* of D.

In the following, we denote by p the number of vertices in a longest directed path and by c the number of vertices in a longest directed cycle of a digraph D. For every strongly connected semicomplete n-partite digraph D, Gutin and Yeo [3] have proved recently that $c \geqslant (p+1)/2$, which confirms a conjecture of the second author [7]. In this paper we present for the special class of semicomplete n-partite digraphs D with connectivity $\kappa(D) = \alpha(D) - 1 \geqslant 1$ the better bound

$$c \geqslant \frac{\kappa(D)}{\kappa(D)+1}(p+1).$$

Examples will show that this inequality is best possible. Both results show that the next conjecture of the second author [8] is valid for $\kappa(D) = 1$ as well as for $\kappa(D) = \alpha(D) - 1$.

Conjecture (Volkmann [8]). Let D be a strongly connected semicomplete multipartite digraph with $\kappa(D) < \alpha(D)$. If p is the number of vertices in a longest path and c the number of vertices in a longest cycle in D, then $\kappa(D)p \leqslant (\kappa(D)+1)c - \kappa(D)$.

It may be noted that Yeo [9] has proved in 1997 that a semicomplete multipartite digraph D with $\kappa(D) \geqslant \alpha(D)$ is even Hamiltonian and thus, we have $p = c = |V(D)|$ in this case.

2. Preliminary results

Theorem 2.1 (Bondy [1]). *Each strongly connected semicomplete n-partite digraph D contains an m-cycle for each* $m \in \{3, 4, \ldots, n\}$.

Theorem 2.2 (Ayel (cf. [4])). *If C is a longest cycle in a strongly connected semicomplete multipartite digraph D, then $D - V(C)$ contains no cycle.*

Originally, Bondy and Ayel have presented their results under the stronger condition that D is a multipartite tournament. For proofs concerning the more general case when D is a semicomplete multipartite digraph, we refer the reader to Volkmann [7] (for Theorem 2.1) and Gutin [2] (for Theorem 2.2).

Theorem 2.3 (Ore [5]). *A digraph D contains a 1-factor if and only if $|S| \leqslant |N_D^+(S)|$ for every subset $S \subseteq V(D)$.*

Theorem 2.4 (Yeo [9]). *Let D be a $\lfloor q/2 \rfloor$-connected semicomplete multipartite digraph such that $\alpha(D) \leqslant q$. If D has a 1-factor, then D is Hamiltonian.*

Theorem 2.5 (Yeo [9]). *Let D be a k-connected semicomplete multipartite digraph, and let X be an arbitrary set of vertices in D with at most k vertices from each partite set. Then there exists a cycle C in D with $X \subseteq V(C)$.*

3. Main results

In 1997, Yeo [9] has proved that every semicomplete multipartite digraph D with the property $\kappa(D) \geqslant \alpha(D)$ is Hamiltonian. For a special class of semicomplete multipartite digraphs D, we will show firstly that the weaker condition $\kappa(D) \geqslant \alpha(D) - 1$ is sufficient for D to be Hamiltonian.

Theorem 3.1. *Let D be a semicomplete n-partite digraph with the partite sets $V_1, V_2, \ldots, V_n$ such that $|V_1| = |V_2| = k+1$ and $|V_3| + \cdots + |V_n| \leqslant k-1$. If $\kappa(D) = k \geqslant 1$, then D is Hamiltonian.*

Proof. Let S be an arbitrary vertex set in D. In the first step of our proof we show that $|S| \leqslant |N_D^+(S)|$.

Since D is k-connected, we have $d_D^+(x) \geqslant k$ for every $x \in V(D)$. Therefore, it follows at once that $|S| \leqslant |N_D^+(S)|$ for $|S| \leqslant k$.

In the case when $|S| \geqslant 2k+2$, we have $|S| \geqslant |V(D)| - (k-1)$ and we deduce from the hypothesis that $D[S]$ is strong. Hence, we observe that $S = N_{D[S]}^+(S) \subseteq N_D^+(S)$, and thus we obtain the desired inequality $|S| \leqslant |N_D^+(S)|$.

If $k+1 \leqslant |S| \leqslant 2k+1$, then we distinguish two cases. First, we assume that S contains at most k vertices from V_1 and at most k vertices from V_2. Then, by Theorem 2.5, there exists a cycle C in D with $S \subseteq V(C)$. This immediately implies $|S| \leqslant |N_C^+(S)| \leqslant |N_D^+(S)|$.

In the remaining case we assume, without loss of generality, that $S \cap V_1 = V_1$. Since $|S| \leqslant 2k+1$, it is clear that $S \cap V_2 \neq V_2$. In addition, since $|V_3| + \cdots + |V_n| \leqslant k-1$, we see that $d_D^-(x, V_1) \geqslant 1$ as well as $d_D^+(x, V_1) \geqslant 1$ for all vertices $x \in V_2$. Now we investigate two subcases.

If $S \cap V_2 \neq \emptyset$, say $v_2 \in S \cap V_2$, then we choose a vertex $v_1 \in V_1 \subseteq S$ such that $v_2 \rightarrow v_1$, and we define $S' = S - \{v_1\}$. In view of Theorem 2.5, there exists a cycle C' in D with $S' \subseteq V(C')$. In the case when $v_1 \in V(C')$, it follows that $S \subseteq V(C')$, and we are done as above. If $v_1 \notin V(C')$, then we have at least $|S| - 1$ out-neighbors of S on the cycle C' and in addition, the vertex v_1 as an out-neighbor of $v_2 \in S$. Consequently, we obtain $|S| \leqslant |N_D^+(S)|$.

In the remaining subcase when $S \cap V_2 = \emptyset$, we define $R = S - V_1$. Since $d_D^-(x, V_1) \geqslant 1$ for all $x \in V_2$, we see that $V_2 \subseteq N_D^+(S)$. If $R = \emptyset$ or $d_D^-(y, V_1) \geqslant 1$ for every vertex y in R, we conclude that $|S| = |R| + |V_1| = |R| + |V_2| \leqslant |N_D^+(S)|$. Otherwise, there exists a vertex $u \in R$ such that $u \rightarrow V_1$, and we also arrive at the desired inequality by $|S| < 2k + 2 = |V_1| + |V_2| \leqslant |N_D^+(S)|$.

Altogether, we have shown that $|S| \leqslant |N_D^+(S)|$ for an arbitrary set $S \subseteq V(D)$, and hence, according to Theorem 2.3, D contains a 1-factor. Thus, because of $\kappa(D) = k \geqslant \lfloor (k+1)/2 \rfloor$, it follows from Theorem 2.4 that D is Hamiltonian. □

Next, we present different examples which show that Theorem 3.1 is best possible.

Let A_1, A_2, A_3 be the partite sets of a 3-partite tournament H such that $|A_1| = |A_2| = k + 1$, and $|A_3| = k - 1$. If $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_1$, then $\kappa(H) = k - 1 = \alpha(H) - 2$, but H is not Hamiltonian, because the longest cycle of H has length $3(k-1)$.

Let A_1, A_2, A_3 be the partite sets of a 3-partite tournament T such that $|A_1| = |A_2| = k + 1$, and $|A_3| = k$. If $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_1$, then $\kappa(H) = k = \alpha(H) - 1$, but T is not Hamiltonian, because the longest cycle of T has length $3k$.

Let A_1, A_2, A_3, A_4 be the partite sets of a 4-partite tournament G such that $|A_1| = |A_2| = |A_3| = k + 1$, and $|A_4| = k - 1$. In addition, let w be an arbitrary vertex of A_2. If $A_1 \rightarrow (A_2 - \{w\}) \rightarrow A_3 \rightarrow (A_4 \cup \{w\}) \rightarrow A_1$, $A_1 \rightarrow A_3$, and $A_2 \rightarrow A_4$, then $\kappa(G) = k = \alpha(H) - 1$, but G has no Hamiltonian cycle, because the longest cycle of G has length $4k$.

Theorem 3.2. *Let D be a semicomplete n-partite digraph with $\kappa(D) = \alpha(D) - 1 \geqslant 1$. If p is the number of vertices in a longest path and c the number of vertices in a longest cycle in D, then $\kappa(D)p \leqslant (\kappa(D) + 1)c - \kappa(D)$.*

Proof. If $V_1, V_2, \ldots, V_n$ are the partite sets of D such that $|V_1| = \cdots = |V_j| > |V_{j+1}| \geqslant \cdots \geqslant |V_n|$ with $1 \leqslant j \leqslant n$, *then* $\kappa = \kappa(D) = |V_1| - 1$. According to Theorem 2.5, there exists a cycle C in D covering $V_{j+1} \cup \cdots \cup V_n$ and at least κ vertices of each partite set $V_1, \ldots, V_j$. Hence, $|V(C)| \geqslant |V(D)| - j$. Now we distinguish four cases.

Case 1. Let $|V(C)| \geqslant |V(D)| - j + 1$. Since $|V(C)| \geqslant \kappa j$, we obtain the desired result as follows.

$$\begin{aligned}\kappa p &\leqslant \kappa|V(D)| \leqslant \kappa(|V(C)| + j - 1) = \kappa|V(C)| + \kappa j - \kappa, \\ &\leqslant \kappa|V(C)| + |V(C)| - \kappa = (\kappa + 1)|V(C)| - \kappa \leqslant (\kappa + 1)c - \kappa.\end{aligned}$$

Case 2. Let $|V(C)| = |V(D)| - j$ and $|V_{j+1}| + \cdots + |V_n| \geqslant \kappa$. Since $|V(C)| \geqslant \kappa j + \kappa$, we obtain the desired bound as follows:

$$\begin{aligned}\kappa p &\leqslant \kappa|V(D)| = \kappa(|V(C)| + j) = \kappa|V(C)| + \kappa j,\\ &\leqslant \kappa|V(C)| + |V(C)| - \kappa \leqslant (\kappa + 1)c - \kappa.\end{aligned}$$

Case 3. Let $|V(C)| = |V(D)| - j$, $|V_{j+1}| + \cdots + |V_n| \leqslant \kappa - 1$, and $j \geqslant 3$. Then, the subdigraph $D' = D - (V_{j+1} \cup \cdots \cup V_n)$ is strong, and by Theorem 2.1, D' contains a 3-cycle $x_1x_2x_3x_1$. Assume without loss of generality, that $x_i \in V_i$ for $i = 1, 2, 3$. In addition, let $x_i \in V_i$ for $4 \leqslant i \leqslant j$ be arbitrarily chosen. Then, in view of Theorem 2.5, the vertex set $X = (V_1 - \{x_1\}) \cup \cdots \cup (V_j - \{x_j\}) \cup V_{j+1} \cup \cdots \cup V_n$ is contained in a cycle C_1, i.e., $X \subseteq V(C_1)$. Clearly, $|V(C_1)| \geqslant |X| = |V(D)| - j$. If $|V(C_1)| \geqslant |V(D)| - j + 1$, then, analogously to Case 1, we are done. However, if $|V(C_1)| = |V(D)| - j$, then, since the subdigraph $D - V(C_1)$ contains the 3-cycle $x_1x_2x_3x_1$, it follows from Theorem 2.2, that there exists a longer cycle in D, and analogously to Case 1, we are done.

Case 4. Let $|V(C)| = |V(D)| - j$, $|V_{j+1}| + \cdots + |V_n| \leqslant \kappa - 1$, and $j \leqslant 2$. Since $j = 1$ is not possible, it remains the case $j = 2$. But then, it follows from Theorem 3.1 that D is Hamiltonian, and this yields the desired result. □

The next example will show that the bound given in Theorem 3.2 is sharp.

Let $A_1, A_2, \ldots, A_n$ be the partite sets of an n-partite tournament H such that $n \geqslant 3$, $|A_1| = |A_2| = \cdots = |A_{n-1}| = k+1$, and $|A_n| = k$. If $A_i \to A_j$ for $1 \leqslant i < j \leqslant n-1$, $A_i \to A_n$ for $2 \leqslant i \leqslant n$, and $A_n \to A_1$, then $\kappa(H) = k = \alpha(H) - 1$, $c = nk$, and $p = nk + n - 1$. Consequently, $\kappa(H)p = k(nk + n - 1) = (\kappa(H) + 1)c - \kappa(H)$, and therefore equality in the bound of Theorem 3.2.

The same example with $|A_n| = q$ for $1 \leqslant q \leqslant k$, shows that the conjecture mentioned in Section 1 would be best possible. Finally, it may be noted that the second author [6] has observed in 1999 that every semicomplete n-partite digraph D, which fulfills the condition $\kappa(D) = \alpha(D) - 1$ of Theorem 3.2, has a Hamiltonian path.

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