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On the Fundamental Solutions to Stokes Equations

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We study the behavior of solutions to the stationary Stokes equations near singular points. Employing the power series expansions of harmonic and biharmonic functions, we have local power series expansions of solutions near singular points. Then we find the precise structures of homogeneous solutions near singular

under an assumption on directions of velocities.

1. INTRODUCTION

When the viscous fluid flows slowly, the effect of convection is negligible. Thus the essential feature of viscous incompressible flow can be modeled by Stokes equations instead of Navier–Stokes equations. In this note we study the behavior of solutions to the stationary Stokes equations which have an isolated singularity.

We assume that $\Omega \subset R^n$ is a bounded domain and the singular point $\{0\}$ lies in Ω. We also assume that $u: \Omega\backslash\{0\} \to R^n$ is the velocity vector and $p: \Omega\backslash\{0\} \to R$ is the pressure. Moreover we assume that (u, p) are smooth in $\Omega\backslash\{0\}$ and satisfy the Stokes equations

$$\Delta u - \nabla p = 0, \qquad \nabla \cdot u = 0$$

in $\Omega\backslash\{0\}$. We observe that u is biharmonic and p is harmonic. Hence considering the power series expansion of biharmonic functions, we can decide the precise structures of solutions near singular points. The regular part corresponds to homogeneous polynomial solutions to Stokes equations and the singular part consists of homogeneous solutions of negative degree. We find the structure of homogeneous solutions of the fixed degree and they form a finite dimensional vector space. Using the information of structure we can construct systematically the celebrated Stokes example of

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a flow in the exterior of a ball moving at constant velocity through a viscous fluid (see p. 239, Ch. 5 in [6]).

Once we know the precise structure of solutions near isolated singular points, we can find a necessary and sufficient condition that a fundamental solution satisfies. In the case of harmonic function we know that if $u \geqslant c$ for some c near a point x_0, then $u(x) = v(x) + cN(x - x_0)$ for a harmonic function v across x_0 and the fundamental solutions $N(x - x_0)$ to Laplace equation. This is originally proved by Bôcher in 1903 (see [1]). We prove an equivalent theorem for Stokes equations. We can say that if the velocity vectors near the singular point have similar directions, then the solution is a fundamental solution.

Suppose that the space dimension n is greater than or equal to 3. We define the fundamental solutions with singularity at the origin by

$$V^i_m(x) = \frac{1}{n-2} \frac{\delta_{mi}}{|x|^{n-2}} + \frac{x_i x_j}{|x|^n}$$

for $i = 1, ..., n$ and

$$Q_m(x) = \frac{2x_m}{|x|^n}$$

for $m = 1, 2, ..., n$. When the dimension is 2, we define

$$V^i_m(x) = \frac{1}{2} \frac{x_m x_i}{|x|^2} - \frac{1}{4} \delta_{mi} - \frac{1}{2} \delta_{mi} \ln |x|$$

for $i = 1, 2$ and

$$Q_m(x) = \frac{x_m}{|x|^2}$$

for $m = 1, 2$.

Define the ball $B_r(x_0) = \{x: |x - x_0| < r\}$ and as usual double indices mean summation up to n. The following theorem is a generalization of Bôcher's theorem to the Stokes equations.

THEOREM 1.1. *Suppose (u, p) are solutions to Stokes equations in $\Omega \backslash \{0\}$. We fix a constant L and define*

$$E_L = \{e \in R^n : e \cdot u(x) > L \text{ for all } x \in B_\delta(0)\}.$$

Then the interior of E_L is nonempty for some L if and only if

$$u(x) = v(x) + \sum_{m=1}^{n} a_m V_m(x)$$

and

$$p(x)=q(x)+\sum_{m=1}^{n} a_m Q_m(x)$$

for some constants a_m, *where* (v, q) *are smooth solutions of Stokes equations in* Ω.

Here the assumption that E_L has nonempty interior means that the velocity vectors u are away to a certain fixed direction near singular point.

2. CHARACTERIZATION OF FUNDAMENTAL SOLUTIONS

The following power series expansion is essential.

LEMMA 2.1. *Let* $\{0\} \in \Omega$. *Suppose that* v *is a smooth biharmonic function in* $\Omega \backslash \{0\}$. *Then,* v *can be written*

$$v(x)=h(x)+|x|^2\, k(x)+|x|^2\, k_1(x)+\delta_{2n}(A\cdot x)\ln|x|,$$

where A *is a constant vector,* $h(x)$ *and* $k(x)$ *are harmonic in* $\Omega \backslash \{0\}$, *the power series expansions of* $h(x)$ *and* $k(x)$ *in terms of homogeneous harmonic polynomials do not involves with* ln *and*

$$\begin{aligned} k_1(x) = c\ln|x| & \quad \textit{for} \quad n=2 \\ c\,|x|^{-2}\ln|x| & \quad \textit{for} \quad n=4 \\ 0 & \quad \textit{for} \quad n\neq 2, 4. \end{aligned}$$

Suppose v is biharmonic in $R^n \backslash \{0\}$. We let m_k be the dimension of spherical harmonics of degree k and denote $\{S_{km}: m=1, ..., m_k\}$ as the orthonormal basis of spherical harmonics of degree k in $L^2(S_1^{n-1})$. Here S_R^{n-1} is the sphere of radius R in R^n. Hence taking power series expansion of h and k at the origin we find

$$\begin{aligned} v(x) = v_1(x)+|x|^2 v_2(x) &+ \sum_{k=0}^{\infty}\sum_{m=1}^{m_k} a_{km}\,|x|^{-k-n+2}\, S_{km}\left(\frac{x}{|x|}\right) \\ &+ \sum_{k=0}^{\infty}\sum_{m=1}^{m_k} b_{km}\,|x|^{-k-n+4}\, S_{km}\left(\frac{x}{|x|}\right) \\ &+ \delta_{2n}(c_1\ln|x|+(c_2\cdot x)\ln|x|+c_3\,|x|^2\ln|x|)+\delta_{4n}(c_4\ln|x|), \end{aligned}$$

where v_1 and v_2 are harmonic in whole Ω, and δ_{ij} is the Kronecker δ-function.

Now considering the power series expansions of harmonic functions $v_1(x)$ and $v_2(x)$, we can write that for homogeneous harmonic polynomials $H^i_{jm}(x)$

$$
\begin{aligned}
v(x) = & \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} H^1_{km}(x) + |x|^2 H^2_{(k-2)m} \\
& + \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} |x|^{-k-n+2} \left(a_{km} S_{km}\left(\frac{x}{|x|}\right) + b_{(k+2)m} S_{(k+2)m}\left(\frac{x}{|x|}\right)\right) \\
& + \sum_{m=1}^{m_0} b_{0m} |x|^{-n+4} S_{0m}\left(\frac{x}{|x|}\right) + \sum_{m=1}^{m_1} b_{1m} |x|^{-n+3} S_{1m}\left(\frac{x}{|x|}\right) \\
& + \delta_{2n}(c_1 \ln |x| + (c_2 \cdot x) \ln |x| + c_3 |x|^2 \ln |x|) + \delta_{4n} c_4 \ln |x|, \qquad (2.1)
\end{aligned}
$$

where m_k is the dimension of homogeneous harmonic polynomials of degree k. We know that the spherical harmonics $S_j(x/|x|)$ can be written $S_j(x) = |x|^{-j} P_j(x)$ for some homogeneous harmonic polynomial $P_j(x)$ of degree j. Therefore we have

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} |x|^{-k-n+2} (a_{km} S_{km} + b_{(k+2)m} S_{(k+2)m}) \\
& \qquad = \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} |x|^{-2k-n} (|x|^2 P^1_{km} + P^2_{(k+2)m}),
\end{aligned}
$$

where $P^j_{km}(x)$ are homogeneous harmonic polynomials of degree k.

If (u, p) are solutions to Stokes equations

$$\Delta u - \nabla p = 0, \qquad \nabla \cdot u = 0$$

in $\Omega \backslash \{0\}$, then u is biharmonic and p is harmonic in $\Omega \backslash \{0\}$. Thus u has a power series expansion of the form (2.1) and p has power series expansion in terms of homogeneous harmonic polynomials near $\{0\}$.

First we decide the structure of homogeneous polynomial solutions to Stokes equations.

LEMMA 2.2. *Let $k \geqslant 2$. Suppose $v(x) = P_k(x)$ and $q(x) = P_{k-1}(x)$ are solutions to Stokes equations, where $P_k(x)$ and $P_{k-1}(x)$ are homogeneous polynomials of degree k and $k-1$ respectively. Then $v(x)$ and $q(x)$ can be written as*

$$v(x) = H_k(x) - \frac{1}{2(k-1)} |x|^2 \nabla(\nabla \cdot H_k(x))$$

$$q(x) = -\frac{n+k-2}{k-1} \nabla \cdot H_k(x)$$

for some homogeneous harmonic polynomial $H_k(x)$ of degree k.

Proof. We do not make any specific notation for vector valued functions and scalar valued functions. From Lemma 2.1 for the characterization of biharmonic function, we know that

$$v(x) = H_k(x) + |x|^2 H_{k-2}(x)$$

and

$$q(x) = H_{k-1}(x)$$

for some homogeneous harmonic polynomials H_k, H_{k-1} and H_{k-2}. Since $\nabla \cdot v = 0$, we have

$$\nabla \cdot H_k(x) + 2x \cdot H_{k-2}(x) + |x|^2 \nabla \cdot H_{k-2}(x) = 0. \tag{2.2}$$

On the other hand $\Delta v = \nabla p$ implies

$$2nH_{k-2}(x) + 2(x \cdot \nabla) H_{k-2}(x) = \nabla H_{k-1}(x).$$

Since H_{k-2} are homogeneous polynomial of degree $k-2$ we have

$$(x \cdot \nabla) H_{k-2}(x) = (k-2) H_{k-2}(x).$$

Hence we have

$$H_{k-2}(x) = \frac{1}{2(n+k-2)} \nabla H_{k-1}(x).$$

Considering the above relation and (2.2) we get

$$\begin{aligned}\nabla \cdot H_k(x) &= -2x \cdot H_{k-2}(x) - |x|^2 \nabla \cdot H_{k-2}(x) \\ &= -\frac{1}{n+k-2} x \cdot \nabla H_{k-1}(x) - \frac{1}{2(n+k-2)} |x|^2 \Delta H_{k-2}(x) \\ &= -\frac{k-1}{n+k-2} H_{k-1}.\end{aligned}$$

Therefore we have

$$H_{k-1} = -\frac{n+k-2}{k-1} \nabla \cdot H_k \qquad \text{and} \qquad H_{k-2}(x) = -\frac{1}{2(k-1)} \nabla(\nabla \cdot H_k)$$

and this completes the proof.

Recall that the dimension of the set of all homogeneous harmonic polynomials is

$$\binom{n+k-1}{k} - \binom{n+k-3}{k-2}.$$

Thus we have the following corollary.

COROLLARY 2.3. *Let $\mathcal{S}(n,k) = \{(v_k(x), q_{k-1}(x))\}$, $k \geqslant 2$ be the set of all homogeneous polynomial solutions to Stokes equations in R^n of degree k and $k-1$ respectively. Then the dimension of $\mathcal{S}(n,k)$ is $n[\binom{n+k-1}{k} - \binom{n+k-3}{k-2}]$.*

Now we decide the structure of singular homogeneous solutions.

LEMMA 2.4. *Let $k \geqslant 0$. Suppose that*

$$v(x) = |x|^{-2k-n} P_{k+2}(x) \qquad \text{and} \qquad q(x) = |x|^{-2k-n} P_{k+1}(x)$$

are solutions to Stokes equations for homogeneous polynomials P_{k+2} and P_{k+1} with degree $k+2$ and $k+1$ respectively. Then we have

$$v(x) = |x|^{-2k-n+2} H_k(x) + \frac{2k+n}{2(k+n-1)} |x|^{-2k-n} W_k(x) x$$

$$- \frac{1}{2(k+n-1)} |x|^{-2k-n+2} \nabla W_k(x)$$

and

$$q(x) = \frac{2k+n}{k+n-1} |x|^{-2k-n} W_k(x)$$

for some homogeneous harmonic polynomial $H_k(x)$ of degree k, where we defined $W_k(x)$ by

$$W_k(x) = (2k+n-2)\, x \cdot H_k(x) - |x|^2 \nabla \cdot H_k(x).$$

Proof. From Lemma 2.1 for the power series expansion of biharmonic polynomials, we know that

$$v(x) = |x|^{-2k-n} (|x|^2 H_k(x) + H_{k+2}(x))$$

and

$$q(x) = |x|^{-2k-n} H_{k+1}(x)$$

for some homogeneous harmonic polynomials H_k, H_{k+1} and H_{k+2} of degree $k, k+1$ and $k+2$ respectively. Thus from direct calculations

$$\Delta v = (\Delta\, |x|^{-2k-n})\, H_{k+2} + 2(\nabla\, |x|^{-2k-n} \cdot \nabla)\, H_{k+2}(x).$$

From homogeneity we know that

$$x \cdot \nabla H_{k+2}(x) = (k+2)\, H_{k+2}(x).$$

Hence we have

$$\Delta v = -2(2k+n)\, |x|^{-2k-n-2}\, H_{k+2}(x) = \nabla q$$

and

$$H_{k+2}(x) = \frac{1}{2} H_{k+1}(x) x - \frac{1}{2(2k+n)} |x|^2 \nabla H_{k+1}. \tag{2.3}$$

Also taking the divergence of v we have

$$(-2k-n+2)\, |x|^{-2k-n}\, x \cdot H_k(x) + |x|^{-2k-n+2}\, \nabla \cdot H_k(x)$$
$$+(-2k-n)\, |x|^{-2k-n-2}\, x \cdot H_{k+2}(x) + |x|^{-2k-n}\, \nabla \cdot H_{k+2}(x) = 0$$

and

$$(-2k-n+2) x \cdot H_k(x) + |x|^2\, \nabla \cdot H_k(x)$$
$$= (2k+n) \frac{x}{|x|^2} \cdot H_{k+2}(x) - \nabla \cdot H_{k+2}.$$

On the other hand from (2.3)

$$x \cdot H_{k+2}(x) = \frac{1}{2} |x|^2 H_{k+1}(x) - \frac{k+1}{2(2k+n)} |x|^2 H_{k+1}(x)$$
$$= \frac{k+n-1}{2(2k+n)} |x|^2 H_{k+1}$$

and apply the Laplacian to both sides

$$\nabla \cdot H_{k+2}(x) = \frac{(k+n-1)(n+2k+2)}{2(2k+n)} H_{k+1}(x).$$

Thus we have

$$(-2k-n+2)x \cdot H_k(x) + |x|^2 \nabla \cdot H_k = -\frac{k+n-1}{2k+n} H_{k+1}(x)$$

and solving for H_{k+1} gives

$$H_{k+1}(x) = \frac{(2k+n-2)(2k+n)}{k+n-1} x \cdot H_k(x) - \frac{2k+n}{k+n-1} |x|^2 \nabla \cdot H_k(x). \tag{2.4}$$

From the expression of H_{k+1} in terms of H_k we conclude that

$$q(x) = \frac{2k+n}{k+n-1} |x|^{-2k-n} W_k(x).$$

Also plugging (2.4) to (2.3) we find that

$$H_{k+2}(x) = \frac{(2k+n-2)(2k+n)}{2(k+n-1)} (x \cdot H_k(x))\, x - \frac{2k+n-2}{2(k+n-1)} |x|^2 \nabla \cdot H_k(x) x - \frac{2k+n-2}{2(k+n-1)} |x|^2 \nabla(x \cdot H_k(x)) + \frac{1}{2(k+n-1)} |x|^4 \nabla(\nabla \cdot H_k(x)).$$

We observe that

$$|x|^2 \nabla(\nabla \cdot H_k(x)) = \nabla(|x|^2 \nabla \cdot H_k(x)) - 2(\nabla \cdot H_k(x))x$$

and

$$H_{k+2}(x) = \frac{2k+n}{2(k+n-1)} ((2k+n-2)(x \cdot H_k(x)) - |x|^2 \nabla \cdot H_k(x))x - \frac{1}{2(k+n-1)} |x|^2 \nabla((2k+n-2)x \cdot H_k - |x|^2 (\nabla \cdot H_k(x))).$$

Now recall that we defined

$$W_k(x) = (2k+n-2)x \cdot H_k(x) x - |x|^2 \nabla \cdot H_k(x).$$

Therefore we conclude that

$$H_{k+2}(x) = \frac{2k+n}{2(k+n-1)} W_k(x)x - \frac{1}{2(k+n-1)} \nabla W_k(x).$$

From this we can express the velocity v in terms of $H_k(x)$. This completes the proof.

Now we study the structure of solutions according to dimensions. The power series expansion of the solution to Stokes equations are different from biharmonic functions, in fact, more restrictive. Also the series expansion appears differently according to dimension. Indeed the cases $n=2$ and 4 are different from the other dimensions.

First we let $n=4$. Since the pressure $p(x)$ is harmonic, no logarithmic function is involved in the expansion and we have

$$p(x) = \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} H_{km}(x) + \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} |x|^{-2k-2} J_{km}(x)$$

for some homogeneous harmonic polynomials $H_{km}(x)$ and $J_{km}(x)$ of degree k. Moreover we know that

$$\nabla(|x|^{-2k-2} J_{km}(x)) = o(|x|^{-2})$$

for all $k \geqslant 0$. Since $\Delta \ln |x| = c\,|x|^{-2}$ and $\Delta u = \nabla p$, we find that $c_4 = 0$ in the expression (2.1) of u. Since $c_4 = 0$, there are no differences between the case $n=4$ and the case $n \geqslant 3$. Thus we assume $n \geqslant 3$ and there is no term involving $\ln |x|$. Now we consider the terms with the same growth of fundamental solution. We assume that $v(x) = |x|^{-n} P_2(x)$ and $q(x) = |x|^{-n} J_1(x)$, which appear in the power series expansion of u and p, are solutions to Stokes equations, where $P_2(x)$ and $J_1(x)$ are homogeneous polynomials of degree 2 and 1 respectively. Therefore taking $k=0$ in Lemma 2.4, we have the following corollary which is useful in deciding the structure of fundamental solutions.

COROLLARY 2.5. *Let $n \geqslant 3$. Suppose that $v(x) = |x|^{-n} P_2(x)$, $q(x) = |x|^{-n} J_1(x)$ are solutions to Stokes equations, then*

$$v(x) = \frac{1}{|x|^n} (A \cdot x)x + \frac{1}{n-2} \frac{1}{|x|^{n-2}} A$$

and

$$q(x) = 2\frac{1}{|x|^n} A \cdot x$$

for some constant vector A.

Define $V_m(x) = (V_m^1(x), V_m^2(x), ..., V_m^n(x))$ and $Q_m(x)$ by

$$V_m^i(x) = \frac{x^m x^i}{|x|^n} + \frac{\delta_{mi}}{n-2}\frac{1}{|x|^{n-2}}$$

$i = 1, 2, ..., n$ and

$$Q^m(x) = \frac{2x_m}{|x|^n}.$$

Then we choose A as the m-the unit vector in Lemma 2.4 and hence we have the following corollary.

COROLLARY 2.6. *Let $n \geqslant 3$. The set of all homogeneous solutions (v, q) of degree $-n+2$ and $-n+1$ is n-dimensional vector space and $\{(V_m, Q_m), m = 1, ..., n\}$ is the basis.*

Let $n = 2$. Since the pressure $p(x)$ is harmonic in $\Omega \backslash \{0\}$, we can write as

$$p(x) = \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} H_{km}(x) + \sum_{k=1}^{\infty} \sum_{m=1}^{m_k} |x|^{-2k} J_{km}(x) + c_4 \ln |x| \qquad (2.5)$$

for some homogeneous harmonic polynomials H_{km} and J_{km} and some constant c(see the book by Kellog [8]). Also we know that

$$\Delta[c_1 \ln |x| + (c_2 \cdot x) \ln |x| + c_3 |x|^2 \ln |x|] = 4c_3(1 + \ln |x|).$$

Hence considering the order of growth of $\nabla p = \Delta u$ as in the case $n = 4$, we conclude

$$c_3 = 0$$

in the power series expansion (2.1) of u and (2.5) of p. From direct computation we find

$$\Delta(x \ln |x|) - \nabla(2 \ln |x|) = 0.$$

Consequently, if $(c_2 \cdot x) \ln |x|$ satisfies Stokes equations, we have

$$c_2 = \tfrac{1}{2} c_4 I,$$

where I is the identity matrix. But, $\nabla \cdot (x \ln |x|) \neq 0$ and

$$c_2 = 0.$$

Now we decide the structure condition of the terms of the same growth to fundamental solutions. From the order of growth

$$\begin{aligned} v(x) &= \sum_{m=1}^{m_2} b_{2m} S_{2m} \left(\frac{x}{|x|} \right) + c_1 \ln |x| \\ &= |x|^{-2} H_2(x) + c_1 \ln |x| \\ q(x) &= |x|^{-2} J_1(x) \end{aligned}$$

is a solution to Stokes equation, where $H_2(x)$ and $J_1(x)$ are homogeneous harmonic polynomials of degree 2 and 1 respectively and c_1 is a constant vector. Now we find that from $\Delta v = \nabla q$

$$-\frac{4}{|x|^4} H_2(x) = -\frac{2x}{|x|^4} J_1(x) + |x|^{-2} \nabla J_1(x)$$

and

$$H_2(x) = \tfrac{1}{2} J_1(x) x - \tfrac{1}{4} |x|^2 \nabla J_1(x).$$

Since $\nabla \cdot v = 0$, we see that

$$\nabla \cdot H_2(x) \, |x|^{-2} - 2H_2(x) \cdot x \, |x|^{-4} + c_1 \cdot x \, |x|^{-2} = 0.$$

Hence we find that

$$c_1 \cdot x = 2H_2 \cdot x \, |x|^{-2} - \nabla \cdot H_2(x).$$

Now since $J_1(x)$ is a linear function, we can set $J_1(x) = A \cdot x$ for some constant vector A. Consequently we conclude that

$$H_2(x) = \tfrac{1}{2} (A \cdot x) x - \tfrac{1}{4} \, |x|^2 \, A$$

and

$$c_1 = -\tfrac{1}{2} A.$$

LEMMA 2.7. *Let* $n = 2$. *Suppose that* $v(x) = |x|^{-2} H_2(x) + c_1 \ln |x|$ *and* $q(x) = |x|^{-2} J_1(x)$ *are solutions to Stokes equations, then*

$$v(x) = \tfrac{1}{2} \, |x|^{-2} \, (A \cdot x) x - \tfrac{1}{4} A - \tfrac{1}{2} \, A \ln |x|$$

and

$$p(x) = |x|^{-2} A \cdot x$$

for some constant vector A.

Define (V_m, Q_m), $m = 1, 2$ by

$$V_m^i(x) = \frac{1}{2} \frac{x_m x_i}{|x|^2} - \frac{1}{4} \delta_{mi} - \frac{1}{2} \delta_{mi} \ln |x|,$$

$i = 1, 2$ and

$$Q_m(x) = \frac{x_m}{|x|^2}.$$

COROLLARY 2.8. *Let* $n = 2$. *Then, the set of all non-constant solutions which are homogeneous degree* 0, *modulo constant vector, is* 2-*dimensional vector space and* $\{(V_m, Q_m)\colon m = 1, 2\}$ *are the basis.*

Now we can characterize the fundamental solutions. The theorem for harmonic functions is known as Bôcher's theorem.

THEOREM 2.9. *Let* $n \geqslant 2$ *and* $B_\delta(0) \subset \Omega$. *Suppose that* (u, p) *are solutions to Stokes equations in* $\Omega \backslash \{0\}$. *We fix L a constant and define*

$$E_L = \{e \in R^n \colon e \cdot u(x) > L \text{ for all } x \in B_\delta(0)\}.$$

Then the interior of E_L *is nonempty for some L if and only if* (u, p) *satisfy*

$$u(x) = v(x) + \sum_{m=1}^{n} a_m V_m(x)$$

and

$$p(x) = q(x) + \sum_{m=1}^{n} a_m Q_m(x)$$

for some constants a_m, $m = 1, ..., n$, *where* (v, q) *are solutions to Stokes equations in whole domain* Ω.

Proof. For simplicity we assume $n \geqslant 3$. The case $n = 2$ follows under the same argument with little modifications. Now we prove the sufficient part. Suppose that u can written by

$$u(x) = v(x) + \sum_{m=1}^{n} a_m V_m(x)$$

for some smooth solution v in $B_\delta(0)$. We assume that $(a_1, a_2, ..., a_n) \neq 0$, otherwise it holds trivially. We choose $e = (a_1, a_2, ..., a_n)$, then

$$\sum_{m=1}^{n} a_m V_m \cdot e = \frac{1}{|x|^{n-2}} \sum_{m=1}^{n} \sum_{i=1}^{n} a_m a_i \left(\frac{x_m x_i}{|x|^2} + \frac{\delta_{mi}}{n-2} \right) > \frac{|a|}{(n-2)\,|x|^{n-2}}.$$

Since v is bounded in $B_\delta(0)$, E_L has nonempty interior for some fixed constant L.

We prove the necessary part. Since E_L has nonempty interior, we can find a constant vector e and a small ball $B_\varepsilon(e)$ centered at e such that $B_\varepsilon(e) \subset E_L$. Adding a constant vector to u we can assume $L = 0$. We know that u can be written as

$$u(x) = v(x) + \sum_{k=1}^{\infty} \sum_{m=1}^{m_k} |x|^{-k-n+2} \left[a_{km} S_{km}\left(\frac{x}{|x|}\right) + b_{(k+2)m} S_{(k+2)m}\left(\frac{x}{|x|}\right) \right]$$
$$+ \frac{1}{|x|^n} (A \cdot x) x + \frac{1}{(n-2)\,|x|^{n-2}} A$$

and

$$p(x) = q(x) + \sum_{k=2}^{\infty} \sum_{m=1}^{m_k} |x|^{-k-n+2} c_{km} S_{km}\left(\frac{x}{|x|}\right) + 2 \frac{1}{|x|^n} A \cdot x$$

for some constant vector A. Moreover (v, q) are solutions to Stokes equations in whole Ω. Thus (v, q) are bounded in $B_\delta(0)$. Define $\omega = x/|x|$. Now we choose ε so small that $e + \varepsilon S_{jl}(\omega) \subset E_0$. Recall S_{jl} is an element of orthonormal basis of spherical harmonics of degree j. Then we multiply $e + \varepsilon S_{jl}(\omega)$ to u and integrate on sphere S_ρ^{n-1} centered at the origin with small radius $\rho \leqslant \delta$. Thus from the orthogonality and the fact that $\int_{S_1^{n-1}} S_{jl}(\omega)\, d\omega = 0$ for $j \geqslant 1$, we have

$$0 \leqslant \int_{S_\rho^{n-1}} u \cdot (e + \varepsilon S_{jl})\, d\sigma_x = \rho^{n-1} \int_{S_1^{n-1}} u \cdot (e + \varepsilon S_{jl})\, d\omega$$
$$= \rho^{n-1} \int_{S_1^{n-1}} v \cdot (e + \varepsilon S_{jl})\, d\omega + \varepsilon(\rho^{-j+1} a_{jl} + \rho^{-j+3} b_{jl}),$$

where $b_{jl} = 0$ for $j = 0$, 1 and 2. Hence if $a_{jl} \neq 0$, then we can choose ρ very small and $\varepsilon a_{jl} < 0$. This contradicts to

$$0 \leqslant \int_{S_\rho^{n-1}} u \cdot (e + \varepsilon S_{jl})\, d\sigma_x$$

for small $\rho > 0$. Similarly we conclude that $b_{jl} = 0$. This implies $a_{km} = b_{km} = 0$ for all $k \geqslant 1$. Thus from the structure theorem we also have $c_{km} = 0$. Therefore

$$u(x) = v(x) + \frac{1}{|x|^n}(A \cdot x)x + \frac{1}{(n-2)\,|x|^{n-2}} A$$

and

$$p(x) = q(x) + 2\frac{A \cdot x}{|x|^n}$$

and this completes the proof.

3. EXTERIOR PROBLEMS FOR STOKES EQUATIONS

In this section we consider the exterior problems for Stokes equations. Since we are interested in the behavior of solutions at infinity, we assume that $\Omega = R^n \backslash B_1(0)$.

We study the asymptotic behavior of solutions at infinity. We list a lemma which describe the asymptotic growth of higher derivatives of solutions. (See Lemma 1 in [3].) Define $\Omega_R = \Omega \cap B_R$.

Lemma 3.1. *Suppose that u is a solution to Stokes equations and for sufficiently large R*

$$\int_{\Omega_R} |u|^2\, |x|^{-s}\, dx < K(R)$$

for some real number s. Then for any nonnegative integer k

$$|\nabla^k u(x)| = O(|x|^{s/2}\, K^{1/2}(|x|)).$$

The following theorem shows the asymptotic behavior of solutions at infinity.

Theorem 3.2. *Let (u, p) be solutions to Stokes equations in Ω. Suppose that as $|x| \to \infty$, $u(x) = o(|x|^m)$ for some nonnegative integer m. Then there exist homogeneous polynomial solutions $(H^1_k(x), H^2_{k-1}(x))$ with nonnegative degree $0 \leqslant k \leqslant m_0 < m$ and homogeneous solutions $(J^1_{-k-n+2}(x), J^2_{-k-n+1}(x))$ with negative degree for $k \geqslant 0$ to Stokes equations such that*

$$u(x)=\sum_{k=0}^{m_0} H_k^1(x)+\sum_{k=0}^{\infty} J^1_{-k-n+2}(x)$$

$$p(x)=\sum_{k=1}^{m_0} H_{k-1}^2(x)+\sum_{k=0}^{\infty} J^2_{-k-n+1}(x),$$

where $c=0$ *if* $m\leqslant 1$. *Moreover we have*

$$\left|\sum_{k=0}^{\infty} J^1_{-k-n+2}\right|\leqslant c\,|x|^{-n+2}, \qquad n\geqslant 3$$

$$\leqslant c\ln|x|, \qquad n=2$$

for some c *for all* $x\in\Omega$ *and*

$$\left|\sum_{k=0}^{\infty} J^2_{-k-n+1}(x)\right|\leqslant c\,|x|^{-n+1}$$

for some c *and for all* $x\in\Omega$.

Proof. The solutions in the annulus Ω_R can be represented by surface potentials. We let ω_n be the surface area of unit sphere in R^n. First we list the single layer and double layer surface potentials of the solutions of Stokes equations:

$$\begin{aligned} u^k(x)&=V_R^k(x, T_{ij}(u))+W_R^k(x,u)-V_1^k(x, T_{ij}(u))-W_1^k(x,u)\\ &=u_R^k(x)-u_1^k(x)\\ p(x)&=E_R(x, T_{ij}(u))+F_R(x,u)-E_1(x, T_{ij}(u))-F_1(x,u)\\ &=p_R(x)-p_1(x)\\ T_{ij}(u)&=-\delta_{ij}p+\left(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\right), \end{aligned}$$

where

$$V_R^k(x,u)=\frac{1}{2\omega_n}\int_{S_R}\left[\frac{\delta_{ij}}{n-2}\frac{1}{|x-y|^{n-2}}+\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^n}\right]T_{ik}(u)\frac{y_j}{R}\,d\sigma_y$$

$$W_R^k(x,u)=-\frac{n}{\omega_n}\int_{S_R}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^{n+2}}\frac{y_j}{R}\,u^i(y)\,d\sigma_y$$

$$E_R(x,T)=\frac{1}{\omega_n}\int_{S_R}\frac{x_k-y_k}{|x-y|^n}\,T_{kj}(u)\frac{y_j}{R}\,d\sigma_y$$

$$F_R(x,u)=\frac{2}{\omega_n}\int_{S_R}\left[\frac{\delta_{ij}}{|x-y|^n}-n\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{n+2}}\right]\frac{y_j}{R}\,u^i(y)\,d\sigma_y$$

when $n \geqslant 3$ and

$$V_R^k(x, u) = -\frac{1}{4\pi} \int_{S_R} \left[\delta_{ij} \ln |x-y| - \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} \right] T_{ik}(u) \frac{y_j}{R} \, d\sigma_y$$

$$W_R^k(x, u) = -\frac{1}{\pi} \int_{S_R} \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|x-y|^4} \frac{y_j}{R} u^i(y) \, d\sigma_y$$

$$E_R(x, T) = -\frac{1}{2\pi} \int_{S_R} \frac{x_k - y_k}{|x-y|^2} T_{kj}(u) \frac{y_j}{R} \, d\sigma_y$$

$$F_R(x, u) = -\frac{1}{\pi} \int_{S_R} \left[\frac{\delta_{ij}}{|x-y|^2} - 2 \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^4} \right] \frac{y_j}{R} u^i(y) \, d\sigma_y$$

when $n = 2$. From the growth condition we have

$$\int_{\Omega_R} |u|^2 \, |x|^{-2m-n-\varepsilon} \, dx \leqslant c$$

for all $\varepsilon > 0$ and for some c. Thus from Lemma 3.1 $\nabla^k u(x) = o(|x|^{m+(n/2)+1})$ for all k and $p(x) = o(|x|^{m+(n/2)+1})$. Thus from the integral representation we have

$$|\nabla^k V_R(x)|, \qquad |\nabla^k W_R(x)| \leqslant c(x) \, R^{m+(n/2)+2-k}$$

and for all $k > m + 2 + (n/2)$ we have

$$\lim_{R \to \infty} |\nabla^k V_R(x)| = 0 \qquad \text{and} \qquad \lim_{R \to \infty} |\nabla^k W_R(x)| = 0.$$

Moreover since $u_R(x) = u(x) + u_1(x)$ is independent of R, we find that $u_R(x)$ is a polynomial. Then from the growth condition $u_R(x)$ is a polynomial of degree $m_0 < m$. From the characterization of homogeneous solutions we find the expression of homogeneous solutions of nonnegative degree.

From the definition of surface potential V_1 and W_1 we have

$$\begin{aligned} |u_1(x)| &\leqslant c \, |x|^{-n+2}, && n \geqslant 3 \\ &\leqslant \ln |x|, && n = 2 \end{aligned}$$

for some c and for all $x \in \Omega$. Similarly we have a bound for p_1. It remains to find a power series expansion of u_1 and p_1 in terms of homogeneous solutions of negative degree. The expression of the part of negative degree can be obtained by Taylor series expansion of fundamental solution. We let Γ_{ij} be the fundamental tensor such that

$$\Gamma_{ij}(x) = \frac{\delta_{ij}}{n-2} \frac{1}{|x|^{n-2}} + \frac{x_i x_j}{|x|^n} \qquad n \geqslant 3$$

$$\Gamma_{ij}(x) = -\delta_{ij} \ln |x| + \frac{x_i x_j}{|x|^2} \qquad n = 2.$$

Then since $\Gamma_{ij}(x)$ is analytic for all large $|x|$, we can have for $|y| = 1$ and large $|x|$

$$\Gamma_{ij}(x-y) = \Gamma_{ij}(x) - \nabla(\Gamma_{ij}(x)) \cdot (y) + \frac{1}{2!} \nabla^2(\Gamma_{ij}(x)) \cdot (-y) \cdot (-y)$$
$$+ \cdots + \frac{(-1)^k}{k!} \nabla^k(\Gamma_{ij}) \cdot y \cdot y \cdot \cdots \cdot y + O(|x|^{-n-k+1}).$$

Therefore from the integral expression we have

$$V_1^k(x) = \Gamma_{ij}(x) \int_{S_1} T_{ik}(u)\, y_j \, d\sigma_y - \nabla(\Gamma_{ij}(x)) \int_{S_1} \cdot y T_{ik}(u)\, y_j \, d\sigma_y$$
$$+ \cdots + \frac{(-1)^k}{k!} \nabla^k(\Gamma_{ij}(x)) \int_{S_1} \cdot y \cdot y \cdot \cdots \cdot y T_{ik}(u)\, y_j \, d\sigma_y$$
$$+ O(|x|^{-n-k+1}).$$

Here we note that $\nabla^k(\Gamma_{ij}(x))$ is homogeneous Stokes solution of degree $-n-k+2$. Similarly we can express $W_R(x)$, $E_R(x)$ and $F_R(x)$ by Taylor series expansions. Combining all these together we find the series expansion in terms of homogeneous solutions.

The case $m = 1$ in Theorem 3.2 has been considered by Chang and Finn (see Theorem 1 in [3]).

THEOREM 3.3. *We fix a constant L and define*

$$F_L = \{ e \in R^n : e \cdot u(x) > L \text{ for all } x \in R^n \backslash B_R(0) \}.$$

Suppose $u(x) = O(|x|^m)$ for some nonnegative integer m. Then the interior of F_L is nonempty for some large R if and only if u is bounded when $n \geqslant 3$ and $u = O(\ln |x|)$ when $n = 2$.

Proof. First we assume $n \geqslant 3$. The case $n = 2$ follows under the same argument with little modifications. The sufficient part is rather obvious.

Indeed, if u is bounded in Ω, then F_L has nonempty interior for sufficiently large negative number L.

We prove the necessary part. Suppose F_L has nonempty interior such that $B_\delta(e) \subset F_L$, where e is a constant vector. Adding a constant vector we can assume $L=0$. Since $u = O(|x|^m)$, from Theorem 3.2 we know that u can be written as

$$u(x) = \sum_{k=0}^{m_0} H_k^1(x) - u_1(x)$$

$$p(x) = \sum_{k=1}^{m_0} H_{k-1}^2(x) - p_1(x)$$

for some homogeneous polynomial solutions H_k^1 and H_k^2 of degree k. Also $u_1(x) = O(|x|^{-n+2})$ and $p_1(x) = O(|x|^{-n+1})$. As we know, (H_k^1, H_{k-1}^2), $k = 1, ..., m_0$ are homogeneous solutions and their structure is decided by

$$H_k^1(x) = \sum_j a_{jk}\, |x|^k\, S_{jk}(\omega) + b_{j(k-2)}\, |x|^k\, S_{j(k-2)}(\omega),$$

where we defined $\omega = x/|x|$. Now we choose $|\varepsilon|$ so small that $e + \varepsilon S_{jl}(\Omega) \subset F_0$, where S_{jl} is the spherical harmonic. Then we multiply $e + \varepsilon S_{jl}(\omega)$, $l \geqslant 1$ to u and integrate on sphere S_ρ^{n-1} centered at the origin with large radius $\rho \geqslant 1$. Thus from the orthogonality and the fact that $\int_{S_1^{n-1}} S_{jl}(\omega)\, d\omega = 0$ for $j \geqslant 1$,

$$0 \leqslant \int_{S_\rho^{n-1}} u \cdot (e + \varepsilon S_{jl})\, d\sigma_x = \rho^{n-1} \int_{S_1^{n-1}} u \cdot (e + \varepsilon S_{jl})\, d\omega$$

$$= \rho^{n-1} \int_{S_1^{n-1}} (H_0^1 - u_1) \cdot (e + \varepsilon S_{jl})\, d\omega + \varepsilon(\rho^{l+n-1} a_{jl} + \rho^{l+n+1} b_{jl}).$$

Hence if $b_{jl} \neq 0$, then we choose ρ very large and $\varepsilon b_{jl} < 0$. This contradicts to

$$0 \leqslant \int_{S_\rho^{n-1}} u \cdot (e + \varepsilon S_{jl})\, d\sigma_x$$

for all $\rho > 0$. Similarly we conclude that $a_{jl} = 0$ for all $l \geqslant 1$. Hence we get $a_{jl} = 0$ for all $l \geqslant 1$ and $b_{jl} = 0$ for all $l \geqslant 0$. Therefore

$$u(x) = c_0 - u_1(x) \qquad \text{and} \qquad p(x) = -p_1(x)$$

and this completes the proof.

Now we discuss the celebrated Stokes example. When $n=3$, Stokes derived in 1851 a remarkable explicit solution (I_l, J_l) given by

$$I_l^i(x) = \frac{3}{4}\frac{x_i x_l}{|x|^3}\left(\frac{1}{|x|^2}-1\right) + \left(1-\frac{3}{4}\frac{1}{|x|}-\frac{1}{4}\frac{1}{|x|^3}\right)\delta_{il}$$
$$J_l(x) = -\frac{3}{2}\frac{x_l}{|x|^3} \tag{3.1}$$

for each $l=1, 2, 3$. (See p. 239, Ch. 5 in [6].) We note $I_l=0$ on the unit sphere and $\lim_{|x|\to\infty} I_l(x)=e_l$, e_l is the lth unit vector. Now we show this Stokes solution is a direct consequence of Lemma 2.4 and Corollary 2.5. Indeed we assume $n=3$ in Lemma 2.4. We choose the homogeneous harmonic vector valued function $H_2(x)$ by

$$H_2(x) = \tfrac{1}{4}(2x_1^2 - x_2^2 - x_3^2,\ 3x_1x_2,\ 3x_1x_3)^t.$$

From Lemma 2.4 we find that

$$(U_1(x), P_1(x)) = (|x|^{-5}\, H_2(x), 0)$$

are solutions to Stokes equations. Taking the fundamental solution $(V_1(x), Q_1(x))$ in Corollary 2.6, we find that

$$(I_1(x), J_1(x)) = (U_1(x), P_1(x)) + \tfrac{3}{4}(V_1(x), Q_1(x)) + (e_1, 0)$$

are the Stokes example, where (V_1, Q_1) are the fundamental solutions defined in Corollary 2.6.

When the dimension is 2 and $u=0$ on $\partial B_1(0)$,

$$\limsup_{|x|\to\infty} |u(x)| = \infty \qquad \text{or} \quad u(x)=0.$$

This is known as Stokes paradox and investigated in great detail by many authors. Here we show that when the dimension is 3, the Stokes examples characterize completely the solutions satisfying the boundary conditions that $u=0$ on $\partial\Omega$. The following theorem can be considered as Stokes paradox for three dimension.

THEOREM 3.4. *Let $n=3$. Suppose that (u, p) are solutions to Stokes equations in Ω and $u=0$ on S_1^2 and bounded. Then u is in the span of the Stokes examples $\{I_l, l=1, 2, 3\}$ which are defined in* (3.1).

Proof. Since u is bounded, we find that from Theorem 3.2

$$u(x) = C_0 - V_1(x) - W_1(x).$$

Since $u=0$ on S_1^2, $V_1(x)=0$. Hence $W_1(x)=C_0-u(x)$ and $W_1(x)=C_0$ for $x\in S_1^2$ and

$$\lim_{|x|\to\infty} W_1(x)=0.$$

Then $\alpha^i(x)=I_l^i(x)C_0^l$ satisfies that $\alpha^i(x)=0$ for all $x\in S_1^2$ and

$$\lim_{|x|\to 0} \alpha^i(x)=\delta_{il}C_0^l=C_0^i.$$

We let $\beta(x)=W_1(x)-C_0+\alpha(x)$, then $\beta(x)=0$ on S_1^2 and

$$\lim_{|x|\to\infty} \beta(x)=0.$$

From the uniqueness of Finn and Noll (see [5]) we conclude that

$$\beta(x)=0 \qquad \text{for all} \quad x\in\Omega$$

and $\beta(x)=0$ for all x. Therefore $u(x)=\alpha(x)$ and completes the proof.

4. EXTERIOR PROBLEMS FOR NAVIER–STOKES EQUATIONS

In this section we study the behavior of solutions to Navier–Stokes equations for exterior domain in R^3. For simplicity we assume $\Omega=R^3\backslash B_1(0)$. We define a homogeneous Sobolev space $H^0(\Omega)$ by completion of $C_0^\infty(\Omega)$ under the seminorm $\|\phi\|=\int_\Omega |\nabla u|^2\,dx$ and $H_\sigma^0(\Omega)$ by the set of all solenoidal vector field in $H^0(\Omega)$. We list a Poincare type inequality suggested by Finn. (See Corollary 2.2 in [4].) Define $\Omega^R=R^3\backslash B_R(0)$.

LEMMA 4.1. *Let $\phi(x)$ be a continuous function and have a generalized first derivative in Ω^R. Let $x=0$ or $|x|\geqslant 2R$. Then there is a constant vector ϕ_0 such that*

$$\int_{\Omega^R} \frac{|\phi(y)-\phi_0|}{|x-y|^2}\,dy\leqslant K\int_{\Omega^R} |\nabla\phi(y)|^2\,dy,$$

where $K=3+2\sqrt{2}<6$. Moreover if $\phi\in H^0(\Omega^R)$, then $\phi_0=0$.

The following lemma is useful in studying asymptotic behavior of solutions.

LEMMA 4.2. *Suppose that* $\phi \in H^0(\Omega)$. *Then for any given* ε_0 *there exists* R_0 *such that for all* $|x| > R_0$

$$\int_\Omega \frac{|\phi(y)|^2}{|x-y|^2}\, dy \leqslant \varepsilon_0.$$

Proof. Let $\varepsilon > 0$. Then we can find a large number R such that

$$\int_{\Omega^R} |\nabla \phi(y)|^2\, dy < \varepsilon.$$

From Lemma 4.1 we have that for $|x| \geqslant 2R$

$$\int_{\Omega^R} \frac{|\phi(y)|^2}{|x-y|^2}\, dy \leqslant 6 \int_{\Omega^R} |\nabla \phi(y)|^2\, dy \leqslant 6\varepsilon.$$

If $|x| \geqslant 2R/\sqrt{\varepsilon}$, then for sufficiently small ε we have $|y|^2/|x-y|^2 \leqslant \varepsilon$ for all $y \in B_R$. Hence we get

$$\int_{B_R \backslash B_1} \frac{|\phi(y)|}{|x-y|^2}\, dy \leqslant \varepsilon \int_{B_R \backslash B_1} \frac{|\phi(y)|}{|y|^2}\, dy$$

$$\leqslant 6\varepsilon \int_\Omega |\nabla \phi(y)|^2\, dy.$$

Since ε is chosen arbitrarily and $\int_\Omega |\nabla \phi(y)|^2\, dy$ is bounded, we complete the proof.

DEFINITION 4.3. A field $u(x) \in L^2(\Omega)$ is said to be a generalized solution to Navier–Stokes equations if

$$\int_\Omega u \cdot \Delta \phi\, dx + \int_\Omega u^i u^j \phi^j_{x_i}\, dx = 0$$

for all $\phi \in C^1_0(\Omega \to R^3)$ with $\nabla \cdot \phi = 0$ and

$$\int_\Omega u \cdot \nabla \phi\, dx = 0$$

for all $\phi \in C^\infty_0(\Omega)$.

From the local regularity and boundary regularity theory we can assume that $u \in H^0_\sigma(\Omega)$ is smooth in Ω. Now from integration by parts we have potential expression such that for $R > |x| > 1$

$$u^k(x) = W^k_R(x) + V^k_R(x) + Y^k_R - W^k_1 - V^k_1(x) - Y^k_1(x) + Z^k_{R,1}(x)$$
$$p(x) = F_R(x) + E_R(x) + G_R(x) - F_1(x) - E_1(x) - G_1(x) + H_{R,1}(x),$$

where V_r, W_r, E_r, F_r are defined in the proof of Theorem 3.2 and

$$Y^k_R(x) = \int_{S_R} K_{kj}(x-y)\, u^i(y)\, u^j(y) \frac{y_i}{R}\, d\sigma_y$$

$$G_R(x) = \frac{1}{\omega_3} \int_{S_R} K_j(x-y) \frac{y_i}{R} u^i(y)\, u^j(y)\, d\sigma_y$$

$$Z^k_{R,1} = -\frac{1}{2\omega_3} \int_{B_R \backslash B_1} \left[\frac{\delta_{jk}(x_i - y_i)}{|x-y|^3} - \frac{\delta_{ij}(x_k - y_k)}{|x-y|^3} - \frac{\delta_{ki}(x_j - y_j)}{|x-y|^3} + 3\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|x-y|^5} \right] u^i(y)\, u^j(y)\, dy$$

$$H_{R,1} = -c_3 \int_{B_R \backslash B_1} \left[\frac{\delta_{ij}}{|x-y|^3} - n \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} \right] u^i(y)\, u^j(y)\, dy,$$

where K_{ij} and K_j are fundamental tensors for velocity and pressure.

THEOREM 4.4. *Let $n = 3$. Suppose $u \in H^0_\sigma(\Omega)$ is a solution to Navier–Stokes equations and F_L is nonempty for some L, then*

$$\lim_{R \to \infty} V_R + W_R + Y_R = 0.$$

for all $x \in \Omega$ and hence $u(x) = -W_1(x) - V_1(x) - Y_1(x) + Z_{\infty,1}(x)$.

Proof. Since $\nabla u \in L^2(\Omega)$, we find that $u \in L^6(\Omega)$ and $p \in L^3(\Omega)$. On the other hand from the local regularity theory we know that (u, p) are smooth in each compact subset of Ω. Now we fix $x \in \Omega$ and $|x| < R_0/2$. Thus we have from Hölder inequality

$$\int_{R_0}^\infty \left(\frac{1}{\rho} \int_{S_\rho} |\nabla u(y)|\, d\sigma_y \right)^2 d\rho \leqslant c \int_{R_0}^\infty \int_{S_\rho} |\nabla u|^2\, d\sigma_y\, d\rho \leqslant c\, \|\nabla u\|^2_\Omega$$

and

$$\int_{R_0}^\infty \left(\frac{1}{\rho^{43}} \int_{S_\rho} |p(y)|\, d\sigma_y \right)^3 d\rho \leqslant c \int_{R_0}^\infty \int_{S_\rho} |p(y)|^3\, d\sigma_y\, d\rho \leqslant c\, \|\nabla u\|^6_\Omega$$

for some c. Hence we conclude that

$$\liminf_{R\to\infty} \frac{1}{R^k} \int_{S_R} |\nabla u(y)| + |p(y)| \, d\sigma_y = 0 \tag{4.1}$$

for all $k \geqslant \frac{4}{3}$. Considering the fact that $u \in L^6(\Omega)$, we can obtain

$$\liminf_{R\to\infty} \frac{1}{R^k} \int_{S_R} |u(y)| \, d\sigma_y = 0 \tag{4.2}$$

for all $k \geqslant \frac{5}{3}$. Similarly we have

$$\liminf_{R\to\infty} \frac{1}{R^k} \int_{S_R} |u(y)|^2 \, d\sigma_y \tag{4.3}$$

for all $k \geqslant \frac{4}{3}$.

From the definition of surface potential V_R we have

$$\begin{aligned} |\nabla^k V_R(x, u)| &\leqslant c \int_{S_R} \left| \nabla_x^k \frac{1}{|x-y|} \right| (|\nabla u(y)| + |p(y)|) \, dy \\ &\leqslant \frac{c}{R^{k+1}} \int_{S_R} |\nabla u(y)| + |p(y)| \, d\sigma_y \end{aligned}$$

for $R > R_0$ and from (4.1) we get that for $k \geqslant 1$

$$\liminf_{R\to\infty} \nabla^k V_R(x) = 0.$$

Also from (4.2) and (4.3) we have that

$$\liminf_{R\to\infty} \nabla^k W_R(x) = 0 \qquad \text{and} \qquad \liminf_{R\to\infty} \nabla^k Y_R(x) = 0$$

for all $k \geqslant 0$. Therefore we have for some sequence $R_j \to \infty$

$$\lim_{R_j\to\infty} |\nabla^k (V_{R_j} + W_{R_j} + Y_{R_j}(x))| = 0$$

for $k \geqslant 1$ and $J(x) = \lim_{R_j\to\infty} V_{R_j}(x) + W_{R_j}(x) + Y_{R_j}(X)$ is a polynomial. Moreover since $V_R + W_R + Y_R$ is a solution to Stokes equations, we find that $J(x)$ is a polynomial solution to Stokes equations and satisfies the characterization lemma (see Lemma 2.2).

The volume potential $Z_{R,1}$ can be estimated by

$$|Z_{R,1}(x)| \leqslant c \int_{B_R \setminus B_1} \frac{|u(y)|^2}{|x-y|^2} \, dy$$

and hence from Lemma 4.2 we find that $Z_{R,1}(x)$ goes to zero uniformly as $|x|$ and R go to infinity, that is, for any given ε there exists R_0 such that if $R_0 < |x| < R$, then

$$\lim_{|x| \to \infty} Z_{R,1}(x) = \varepsilon.$$

Since F_L has nonempty interior, we can find a constant vector and a small ball $B_\delta(e) \subset F_L$. We multiply $e + \varepsilon S_{jl}(\omega) \in B_\delta(0)$ to u and integrate on unit sphere. Thus we get

$$0 \leqslant \int_{S_\rho} u \cdot (e + \varepsilon S_{jl})\, d\sigma_x = \rho^2 \int_{S_1} u \cdot (e + \varepsilon S_{jl})\, d\omega$$

$$= \rho^2 \int_{S_1} (V_R + W_R + Y_R) \cdot (e + \varepsilon S_{jl})\, d\omega$$

$$+ \rho^2 \int_{S_1} (V_1 + W_1 + Y_1 - Z_{R,1}) \cdot (e + \varepsilon S_{jl})\, d\omega.$$

Since $(V_1 + W_1 + Y_1)(x) = O(|x|^{-1})$ and $Z_{R,1}(x)$ goes to zero uniformly as $|x|$ and R go to infinity, we have

$$\rho^2 \int_{S_1} (V_1 + W_1 + Y_1 - Z_{R,1}) \cdot (e + \varepsilon)\, d\omega = o(\rho^2).$$

Therefore sending R to infinity at (4.5) we conclude that $J(x)$ is a constant vector C_0. If this is the case, from the assumption that $u \in H^0$ the constant vector C_0 should be zero. This completes the proof.

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