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Latin Squares with Self-Orthogonal Conjugates

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Dedicated to Curt Lindner on the occasion of his 65th birthday

Abstract

In this paper, we investigate the existence of idempotent Latin squares for which each conjugate is orthogonal to precisely its own transpose. We show that the spectrum of Latin squares with this desired property contains all integers $v \geqslant 8$, with the possible exception of 10 and 11. As an application of our results, it is shown that for all integers $v \geqslant 8$, with the possible exception of 10 and 11, there exists an idempotent Latin square of order v that realizes the one-regular graph on 6 vertices as a conjugate orthogonal Latin square graph.

Keywords: Latin square; Orthogonal conjugates; The Schröder quasigroup

1. Introduction

A *quasigroup* is an ordered pair $(Q,\cdot)$, where Q is a set and $(\cdot)$ is a binary operation on Q such that the equations

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b, \tag{1}$$

are uniquely solvable for every pair of elements $a,b \in Q$. A quasigroup is called *idempotent* if the identity

$$x \cdot x = x, \tag{2}$$

is satisfied for all $x \in Q$. If the identity

$$(x \cdot y) \cdot (y \cdot x) = x, \tag{3}$$

holds for all $x, y \in Q$, then it is called a *Schröder quasigroup*. For a finite set Q, it is well known that the multiplication table of the quasigroup defines a Latin square; that is, a Latin square can be viewed as the multiplication table of the quasigroup with the headline and sideline removed. The *order* of the quasigroup is $|Q|$. Two quasigroups of the same order are *orthogonal* if when the two corresponding Latin squares are superposed, each symbol in the first square meets each symbol in the second square exactly once.

If $(Q,\otimes)$ is a quasigroup, we define on the set Q six binary operations $\otimes_{(1,2,3)}$, $\otimes_{(1,3,2)}$, $\otimes_{(2,1,3)}$, $\otimes_{(2,3,1)}$, $\otimes_{(3,1,2)}$, and $\otimes_{(3,2,1)}$ as follows: $a \otimes b = c$ if and only if

$$a \otimes_{(1,2,3)} b = c, \quad a \otimes_{(1,3,2)} c = b, \quad b \otimes_{(2,1,3)} a = c,$$

$$b \otimes_{(2,3,1)} c = a, \quad c \otimes_{(3,1,2)} a = b, \quad c \otimes_{(3,2,1)} b = a.$$

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Table 1
Conjugate-equivalence of orthogonality

	$*$	$*_{132}$	$*_{213}$	$*_{231}$	$*_{312}$	$*_{321}$
$*$	—	P1′	P0	P2′	P2	P1
$*_{132}$	P1′	—	P2	P1″	P0″	P2″
$*_{213}$	P0	P2	—	P1	P1′	P2′
$*_{231}$	P2′	P1″	P1	—	P2″	P0′
$*_{312}$	P2	P0″	P1′	P2″	—	P1″
$*_{321}$	P1	P2″	P2′	P0′	P1″	—

Example 1.1

*	0	1	2	3	4	5	6	7	8
0	0	2	6	4	5	3	7	8	1
1	8	1	3	7	6	4	5	0	2
2	7	6	2	5	1	8	0	4	3
3	6	5	8	3	0	7	2	1	4
4	3	8	7	6	4	2	1	5	0
5	1	7	0	8	3	5	4	2	6
6	2	4	5	1	8	0	6	3	7
7	4	3	1	0	2	6	8	7	5
8	5	0	4	2	7	1	3	6	8

Fig. 1. A CSOLS*(9).

These six (not necessarily distinct) quasigroups $(Q, \otimes_{(i,j,k)})$ are called the *conjugates* of $(Q, \otimes)$ [20]. If L is the multiplication table of a quasigroup $(Q, \otimes)$, then the six Latin squares defined by the multiplication table of its conjugates $(Q, \otimes_{(i,j,k)})$ are called the *conjugates* of L. For more information on Latin squares and quasigroups, the interested reader may refer to the book of Dénes and Keedwell [14].

A quasigroup (Latin square) which is orthogonal to its (i,j,k)-conjugate will be called (i,j,k)-*conjugate orthogonal*. A quasigroup (Latin square) is called *self-orthogonal* if it is orthogonal to its $(2,1,3)$-conjugate.

We say that a property P' is a *conjugate-implicant* of a property P if whenever a Latin square satisfies P, one of its conjugates satisfies P'. We say two constraints are *conjugate-equivalent* if they are conjugate-implicants of each other. A property P is said to be *conjugate-invariant* if whenever it holds for one conjugate, it holds for every conjugate.

The orthogonality of a pair of conjugates can be logically equivalent to, or conjugate-equivalent to, orthogonality of other pairs of conjugates. These relationships are summarized in Table 1 (taken from [21]). Each table entry is a code name for a constraint that is defined to be logically equivalent to the orthogonality of its row and column labels. For example, P1 is defined by orthogonality of $*$ and $*_{321}$, but could have been defined equivalently by orthogonality of $*_{213}$ and $*_{231}$. The constraints P0, P0′ and P0″ are conjugate-equivalent; so are P1, P1′ and P1″, and so are P2, P2′ and P2″.

In this paper, we investigate an open problem relating to the spectrum of Latin squares, where each conjugate is required to be orthogonal to precisely its own transpose from among the other five conjugates. We show that there exist idempotent Latin squares with the desired property for all orders $n \geqslant 8$ with the possible exception of orders $n = 10$ and 11. We shall call a Latin square of order v for which each conjugate is orthogonal to its transpose a *conjugate self-orthogonal Latin square* and denote it by CSOLS(v). In a CSOLS(v), if each conjugate is orthogonal to precisely its own transpose, we denote it by CSOLS*(v).

It is easy to check that the Latin square in Fig. 1 is a CSOLS*(9) because it is self-orthogonal and both its 231- and 132-conjugates are also self-orthogonal. In terms of the notations from Table 1, this Latin square satisfies the properties P0, P0′, and P0″.

To show that each conjugate is orthogonal to precisely its own transpose, we list below the counter-examples ($*$ is the same as $*_{123}$).

$0 *_{123} 8 = 1 *_{123} 1 = 1 \quad 0 *_{231} 8 = 1 *_{231} 1 = 1$

$0 *_{123} 8 = 1 *_{123} 1 = 1 \quad 0 *_{312} 8 = 1 *_{312} 1 = 1$

$0 *_{123} 8 = 2 *_{123} 4 = 1 \quad 0 *_{132} 8 = 2 *_{132} 4 = 7$

$0 *_{123} 8 = 6 *_{123} 3 = 1 \quad 0 *_{321} 8 = 6 *_{321} 3 = 4$

$0 *_{231} 8 = 1 *_{231} 1 = 1 \quad 0 *_{312} 8 = 1 *_{312} 1 = 1$

$0 *_{231} 7 = 4 *_{231} 1 = 2 \quad 0 *_{132} 7 = 4 *_{132} 1 = 6.$

The other counter-examples can be derived from the above six cases because of logical equivalence (see Table 1). The above six cases show exactly that none of the properties P1, P1′, P1″, P2, P2′, and P2″ hold.

To facilitate our investigation, we need the notion of a "holey" quasigroup. Let Q be a set and $\mathbf{H} = \{S_1, S_2, \ldots, S_k\}$ be a set of subsets (not necessarily disjoint) of Q. A *holey idempotent quasigroup* (HIQ) having hole set $\mathbf{H}$ is a triple $(Q, \mathbf{H}, \cdot)$, which satisfies the following properties:

(1) $(\cdot)$ is a binary operation defined on Q, however, when both points a and b belong to the same set S_i, there is no definition for $a \cdot b$,
(2) the Eq. (1) holds when a, b are not contained in the same set $S_i, 1 \leqslant i \leqslant k$,
(3) the identity (2) holds for any $x \notin \cup_{1 \leqslant i \leqslant k} S_i$.

If $\mathbf{H} = \{S_1, S_2, \ldots, S_k\}$ is a partition of Q, then we call this quasigroup a *frame idempotent quasigroup* and the *type* of this frame is defined to be the multiset $\{|S_i| : 1 \leqslant i \leqslant k\}$. We shall use an "exponential" notation $s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$ to describe the type of n_i occurrences of $s_i, 1 \leqslant i \leqslant t$ in the multiset. We briefly denote such a frame idempotent quasigroup of type $s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$ by FIQ($s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$). We note that a FIQ($1^{v-n} n^1$) is called an *incomplete* IQ(v) with one hole of size n, briefly denoted by IIQ(v, n).

The holey Latin square corresponding to a self-orthogonal FIQ($s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$) will be denoted as an FSOLS($s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$). We briefly denote a frame ISQ (idempotent Schröder quasigroup) of type $s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$ by FISQ($s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$). We note that a FISQ($1^{v-n} n^1$) is equivalent to an *incomplete* ISQ(v) with one hole of size n, briefly denoted by IISQ(v, n). If each conjugate of an ISQ(v) is orthogonal to precisely its own transpose, it is denoted by ISQ*(v). Similarly, we adopt the notations FISQ*($s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$) and IISQ*($v, n$).

2. The Schröder quasigroup

For the most part of our investigation, we are able to exploit the conjugate-invariant property of the Schröder identity, which has been studied quite extensively (see, for example, [2,17,13,6–8,20]). It is well known that Schröder quasigroups are self-orthogonal and an idempotent Schröder quasigroup of order n (ISQ(n)) exists only if $n \equiv 0, 1 \pmod 4$. An ISQ(n) does not exist for $n = 5$ and 9, but for other orders $n \equiv 0, 1 \pmod 4$, idempotent Schröder quasigroups are now known to exist. For a Schröder quasigroup, the corresponding Latin square is self-orthogonal with Weisner property [17]. It is not too difficult to show that the Schröder identity is conjugate invariant and so every conjugate of a Schröder quasigroup is self-orthogonal.

Idempotent Schröder quasigroups, or ISQs, are associated with other combinatorial configurations such as a class of edge-coloured block designs with block size 4, triple tournaments and self-orthogonal Latin squares with Weisner property (see [13,2,17,19]). The following theorem gives a complete solution of the existence of ISQs.

Theorem 2.1 (Bennett et al. [8], Colbourn and Stinson [13]). *An idempotent Schröder quasigroup of order v exists if and only if $v \equiv 0, 1 \pmod 4$ and $v \neq 5, 9$.*

Although we cannot construct idempotent Schröder quasigroups for orders $n \equiv 2, 3 \pmod 4$, we may construct incomplete idempotent Schröder quasigroups (IISQs) for these orders with "holes" of various sizes.

*	0	1	2	3	4	x	y
0	0	2	4	x	y	1	3
1	y	1	3	0	x	2	4
2	x	y	2	4	1	3	0
3	2	x	y	3	0	4	1
4	1	3	x	y	4	0	2
x	3	4	0	1	2		
y	4	0	1	2	3		

Counterexample for other orthogonalities:

```
0 *123 6 = 6 *123 4 = 3        0 *231 6 = 6 *231 4 = 1
0 *123 5 = 4 *123 0 = 1        0 *312 5 = 4 *312 0 = 2
0 *123 6 = 5 *123 0 = 3        0 *321 6 = 5 *321 0 = 2
0 *231 6 = 1 *231 1 = 1        0 *312 6 = 1 *312 1 = 1
0 *231 6 = 3 *231 0 = 1        0 *132 6 = 3 *132 0 = 4
0 *123 5 = 2 *123 4 = 1        0 *132 5 = 2 *132 4 = 3
```

Fig. 2. An IISQ$^*(7,2)$.

An ISQ(v) is equivalent to an edge-coloured block design CBD$[G_6; v]$ which is investigated in [13]. Here G_6 is used to represent the special case of the coloured graph K_4, where, if K_4 is based on the vertex set $\{1,2,3,4\}$, the edges $\{1,2\}$ and $\{3,4\}$ receive one colour, say colour 1, the edges $\{1,3\}$ and $\{2,4\}$ receive colour 2, and the edges $\{1,4\}$ and $\{2,3\}$ receive colour 3. An *edge-coloured block design* CBD$[G_6; v]$ on a v-set Q is a partition of the coloured edges of a triplicate complete graph $3K_v$, each K_v receives one colour for its edges from three different colours, into blocks (a,b,c,d) each containing edges $\{a,b\},\{c,d\}$ coloured with colour 1, edges $\{a,c\},\{b,d\}$ with colour 2, and edges $\{a,d\},\{b,c\}$ with colour 3. If we define a binary operation $(\cdot)$ as $a\cdot b=c, b\cdot a=d, c\cdot d=a$ and $d\cdot c=b$ from the block (a,b,c,d) and define $x\cdot x=x$ for every $x\in Q$, an ISQ(v) is obtained on set Q. On the other hand, suppose Q is an ISQ. If $a\cdot b=c$, $b\cdot a=d$, then we must have $c\cdot d=(a\cdot b)\cdot(b\cdot a)=a$ and $d\cdot c=(b\cdot a)\cdot(a\cdot b)=b$. So the block (a,b,c,d) is determined and a CBD$[G_6; v]$ can be obtained in this way.

Suppose the block set **B** of an edge-coloured design is closed under the action of some abelian group G. Then one needs only list part of the blocks, called *starter blocks*, in order to obtain the block set **B**. The difference condition for the starter blocks to satisfy is that the differences $\pm(x-y)$ from all the edges $\{x,y\}$ with colour i in the starter blocks are precisely $G/\{0\}$ for $1\leqslant i\leqslant 3$.

If we consider a $3K_v$ with some edges deleted and the deleted edges form the edges of a $3K_n$, we may denote the incomplete triplicate complete graph by $3K_v-3K_n$. If the edges of the $3K_v-3K_n$ are coloured and can be partitioned into blocks as above, we obtain an incomplete edge-coloured design, from which we further obtain an IISQ(v,n).

Let G be an abelian group of order $v-n$. Let $X=\{x_1,x_2,\dots,x_n\}$, where G and X are disjoint, and $Q=G\cup X$. Suppose $3K_v-3K_n$ is based on Q and the missing $3K_n$ is based on X. We consider the elements x_i as infinite elements for the group G. We may also use starter blocks to generate all blocks under group G, provided that each x_i appears in exactly one starter block and the differences $(x-y)$ from all the edges $\{x,y\}$ with colour i in the starter blocks are precisely $G/\{0\}$ for $1\leqslant i\leqslant 3$.

Example 2.2. Let G be the cyclic group Z_5 and $X=\{x,y\}$. Take the starter blocks $(x,0,3,1)$ and $(y,0,4,3)$. The edges $\{3,1\}$ and $\{4,3\}$ with colour 1 give differences $\pm(3-1)$ and $\pm(4-3)$, which are precisely $G/\{0\}$. The same holds for colour 2 and colour 3. We obtain an incomplete edge-coloured design on $3K_7-3K_2$. This further leads to an IISQ$^*(7,2)$ shown in Fig. 2.

Let G be an abelian group of order hk and H a subgroup of order h. Let $X=\{x_1,x_2,\dots,x_u\}$, where G and X are disjoint, and $Q=G\cup X$. Suppose we have some starter blocks based on Q such that the elements x_i act as infinite elements for the group G, each x_i appearing in exactly one starter block and the differences $\pm(x-y)$ from all the edges $\{x,y\}$ with colour i in the starter blocks are precisely G/H for $1\leqslant i\leqslant 3$. Then we obtain a FISQ(h^ku^1).

Example 2.3. Let G be the cyclic group Z_8, $H=\{0,4\}$ and $X=\{x,y,z\}$. Take the starter blocks $(x,0,5,7)$, $(y,0,6,3)$ and $(z,0,7,6)$. The edges $\{5,7\}$, $\{6,3\}$ and $\{7,6\}$ with colour 1 give differences $\pm(5-7),\pm(6-3)$ and $\pm(7-6)$, which are precisely G/H. The same holds for colours 2 and 3. We then have the FISQ$^*(2^43^1)$ shown in Fig. 3.

*	0	1	2	3	4	5	6	7	x	y	z
0		2	x	5		y	1	z	7	3	6
1	z		3	x	6		y	2	0	4	7
2	3	z		4	x	7		y	1	5	0
3	y	4	z		5	x	0		2	6	1
4		y	5	z		6	x	1	3	7	2
5	2		y	6	z		7	x	4	0	3
6	x	3		y	7	z		0	5	1	4
7	1	x	4		y	0	z		6	2	5
x	5	6	7	0	1	2	3	4			
y	6	7	0	1	2	3	4	5			
z	7	0	1	2	3	4	5	6			

Counterexample for other orthogonalities:

```
0 *123 10  =  10*123 7 = 6        0 *231 10  =  10*231 7 = 1
0 *123  9  =   9 *123 5 = 3       0 *312  9  =   9 *312 5 = 2
0 *123  9  = 10 *123 4 = 3        0 *132  9  = 10 *132 4 = 5
0 *231  9  =   8 *231 2 = 3       0 *132  9  =   8 *132 2 = 5
0 *123 10  =  6
                       0 *321 10 = 2,     <6, 2> is a hole
0 *231 10  =  1
                       0 *312 10 = 1,     <1, 1> is a hole
```

Fig. 3. A FISQ$^*(2^4 3^1)$.

3. Recursive constructions

Recall that we denote by CSOLS$^*(n)$ a CSOLS(n) with the property that every conjugate is orthogonal to precisely its transpose. Similarly, ISQ$^*(n)$ is an ISQ(n) with the property that every conjugate is orthogonal to precisely its transpose and IISQ$^*(v,n)$ is an IISQ(v,n) with the property that every conjugate is orthogonal to precisely its transpose.

The following construction may be called *filling in holes*. The technique is fairly straightforward and is commonly used in constructing designs.

Construction 3.1 (Filling in holes). (1) If there exist IISQ(v,n) and ISQ$^*(n)$, then there exists an ISQ$^*(v)$.

(2) If there exist IISQ$^*(v,n)$ and ISQ(n), then there exists an ISQ$^*(v)$.

(3) If there exist IISQ(v,n) and CSOLS$^*(n)$, then there exists a CSOLS$^*(v)$.

(4) If there exist ICSOLS$^*(v,n)$ and CSOLS(n), then there exists a CSOLS$^*(v)$.

(5) Suppose there exists a FISQ of type $\{s_i : 1 \leqslant i \leqslant n\}$. Let $a \geqslant 0$ be an integer. For $1 \leqslant i \leqslant n-1$, if there exists an IISQ$(a+s_i, a)$, then there is an IISQ$(a+s, a+s_n)$, where $s=\sum_{i=1}^{n} s_i$.

(6) Suppose there exists a FISQ of type $\{s_i : 1 \leqslant i \leqslant n\}$. Let $a \geqslant 0$ be an integer. For $1 \leqslant i \leqslant n-1$, if there exists an IISQ$^*(a+s_i, a)$, then there is an IISQ$^*(a+s, a+s_n)$, where $s=\sum_{i=1}^{n} s_i$.

The next recursive construction for FISQ uses group divisible designs [10]. A group divisible design (or GDD), is a triple $(X, \mathbf{G}, \mathbf{B})$ which satisfies the following properties:

(1) $\mathbf{G}$ is a partition of X into subsets called *groups*,
(2) $\mathbf{B}$ is a family of subsets of X (called *blocks*) such that a group and a block contain at most one common point,
(3) every pair of points from distinct groups occurs in exactly λ blocks.

The *type* of the GDD is the multiset $\{|G| : G \in \mathbf{G}\}$. We also use the notation GD$(K, M; \lambda)$ to denote the GDD when its block sizes belong to K and group sizes belong to M. If $M=\{1\}$, then the GDD becomes a *pairwise balanced design* (PBD). If $K=\{k\}$, $M=\{n\}$ and with the type n^k, then the GDD becomes a *transversal design* TD(k,n). It is well known that the existence of a TD(k,n) is equivalent to the existence of $k-2$ *mutually orthogonal Latin squares of order n* (MOLS(n)). For known results on MOLS and TDs, the reader is referred to [1]. For more detailed information about PBDs, GDDs and related structures, the reader is referred to [12,15,22].

The following construction comes from the weighting construction of GDDs [22].

Construction 3.2 (Weighting). Suppose $(X, \mathbf{H}, \mathbf{B})$ is a GDD with $\lambda=1$ and let $w : X \mapsto Z^+ \cup \{0\}$. Suppose there exist FISQs of type $\{w(x) : x \in B\}$ for every $B \in \mathbf{B}$. Then there exists a FISQ of type $\{\sum_{x \in H} w(x) : H \in \mathbf{H}\}$.

Lemma 3.3. (1) *For any prime power* p, *there exists a* TD(k,p), *where* $3 \leqslant k \leqslant p+1$.
(2) *There exists a* TD$(4,m)$ *for any positive integer* $m \neq 2,6$.

The next construction is another modification of the weighting construction, where the starting GDD is edge-coloured and the input design is a TD$(4,m)$.

Construction 3.4 (Bennett et al. [8, Construction 3.5]). Suppose there exists a FISQ$(h_1^{n_1}h_2^{n_2}\cdots h_k^{n_k})$ of order v. Then there exists a FISQ$((mh_1)^{n_1}(mh_2)^{n_2}\cdots(mh_k)^{n_k})$ of order mv, where $m \neq 2$ or 6.

The following construction, which provides a way to get FISQ from FSOLS, can be found in [13].

Construction 3.5. Suppose there exists an FSOLS$(h_1^{n_1}h_2^{n_2}\cdots h_k^{n_k})$ of order v. Then there exists a FISQ$((4h_1)^{n_1}(4h_2)^{n_2}\cdots(4h_k)^{n_k})$ of order $4v$.

Theorem 3.6. *There exists a* TD$(5,n)$ *for any positive integer* $n \neq 2,3,6$ *and* 10.

The following result is proved in [8]. Note that $u \leqslant h(k-1)/2$ is a necessary condition for the existence of a FISQ$(h^k u^1)$.

Theorem 3.7. *For* $1 \leqslant u \leqslant 4$, *a* FISQ$(2^n u^1)$ *exists if and only if* $n \geqslant u+1$ *with the exception of* $(n,u) \in \{(2,1),(3,1),(3,2)\}$.

An immediate consequence of the preceding theorem is the following useful result, which can also be found in [23].

Theorem 3.8. *There exist* FSOLS(2^n) *and* FSOLS$(2^n 3^1)$ *for* $n \geqslant 4$.

The following result is established in [6]. Note that $u \geqslant 4$ is a necessary condition for the existence of a FISQ(h^u).

Theorem 3.9. *A* FISQ(h^u) *exists if and only if* $h^2u(u-1) \equiv 0 \pmod 4$ *with the exception of the pairs* $(h,u) = (1,5),(1,9),(2,4)$ *and the possible exception of* $(h,u) = (6,4)$.

4. Main lemmas

Lemma 4.1. *There exists a* CSOLS(n) *for all prime powers* $n \neq 2,3,5$.

Proof. For $n = 4$ and 7, these can easily be constructed, see for example [5]. For $n \geqslant 8$, see for example [16]. □

Lemma 4.2. *There exists an* ISQ$^*(v)$ *for* $8 \leqslant v \leqslant 40$ *and* $v \equiv 0 \pmod 4$.

Proof. See [9]. □

Lemma 4.3. *For all integers* $n \geqslant 4$, *there exist* FISQ(8^n) *and* FISQ$(8^n 12^1)$. *For all integers* $n \geqslant 5$, *there exists a* FISQ$(8^n 16^1)$.

Proof. This follows from Construction 3.4 or 3.5 and Theorem 3.7 or 3.8. □

Lemma 4.4. *There exist* IISQ$^*(v,n)$ *for* $(v,n) \in \{(7,2),(10,3),(11,3),(13,4),(17,4),(21,4),(22,7),(23,7),(25,4),(26,7),(29,4),(30,7),(33,4),(34,7),(37,4),(38,7),(41,4),(46,15),(51,14),(54,15)\}$.

Proof. See Example 2.2 and [9]. □

Lemma 4.5. *There exist* FISQ* *of types* 2^5, $2^5 4^1$, 4^4, $4^4 2^1$, $4^4 6^1$, 4^5, 4^6, $4^5 6^1$, $4^5 7^1$, 5^9, $5^9 6^1$, 6^5, 6^7, 7^5, $7^5 4^1$, 11^5.

Proof. See [8,9]. □

Lemma 4.6. *Suppose there exists a* TD$(6,t)$. *Then there exists a* FISQ$^*((4t)^4\ (4x)^1(2u)^1)$, *where* $0 \leqslant x \leqslant t$ *and* $2 \leqslant u \leqslant 3t$.

Proof. In all but the last two groups of the TD$(6,t)$, give the points weight 4. In the next to last group, give x points weight 4 and give the remaining points a weight of zero. In the last group, we give each point a weight of 0, 2, 4, or 6. We then apply Construction 3.2 with the necessary input designs supplied in Lemma 4.5. □

Lemma 4.7. *Suppose there is a* TD$(5,t)$. *Then there exists a* FISQ$^*((4t)^4(2u)^1)$, *where* $0 \leqslant u \leqslant 3t$.

Proof. In all except the last group of the TD$(5,t)$, give the points weight 4. In the last group we give each point a weight of 0, 2, 4, or 6. We then apply Construction 3.2 with the necessary input designs supplied in Lemma 4.5. □

Lemma 4.8. *Suppose there is a* TD$(5,t)$. *Then there exists a* FISQ$^*((2t)^4u^1)$, *where* $2t \leqslant u \leqslant 3t$.

Proof. In a manner similar to the above, apply Construction 3.2 with the necessary input designs supplied in Lemma 4.5 and Fig. 3. □

Combining the main lemmas with the direct constructions for small orders, we are now in a position to establish the main results of this section.

Theorem 4.9. *There exists an* ISQ$^*(v)$ *for all integers* $v \equiv 0,1 \pmod 4$, *where* $v \neq 4,5,9$.

Proof. For the case $v \equiv 0 \pmod 4$, where $v \geqslant 44$, we can apply Lemma 4.3 with Construction 3.1 (5). Here we first obtain for $n \geqslant 4$, an IISQ$(8n,8)$ or an IISQ$(8n+12,8)$ and then fill in the hole of size 8 with an ISQ$^*(8)$. For $v \leqslant 40$, the constructions can be found in Lemma 4.2.

For the case $v \equiv 1 \pmod 4$, where $v \leqslant 41$, the constructions can be derived from IISQ$^*(v,4)$ in Lemmas 4.4 by filling in an ISQ(4) in the hole. For $v \geqslant 45$, we can apply some recursive constructions as follows. First of all, from [6], we have a FISQ(h^4) for $h = 11, 12, 15$. So we can adjoin an infinite point to fill in the holes and apply Construction 3.1 (5) to cover the cases $v = 45, 49, 61$. For $v = 53$, a FISQ$(5^9 6^1)$ is provided in Lemma 4.5. So we can adjoin two infinite points and fill in the holes by using the IISQ$^*(7,2)$ and an ISQ$^*(8)$ in Construction 3.1 (6). For $v = 57$, we have a FISQ(7^8) from [6]. To this we can adjoin an infinite point and fill in the holes with an ISQ$^*(8)$. For $v = 69$, an IISQ$(69,20)$ is constructed in [8]. So we can fill in the hole with an ISQ$^*(20)$ for the desired result. For $v = 73$, we have a FISQ(12^6) from [6]. To this we adjoin an infinite point and fill in the holes with an ISQ$^*(13)$.

For $v = 65, 77, 81, 85, 89$, we can first apply Lemma 4.7 with the parameters $t = 4$, $u = 0, 6, 8, 10, 12$, then fill in the holes of the resulting FISQ by adjoining an infinite point and using ISQ$^*(17)$ and ISQ$^*(2u+1)$.

For v in the interval [93, 221], a similar approach is taken, where we first apply Lemma 4.6 with the parameters t, x and even values of u as indicated in Table 2. An infinite point is then used to fill in the holes of the resulting FISQ, using ISQ$^*(4t+1)$, ISQ$^*(4x+1)$ and ISQ$^*(2u+1)$ in order to obtain an ISQ$^*(v)$, where $v = 16t + 4x + 2u + 1$.

For $v \equiv 1 \pmod 4$, where $v \geqslant 221$, we have $v = 16t + w$, where $t \geqslant 13$ and $w = 13, 17, 21, 25$. So we can first apply Lemma 4.7 with the parameters $t \geqslant 13$ and $u = 6, 8, 10, 12$, then adjoin an infinite point to fill in the holes of the resulting FISQs to get the desired results. □

Combining Theorem 4.9 with Example 1.1, we have proved the following:

Theorem 4.10. *There exists a* CSOLS$^*(v)$ *for all integers* $v \equiv 0,1 \pmod 4$, *where* $v \neq 4,5$.

Table 2
Parameters t, x, u for v in [93, 221]

t	x	u	$v = 16t + 4x + 2u + 1$
5	0, 3–5	6–14	93–129
7	0, 3–7	6–20	125–181
9	0, 3–9	6–20	157–221

Table 3
Parameters t, x, u for v in [94, 222]

t	x	u	$v = 16t + 4x + 2u$
5	0–5	7–15	94–130
7	0–7	7–21	126–182
9	0–9	7–21	158–222

Theorem 4.11. *There exists a* CSOLS$^*(v)$ *for all integers* $v \equiv 2, 3 \pmod 4$, *where* $v \geqslant 14$.

Proof. A CSOLS$^*(v)$, where $v = 14, 15$, can be found in [9]. A CSOLS$^*(v)$, where $v = 18, 19$, can be obtained from ICSOLS$^*(v, 4)$ in [9] by filling in an ISQ(4).

For the case $v \equiv 2 \pmod 4$, where $22 \leqslant v \leqslant 38$, the constructions can be derived from IISQ$^*(v, 7)$ in Lemma 4.4 by filling in a CSOLS(7).

For the case $v \equiv 2 \pmod 4$, where $v \geqslant 42$, we can apply recursive constructions as follows. For $v = 42$, a FISQ(5^8) is constructed in [6]. So we can adjoin two infinite points and fill in the holes by using the IISQ$^*(7, 2)$ in Construction 3.1 (6) to first get an IISQ$^*(42, 7)$. We then fill in the hole of size 7 with a CSOLS(7) to get a CSOLS$^*(42)$.

For the case $v = 50$, we first construct a FISQ($7^5 14^1$) from the FISQ($1^5 2^1$) in Lemma 4.4 and Construction 3.4. We then adjoin an infinite point to this and fill in the holes using CSOLS$^*(n)$ for $n = 8$ and 15. For $v = 46, 54$, we can fill in the hole of size 15 in the IISQ$^*(v, 15)$ given in Lemma 4.4.

For $v = 58, 62, 66, 70, 74$, we have FISQs of types 11^5, 5^{12}, 16^4, 17^4, 9^8 from [6]. We can therefore adjoin 3 infinite points to the first type and two infinite points to each of the other types to get the desired results by applying Construction 3.1.

For the case $v \equiv 2 \pmod 4$, where $v \geqslant 78$, the constructions follow similar lines to those for the case $v \equiv 1 \pmod 4$, where $v \geqslant 77$. Here we can construct an IISQ(v, n) for values of n in the interval [14, 42], where $n \equiv 2 \pmod 4$. We can then fill in the hole of size n with a CSOLS$^*(n)$ for the desired result. To be more specific, for v in the interval [78, 86], we first apply Lemma 4.7 with parameters $t = 4$ and $u = 7, 9, 11$, then fill in the holes of the resulting FISQ by using ISQ$^*(16)$ and CSOLS$^*(2u)$. For $v = 90$, we have a FISQ(18^5) from [6]. We then fill in the holes with a CSOLS$^*(18)$ for the desired result.

For v in the interval [94, 222], we first apply Lemma 4.6 with the parameters t, x and odd values of u as indicated in Table 3. We then fill in the holes of the resulting FISQ, using ISQ$^*(4t)$, ISQ$(4x)$ and CSOLS$^*(2u)$ in order to obtain a CSOLS$^*(v)$, where $v = 16t + 4x + 2u$.

For $v \equiv 2 \pmod 4$, where $v \geqslant 222$, we have $v = 16t + w$, where $t \geqslant 13$ and $w = 14, 18, 22, 26$. So we can first apply Lemma 4.7 with the parameters $t \geqslant 13$ and $u = 7, 9, 11, 13$, then fill in the holes of the resulting FISQs to get the desired results.

For the case $v \equiv 3 \pmod 4$, where $23 \leqslant v \leqslant 39$, a CSOLS$^*(23)$ can be obtained from IISQ$^*(23, 7)$ in Lemma 4.4 by filling in a CSOLS(7); a CSOLS$^*(27)$ can be obtained from FISQ(5^5) in Theorem 3.9 by adjoining two infinite points and then filling in the holes using IISQ$^*(7, 2)$ and CSOLS(7); a CSOLS$^*(31)$ can be obtained from FISQ$^*(6^5)$ in Lemma 4.5 by adjoining one point and then filling in CSOLS(7); a CSOLS$^*(35)$ can be obtained from FISQ$^*(7^5)$ in Lemma 4.5 by filling in CSOLS(7); a CSOLS$^*(39)$ can be obtained from FISQ$^*(7^5 4^1)$ in Lemma 4.5 by filling in the holes with CSOLS(7) and ISQ(4).

For the case $v \equiv 3 \pmod 4$, where $v \geqslant 43$, we can apply Lemmas 4.3 and 4.4 with Construction 3.1 (6). Here for $n \geqslant 4$, starting with FISQ(8^n), we adjoin three infinite points and fill in all but one hole of size 8 using an IISQ$^*(11, 3)$. We then fill the last hole of size 8 together with the three infinite points with a CSOLS(11) to get a CSOLS($8n + 3$). Starting with a FISQ($8^n 12^1$), we adjoin three infinite points and fill in all the holes of size 8 using an IISQ$^*(11, 3)$. We then fill in the hole of size 12 together with the three infinite points with a CSOLS$^*(15)$ to get a CSOLS$^*(8n + 15)$. □

Summarizing the results of this section, we have proved the main result:

Theorem 4.12. *A* CSOLS$^*(v)$ *exists if and only if* $v \geqslant 8$, *except possibly for* $v = 10, 11$.

Proof. A computer search confirms that CSOLS$^*(v)$ does not exist for $v \leqslant 7$. The rest of the proof follows from Theorems 4.10 and 4.11. □

5. Conjugate orthogonal latin square graphs

The result of Theorem 4.12 has an application in the study of *orthogonal Latin square graphs* (OLSG). This notion was first introduced in [16] and further studied in [5,11]. An OLSG is a graph in which the vertices are Latin squares defined on the same set of elements and two vertices are joined if and only if the Latin squares are orthogonal. If G is a finite graph, we say that G is *realizable* if there is an OLSG isomorphic to G. Some special cases of graphs that are realizable by all six conjugates of a Latin square have been studied with varying degrees of success (see, for example, [16,5,3,4,11]). It is indeed an interesting situation when we represent the orthogonal relationships among the distinct conjugates of a Latin square L by a graph where the conjugates are the vertices and two vertices are joined if and only if the corresponding conjugates are orthogonal. Such a graph is called *the conjugate orthogonal Latin square graph* (briefly COLSG) of L. It is still an open problem to determine completely the spectrum of Latin squares realizing either the 6-cycle, C_6 or the complete graph K_6. In this paper, we have made considerable progress towards obtaining an almost complete determination of the spectrum of Latin squares that realize the one-regular COLSG. In what follows, the results are summarized.

Theorem 5.1. *For every integer $v \geqslant 8$, except possibly for $v = 10, 11$, there exists an idempotent Latin square of order v realizing a one-regular COLSG.*

Proof. If $C(L)$ denotes the set of conjugates of a Latin square L, then it is known [18] that $|C(L)| = 1, 2, 3$ or 6. It is easy to check that $|C(L)| = 6$ for every Latin square L of order v for which a CSOLS$^*(v)$ has been constructed in this paper. The fact that the COLSG is one-regular comes from Theorem 4.12. □

Lemma 5.2. *For $v = 5, 7$, there exists a Latin square of order v realizing a one-regular* COLSG.

Proof. A computer search found the two Latin squares in Fig. 4. Both of them are self-orthogonal, their (231)-conjugates are orthogonal to their (132)-conjugates, and their (312)-conjugates are orthogonal to their (321)-conjugates, respectively. No other orthogonality exists between their conjugates. □

Combining Theorem 5.1 with Lemma 5.2, we have the following result.

Theorem 5.3. *For every positive integer v, except for $v = 2, 3, 4, 6$ and except possibly for $v = 10, 11$, there exists a Latin square of order v realizing a one-regular* COLSG.

Proof. The cases $v = 2, 3, 6$ are obviously impossible. For $v = 4$, non-existence is easily established by an exhaustive computer search. The rest follows from Theorem 5.1 and Lemma 5.2 (see Fig. 5). □

In view of Table 1, we have the following result.

Theorem 5.4. (*a*) *A Latin square realizes a one-regular* COLSG *if and only if the satisfied constraint is conjugate-equivalent to one of the following*: $\{P0, P0', P0''\}$, $\{P0, P1''\}$, *or* $\{P0, P2''\}$.

(*b*) *A Latin square realizes C_6 as a* COLSG *if and only if the satisfied constraint is conjugate-equivalent to one of the following*: $\{P1, P1', P1''\}$, $\{P0, P0', P1', P1''\}$, *or* $\{P0, P0', P2, P2''\}$.

x\y	0	1	2	3	4
0	1	0	4	2	3
1	2	4	3	0	1
2	3	2	0	1	4
3	4	1	2	3	0
4	0	3	1	4	2

(a)

x\y	0	1	2	3	4	5	6
0	0	3	5	1	2	4	6
1	2	4	6	0	3	5	1
2	3	5	1	2	4	6	0
3	4	6	0	3	5	1	2
4	5	1	2	4	6	0	3
5	6	0	3	5	1	2	4
6	1	2	4	6	0	3	5

(b)

Fig. 4. Two Latin squares realizing one-regular COLSG.

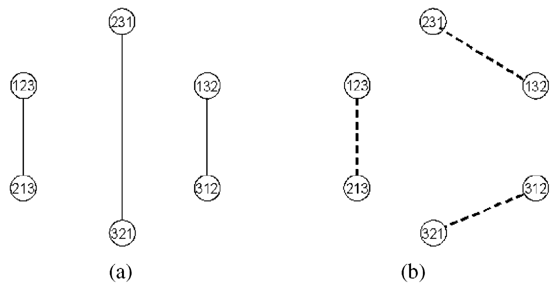

Fig. 5. (a) The one-regular graph produced by Theorem 5.1; (b) the one-regular graph by Lemma 5.2.

(*c*) *A Latin square realizes* K_6 *as a COLSG if and only if all of the properties* $P0, P0', P0'', P1, P1', P1'', P2, P2'$, *and* $P2''$ *hold.*

Proof. It is easy to check that all three necessary conditions hold. An exhaustive search confirms that they are also sufficient conditions. □

6. Concluding remarks

We have investigated the existence of CSOLS$^*(v)$ for $v \geqslant 4$. Most recursive constructions used in this paper are standard in combinatorial designs and many of the direct constructions of CSOLS in this paper are carried out by computer. As an application of our main result in the preceding section, we are able to obtain fairly conclusive results relating to the spectrum of Latin squares that realize the one-regular COLSG, thereby answering one of the open questions raised in [11]. Evidently, there is still a lot of work to be done to address the more general question of which admissible graphs can be realized by COLSGs. With regards to our result in Theorem 4.9, it is perhaps worth mentioning that, for all orders $v \equiv 1, 4 \pmod{12}$, this supplies a variety of idempotent Schröder quasigroups different from those that would normally be obtained from BIBDs with block size four (see, for example, [17]).

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