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Comparison results for parallel multisplitting methods with applications to AOR methods ☆

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Abstract

In this paper we present some comparison theorems between two general parallel multisplittings. These comparison theorems can be applied to many classical splittings, such as Jacobi, Gauss–Seidel and AOR splittings. Some significant improvements and generalizations of the existing comparison results are obtained. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Consider the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsingular. A multisplitting of A is a collection of matrices (M_l, N_l, E_l), $l = 1, \dots, k$, so that $A = M_l - N_l, l = 1, \dots, k$, where each M_l is

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nonsingular and each E_l is nonnegative diagonal with $\sum_{l=1}^{k} E_l = I$. A so-called (parallel) multisplitting iterative method is defined by

$$
\begin{aligned}
&\text{Given the initial vector } x^0.\\
&\text{For } s = 1, 2 \ldots \text{ until convergence.}\\
&\text{For } l = 1, \ldots, k,\\
&M_l y^{s,l} = N_l x^s + b\\
&x^{s+1} = \textstyle\sum_{l=1}^{k} E_l y^{s,l} = Hx^s + Gb,
\end{aligned}
\tag{1.2}
$$

where

$$H = \sum_{l=1}^{k} E_l M_l^{-1} N_l, \qquad G = \sum_{l=1}^{k} E_l M_l^{-1}. \tag{1.3}$$

This multisplitting method was first introduced by O'Leary and White [15] for solving large scale linear systems. They proved that when A is an inverse positive matrix and each of the splittings in (1.2) is weakly regular, the multisplitting method converges to the solution of (1.1) for any x^0, i.e.,

$$\rho\left(\sum_{l=1}^{k} E_l M_l^{-1} N_l\right) < 1,$$

where $\rho(\cdot)$ denotes the spectral radius. Kavanagh and Neumann [9] and Bru et al. [4] extended the result to singular systems in which the coefficient matrix is an M-matrix. Recently, the parallel multisplitting AOR methods have been discussed by many authors; see e.g. [2,12,17–20].

Let A be an inverse positive matrix, i.e., $A^{-1} \geqslant 0$,

$$A = M_l - N_l = \widetilde{M}_l - \widetilde{N}_l, \quad l = 1, \ldots, k,$$

be weak regular splittings of A, and let

$$H = \sum_{l=1}^{k} E_l M_l^{-1} N_l, \quad \widetilde{H} = \sum_{l=1}^{k} \widetilde{E}_l \widetilde{M}_l^{-1} \widetilde{N}_l \tag{1.4}$$

be their iteration matrices, respectively. The question of interest raised in [14] (or see [6]) is that if

$$M_l \leqslant \widetilde{M}_l, \quad l = 1, \ldots, k, \tag{1.5}$$

is it true that

$$\rho(H) \leqslant \rho(\widetilde{H}). \tag{1.6}$$

Neumann and Plemmons [14] studied this problem and gave some sufficient conditions (see [14, Theorem 2.1]). Their theorem strongly suggests that the rate of convergence of a parallel multisplitting iteration is inherent in the splitting alone and is independent of E_l. Usually, one can choose $\widetilde{E}_l = E_l$. In [6] Elsner provided a comparison theorem in the case $\widetilde{M}_l = \widetilde{M}$, $l = 1, 2, \ldots, k$, i.e., between a multisplitting and a single splitting, and gave a counterexample to illustrate that inequality

(1.6) is not true in general. Later, Climent and Perea extended Elsner's result; see [5, Theorems 3.3 and 3.4]. Comparison results for some classical multisplittings were obtained by many authors, such as Song for AOR multisplittings (see [18, Theorem 4.1]), Wang for AOR, Gauss–Seidel and Jacobi multisplittings (see [20, Theorems 2.1 and 2.2 and Theorem 3.1]), and Frommer and Pohl for block (non)overlapping multisplittings (see [7, Theorem 2.2]). But all the comparison results were established between a multisplitting and a single splitting.

The goal of this paper is to compare multisplittings. We shall give some sufficient conditions for E_l such that (1.6) holds. When our comparison theorem is applied to some classical splittings, e.g., AOR, Gauss–Seidel, and Jacobi, some improvements and generalizations of existing results are obtained.

2. Definitions and notations

Let $A \in \mathbb{R}^{n\times n}$. A matrix A is said to be reducible if there is a permutation matrix P such that

$$P^{\mathrm{T}}AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square. Otherwise A is called irreducible. It is well known that for any matrix there is a permutation matrix P such that

$$P^{\mathrm{T}}AP = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{ss} \end{bmatrix}, \tag{2.1}$$

where A_{ii} is irreducible or 1×1 zero, $i = 1, \dots, s$. Usually, the block upper triangular form (2.1) is called Frobenius normal form. Notice that $s = 1$ if and only if A is irreducible.

A matrix B is nonnegative or positive if each entry of B is nonnegative or positive, respectively. We denote them by $B \geqslant 0$ or $B > 0$, respectively. A matrix $A = (a_{ij})$ is called a Z-matrix if for any $i \neq j,\ a_{ij} \leqslant 0$, and an M-matrix if $A = sI - B,\ B \geqslant 0$ and $s \geqslant \rho(B)$, where $\rho(B)$ denotes the spectral radius of B. A is a nonsingular M-matrix if and only if $\rho(B) < s$. A nonsingular matrix A is said to be inverse positive if $A^{-1} \geqslant 0$. A matrix M is said to be a positive (nonnegative) diagonal matrix if M is a diagonal matrix with positive (nonnegative) elements.

Definition 2.1. Let A be an $n \times n$ matrix. The splitting $A = M - N$ is said to be:
- nontrivial if $N \neq 0$,
- convergent if $\rho(M^{-1}N) < 1$,
- regular if $M^{-1} \geqslant 0$ and $N \geqslant 0$,
- nonnegative [23] if $M^{-1} \geqslant 0$, $M^{-1}N \geqslant 0$ and $NM^{-1} \geqslant 0$,

- weakly nonnegative [23] of the first type (or weakly regular) if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$,
- weakly nonnegative [23] of the second type if $M^{-1} \geqslant 0$ and $NM^{-1} \geqslant 0$,
- a weak splitting [23] of the first type if $M^{-1}N \geqslant 0$; a weak splitting of the second type if $NM^{-1} \geqslant 0$,
- an M-splitting [16] if M is a nonsingular M-matrix and $N \geqslant 0$,
- block upper triangular [10] if there is a permutation similarity bringing A into Frobenius normal form as in (2.1) which also brings M into a block upper triangular form when partitioned conformally to A.

In general,

$$M\text{-splitting} \begin{cases} \Rightarrow \text{regular} \Rightarrow \text{nonnegative} \\ \quad \Rightarrow \text{weakly nonnegative (regular)} \Rightarrow \text{weak}, \\ \\ \Rightarrow \text{block upper triangular (see [10])}. \end{cases}$$

Definition 2.2. Let A be an $n \times n$ matrix and (M_l, N_l, E_l), $l = 1, \ldots, k$, be multisplitting of A, i.e., E_l is a nonnegative diagonal matrix and $\sum_{l=1}^{k} E_l = I$ and $A = M_l - N_l, l = 1, \ldots, k$. The multisplitting $(M_l, N_l, E_l), l = 1, \ldots, k$, is called
- convergent if $\rho(\sum_{l=1}^{k} E_l M_l^{-1} N_l) < 1$,
- nontrivial, (weakly) regular [9], nonnegative, weakly nonnegative [4] of the first (second) type, weak [4] of the first (second) type [5], M-splitting [5] and block upper triangular if each single splitting is nontrivial, (weakly) regular, nonnegative, weakly nonnegative of the first (second) type, weak of the first (second) type, M-multisplitting and block upper triangular, respectively.

3. Comparison results

Let $(M_l, N_l, E_l), l = 1, \ldots, k$ and $(\widetilde{M}_l, \widetilde{N}_l, E_l), l = 1, \ldots, k$, be two multisplittings of A, whose iteration matrices are denoted by H and $\widetilde{H}$ throughout this paper, respectively, i.e.,

$$H = \sum_{l=1}^{k} E_l M_l^{-1} N_l \tag{3.1}$$

and

$$\widetilde{H} = \sum_{l=1}^{k} E_l \widetilde{M}_l^{-1} \widetilde{N}_l. \tag{3.2}$$

In this section, we compare spectral radii of iteration matrices for parallel multisplitting methods. Before we discuss our main theorem, the following lemmas are needed. In the first lemma, the part (i) is Theorem 1.1 of O'Leary and White [15] and the part (ii) is (2.2) of Elsner [6]. The second lemma is Theorem 6.4 of Woźnicki [23].

Lemma 3.1. *Let* (M_l, N_l, E_l), $l = 1, \ldots, k$, *be a multisplitting of* A. *Then*

$$I - H = \left(\sum_{l=1}^{k} E_l M_l^{-1} \right) A. \tag{3.3}$$

In addition, if $A^{-1} \geqslant 0$ *and* $A = M_l - N_l$, $l = 1, \ldots, k$, *are all weakly regular, then*

(i) $\rho(H) < 1$;
(ii) $\sum_{l=1}^{k} E_l M_l^{-1}$ *is nonsingular*, $A = M - N$ *is weakly regular and* $H = M^{-1}N$, *where* $M = (\sum_{l=1}^{k} E_l M_l^{-1})^{-1}$.

Lemma 3.2. *Let* $A^{-1} \geqslant 0$ *and* $A = M_1 - N_1 = M_2 - N_2$ *be two convergent weak splittings of different types. If* $M_1^{-1} \geqslant M_2^{-1}$, *then*

$$\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2) < 1.$$

Lemma 3.2 is a comparison result between two weak single-splittings where A is an inverse positive matrix. The results for some weakly nonnegative and M-splittings are given in the following lemmas.

Lemma 3.3. *Let* (M_l, N_l, E_l) *and* $(\widetilde{M}_l, \widetilde{N}_l, E_l)$, $l = 1, \ldots, k$, *be two weakly regular multisplittings of* A, *and let* $\widetilde{M}_l$ *be a positive diagonal matrix and* $M_l^{-1} \geqslant \widetilde{M}_l^{-1}$, $l = 1, \ldots, k$. *If* A *is not an M-matrix, then* $\rho(H) \geqslant \rho(\widetilde{H}) > 1$, *where* H *and* $\widetilde{H}$ *are defined in* (3.1) *and* (3.2), *respectively.*

Proof. Let $G = \sum_{l=1}^{k} E_l M_l^{-1}$ and $\widetilde{G} = \sum_{l=1}^{k} E_l \widetilde{M}_l^{-1}$. Since $\widetilde{M}_l$ is a positive diagonal matrix for any l, it is readily seen that $\widetilde{G}$ is also a positive diagonal matrix, and thus nonsingular. Let $\widetilde{M} = \widetilde{G}^{-1}$ and $\widetilde{N} = \widetilde{M} - A$. Then $\widetilde{M}$ is a positive diagonal matrix. We have $\widetilde{M}^{-1}\widetilde{N} = I - (\sum_{l=1}^{k} E_l \widetilde{M}_l^{-1})A = \widetilde{H} \geqslant 0$ and therefore, $\widetilde{N} \geqslant 0$, which implies that the splitting $A = \widetilde{M} - \widetilde{N}$ is an M-splitting and A is a Z-matrix. It follows from Lemma 4.2 of [11] that $\rho(\widetilde{H}) > 1$. By the Perron–Frobenius theorem (e.g., see [3]), there is nonzero nonnegative vector x such that $\widetilde{M}^{-1}\widetilde{N}x = \widetilde{\rho}x$, where $\widetilde{\rho} = \rho(\widetilde{M}^{-1}\widetilde{N}) = \rho(\widetilde{H})$. Hence $\widetilde{M}^{-1}Ax = (1 - \widetilde{\rho})x$, which implies that

$$Ax = (1 - \widetilde{\rho})\widetilde{M}x. \tag{3.4}$$

Since $\widetilde{\rho} > 1$ and $\widetilde{M}$ is a positive diagonal matrix, we have $Ax \leqslant 0$ from (3.4). By the assumption that $M_l^{-1} \geqslant \widetilde{M}_l^{-1}$, $l = 1, \ldots, k$, it is easy to see that $G \geqslant \widetilde{M}^{-1}$. Hence $GAx \leqslant \widetilde{M}^{-1}Ax$. From (3.3) we have $(I - H)x \leqslant (1 - \widetilde{\rho})x$, i.e., $Hx \geqslant \widetilde{\rho}x$. Since each $A = M_l - N_l$ is weakly regular, we have $H = \sum_{l=1}^{k} E_l M_l^{-1} N_l \geqslant 0$. It follows from Theorem 2.1.11 of [3] that $\rho(H) \geqslant \widetilde{\rho}$. The proof is completed. □

The following lemma is similar to Theorem 3.2 of [5] but the condition $E_l A = AE_l$ for $l = 1, \ldots, k$ is changed to be $E_l M_l = M_l E_l$ for $l = 1, \ldots, k$.

Lemma 3.4. *Let A be inverse positive and (M_l, N_l, E_l), $l = 1, \ldots, k$, be a weakly nonnegative multisplitting of the second type with $E_l M_l = M_l E_l$ for $l = 1, \ldots, k$. Then $\rho(H) < 1$, where H is defined in* (3.1). *Furthermore, the matrix $G = \sum_{l=1}^{k} E_l M_l^{-1}$ is nonsingular and $A = M - N$ is also weakly nonnegative splitting of the second type, where $M = G^{-1}$ and $N = M - A$.*

Proof. Let $K = \sum_{l=1}^{k} N_l M_l^{-1} E_l$. Then $AG = \sum_{l=1}^{k} A M_l^{-1} E_l = I - K$. By the same argument as Theorem 3.2 in [5] we obtain $\rho(K) < 1$, which implies that G is nonsingular. Since $GA = I - H$ and GA have the same eigenvalues as AG, $\rho(H) < 1$. It is easy to see that $NM^{-1} = I - AG = K \geqslant 0$ and $M^{-1} = G \geqslant 0$, which proves the lemma. □

Remark 3.1. Let (M_l, N_l, E_l) and $(\widetilde{M}_l, \widetilde{N}_l, E_l)$, $l = 1, \ldots, k$, be two weakly nonnegative multisplittings of the second type, and let $E_l M_l = M_l E_l$ or $AE_l = E_l A$ for $l = 1, \ldots, k$. Then one may deduce that Lemma 3.3 is also valid.

Lemma 3.5. *Let A be an irreducible matrix and (M_l, N_l, E_l), $l = 1, \ldots, k$, be a nontrivial M-multisplitting of A . If $\rho(H) = 1$, then $G = \sum_{l=1}^{k} E_l M_l^{-1}$ is nonsingular, where H is defined in* (3.1).

Proof. By Lemma 2.1 of [4] we may assume without loss of generality that the iteration matrix of the multisplitting is as follows:

$$H = \begin{bmatrix} 0 & H_{11} \\ 0 & H_{22} \end{bmatrix},$$

where H_{22} is a nonzero irreducible matrix. Hence 1 is a simple eigenvalue of H by the Perron–Frobenius theorem, which implies that the dimension of the eigenspace associated with the zero eigenvalue of $I - H^{\mathrm{T}}$ is equal to 1. Let x be a nonzero nonnegative eigenvector of $(I - H^{\mathrm{T}})$ corresponding to the zero eigenvalue. We partition x into $x^{\mathrm{T}} = [x_1, x_2]$, where x_2 is a row vector whose dimension is the same as the order of H_{22}. By $x^{\mathrm{T}}(I - H) = 0$ one can deduce that $x_1 = 0$ and $x_2 > 0$. Now we assume that $G = \sum_{l=1}^{k} E_l M_l^{-1}$ is singular. Then there is a nonzero vector y such that $y^{\mathrm{T}} G = 0$. Form (3.3) we have

$$y^{\mathrm{T}} GA = y^{\mathrm{T}}(I - H) = 0.$$

Because the dimension of the eigenspace associated with the zero eigenvalue of $I - H^{\mathrm{T}}$ is 1, x and y are dependent, i.e., there is a nonzero real number r such that $y = rx$, and thus $x^{\mathrm{T}} G = 0$, i.e., $\sum_{l=1}^{k} x^{\mathrm{T}} E_l M_l^{-1} = 0$, which implies that $x^{\mathrm{T}} E_l M_l^{-1} = 0$, $l = 1, \ldots, k$. Notice that $\sum_{l=1}^{k} E_l = I$ and x_2 is positive. Then there is an l for which the last row of M_l^{-1} is zero. This implies that M_l^{-1} is singular, which contradicts that M_l is nonsingular. Hence G is nonsingular. □

Lemma 3.6. *Let* (M_l, N_l, E_l), $l = 1, \ldots, k$, *be an M-multisplitting of A. Then* $\rho(H) = 1$ *if and only if A is a singular M-matrix, where H is defined in* (3.1).

Proof. ($\Leftarrow$) By Lemma 3.1 and Corollary 3.4(i) of [9], $GA = I - H$ is a singular M-matrix, where G is the same as in Lemma 3.5. Thus $\rho(H) = 1$.

($\Rightarrow$) Without loss of generality we may assume that $A = (A_{ij})$ is the block partition given as in (2.1). Since $A = M_l - N_l$ is an M -splitting, it is block upper triangular weakly regular, i.e., $M_l = (M_{ij}^{(l)})$ and $N_l = (N_{ij}^{(l)})$ are block upper triangular matrices with the same partition as in (2.1), $l = 1, \ldots, k$. Clearly $H_{ij} = 0$ for $i \geqslant j$, and H_{ii} is an iteration matrix of the multisplitting $(M_{ii}^{(l)}, N_{ii}^{(l)}, E_{ii}^{(l)}), l = 1, \ldots, k$, of A_{ii}, where $E_l = \operatorname{diag}(E_{11}^{(l)}, \ldots, E_{ss}^{(l)})$, $i = 1, \ldots, s, l = 1, \ldots, k$.

In order to show that A is a singular M-matrix, by (2.1) it suffices to show that the block diagonal matrices A_{ii}, $i = 1, \ldots, s$, are all M-matrices. Since $\rho(H) = 1$, $\rho(H_{ii}) \leqslant 1$. It is noted that H_{ii} is the iteration matrix of the parallel multisplitting $(M_{ii}^{(l)}, N_{ii}^{(l)}, E_{ii}^{(l)})$, $l = 1, \ldots, k$, of A_{ii}. In the case $\rho(H_{ii}) < 1$, by Lemma 3.1 we have

$$A_{ii}^{-1} = (I - H_{ii})^{-1} \sum E_{ii}^{(l)} M_{ii}^{(l)-1} \geqslant 0,$$

which implies that A_{ii} is a nonsingular M-matrix; e.g. see [3]. In the case $\rho(H_{ii}) = 1$, we let $G_{ii} = \sum E_{ii}^{(l)} M_{ii}^{(l)-1}$ and assume that A_{ii} is nonsingular. By (3.3) we have $G_{ii} = (I - H_{ii}) A_{ii}^{-1}$, and hence there exists a nonzero nonnegative vector y such that $y^{\mathrm{T}} G_{ii} = 0$. By the same proof as Lemma 3.5 one can conclude that $M_{ii}^{(l)-1}$ has zero row for some l, which contradicts the nonsingularity of $M_{ii}^{(l)}$. Therefore, A_{ii} is singular. Because $M_{ii}^{(l)}$ is a nonsingular M-matrix, $N_{ii}^{(l)}$ is nonzero. Hence $(M_{ii}^{(l)}, N_{ii}^{(l)}, E_{ii}^{(l)}), l = 1, \ldots, k$, is a nontrivial M-multisplitting of an irreducible matrix A_{ii}. By Lemma 3.5 G_{ii} is nonsingular. Since $\rho(H_{ii}) = 1$, there is a nonzero nonnegative vector x for which $H_{ii} x = x$. By (3.3) $G_{ii} A_{ii} x = 0$ and by the nonsingularity of G_{ii} we have $A_{ii} x = 0$. Because A_{ii} is irreducible, by the Perron–Frobenius theorem one can easily deduce that A_{ii} is an M-matrix. The proof is completed. □

We now provide two sufficient conditions for inequality (1.6) to hold.

Proposition 3.1. *Let* $A^{-1} \geqslant 0$, *and* (M_l, N_l, E_l), $l = 1, \ldots, k$, *and* $(\widetilde{M}_l, \widetilde{N}_l, E_l)$, $l = 1, \ldots, k$, *be two weakly nonnegative multisplittings of A of the first type and the second type, respectively. If* $M_l^{-1} \geqslant \widetilde{M}_l^{-1}$, $l = 1, \ldots, k$, *then the inequalities* $\rho(H) \leqslant \rho(\widetilde{H}) < 1$ *hold provided the each weighting* E_l *satisfies one of the following conditions*:
(i) $AE_l = E_l A$, $l = 1, \ldots, k$.
(ii) $\widetilde{M}_l E_l = E_l \widetilde{M}_l$, $l = 1, \ldots, k$,
where H and $\widetilde{H}$ *are defined in* (3.1) *and* (3.2), *respectively.*

Proof. Let $G = \sum_{l=1}^{k} E_l M_l^{-1}$ and $\widetilde{G} = \sum_{l=1}^{k} E_l \widetilde{M}_l^{-1}$. Assume that the condition (i) or (ii) holds. From Lemma 3.1(ii), Lemma 3.4 and Theorem 3.2 of [5] it follows that G and $\widetilde{G}$ are nonsingular and $A = M - N = \widetilde{M} - \widetilde{N}$ are convergent and weakly nonnegative splittings of the first type and the second type, respectively, where $M = G^{-1}$, $N = M - A$, $\widetilde{M} = \widetilde{G}^{-1}$ and $\widetilde{N} = \widetilde{M} - A$. By the hypothesis that $M_l^{-1} \geqslant \widetilde{M}_l^{-1}$, $l = 1, \dots, k$, we have

$$M^{-1} = \sum_{l=1}^{k} E_l M_l^{-1} \geqslant \sum_{l=1}^{k} E_l \widetilde{M}_l^{-1} = \widetilde{M}^{-1}. \tag{3.5}$$

Hence the inequalities $\rho(H) \leqslant \rho(\widetilde{H}) < 1$ are obtained by (3.5) and Lemma 3.2.

Remark 3.2. In fact, condition (ii) in Proposition 3.1 can be changed to be $M_l E_l = E_l M_l, l = 1, \dots, k$, when the multisplittings (M_l, N_l, E_l), $l = 1, \dots, k$ and $(\widetilde{M}_l, \widetilde{N}_l, E_l), l = 1, \dots, k$, are two weakly nonnegative multisplittings of A of the second type and the first type, respectively.

Remark 3.3. When each $A = \widetilde{M}_l - \widetilde{N}_l$ is a block Jacobi-type splitting, i.e.,

$$\widetilde{M}_l = \mathrm{diag}(\widetilde{M}_{l1}, \widetilde{M}_{l2}, \dots, \widetilde{M}_{lq}) \tag{3.6}$$

and the weighting is

$$E_l = \mathrm{diag}(\alpha_{l1} I, \alpha_{l2} I, \dots, \alpha_{lq} I), \tag{3.7}$$

then condition (ii) in Proposition 3.1 is satisfied and therefore, the convergence of any multisplitting with $M_l^{-1} \geqslant \widetilde{M}_l^{-1}$ is better than that of the Jacobi-type multisplitting.

Corollary 3.1. *Let* $A^{-1} \geqslant 0$, *and* (M_l, N_l, E_l), $l = 1, \dots, k$, *and* $(\widetilde{M}_l, \widetilde{N}_l, E_l)$, $l = 1, \dots, k$, *be nonnegative multisplittings with* $M_l^{-1} \geqslant \widetilde{M}_l^{-1}, l = 1, \dots, k$. *If* (3.6) *and* (3.7) *are satisfied, then* $\rho(H) \leqslant \rho(\widetilde{H}) < 1$, *where* H *and* $\widetilde{H}$ *are defined in* (3.1) *and* (3.2), *respectively.*

Remark 3.4. Elsner in [6] provided a counterexample to illustrate that (1.6) is not true in general, in which

$$A = \begin{bmatrix} 1 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} = M_1 - N_1 = M_2 - N_2$$

and

$$M_1 = \begin{bmatrix} 1+\varepsilon & -1+\sigma \\ -\frac{1}{2} & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} \varepsilon & \sigma \\ 0 & 0 \end{bmatrix}, \quad M_2 = A, \quad N_2 = 0$$

and each E_l is singular. It is noted that the inequality $\rho(H) \leqslant \rho(\widetilde{H})$ is still not true even for nonsingular weighting E_l. Elsner's example can be used here to show that the inequality (1.6) is not true by adding a small perturbation on the singular weighting matrices in [6]. It is also noted that Elsner's example does not satisfy the conditions (i) and (ii) in Proposition 3.1.

Remark 3.5. In [5] Climent and Perea introduced the condition that $AE_l = E_lA$ to show the convergence of a weak nonnegative multisplitting of the second type. In Lemma 3.4 we give another sufficient condition such that the weak nonnegative multisplitting of the second type converges.

The following is a Stein–Rosenberg type theorem for a general parallel multisplitting method and a Jacobi-type multisplitting method.

Theorem 3.1. *Let* (M_l, N_l, E_l), $l = 1, \dots, k$ *and* $(\widetilde{M}_l, \widetilde{N}_l, E_l)$, $l = 1, \dots, k$, *be both M -multisplittings of A with* $M_l^{-1} \geqslant \widetilde{M}_l^{-1}$, $l = 1, \dots, k$. *If* $\widetilde{M}_l$ *is a positive diagonal matrix*, $l = 1, \dots, k$, *then*

(i) $\rho(H) < 1$ *if and only if* $\rho(\widetilde{H}) < 1$, *in which case* $\rho(H) \leqslant \rho(\widetilde{H}) < 1$;

(ii) $\rho(H) = 1$ *if and only if* $\rho(\widetilde{H}) = 1$;

(iii) $\rho(H) > 1$ *if and only if* $\rho(\widetilde{H}) > 1$, *in which case* $\rho(H) \geqslant \rho(\widetilde{H}) > 1$.

Proof. (i) Assume that $\rho(H) < 1$. Then $I - H$ is a nonsingular *M*-matrix. Hence $(I - H)^{-1} \geqslant 0$. Let $G = \sum_{l=1}^{k} E_l M_l^{-1}$. Then $G \geqslant 0$. By Lemma 3.1 we have $GA = I - H$. Hence G and A are both nonsingular, and thus we have $A^{-1} = (I - H)^{-1} G \geqslant 0$. Since $\widetilde{M}_l$ and E_l are diagonal, $l = 1, \dots, k$, it is easy to see that condition (ii) of Proposition 3.1 is satisfied. Hence we have $\rho(H) \leqslant \rho(\widetilde{H}) < 1$ from Proposition 3.1. Conversely, if $\rho(\widetilde{H}) < 1$, then similarly to the case that $\rho(H) < 1$ one can deduce $\rho(H) \leqslant \rho(\widetilde{H}) < 1$.

(ii) If $\rho(H) = 1$, then A is a singular *M*-matrix by Lemma 3.6. Since $(\widetilde{M}, \widetilde{N}, E_l)$, $l = 1, \dots, k$, is an *M*- multisplitting of A, it follows from Corollary 3.4 of [9] that $\rho(\widetilde{H}) = 1$. The proof of sufficiency is analogous.

(iii) From (i) and (ii) of this theorem we have $\rho(H) > 1$ if and only if $\rho(\widetilde{H}) > 1$ and A is not an *M*-matrix. Then the inequalities in (iii) follow immediately from Lemma 3.3. □

Corollary 3.2. *Let* (M_l, N_l, E_l), $l = 1, \dots, k$, *and* $(\widetilde{M}_l, \widetilde{N}_l, E_l)$, $l = 1, \dots, k$, *both be M-multisplittings of A with* $M_l^{-1} \geqslant \widetilde{M}_l^{-1}$, $l = 1, \dots, k$. *If* $\widetilde{M}_l$ *is a positive diagonal matrix*, $l = 1, \dots, k$, *then one and only one of the following mutually exclusive results holds*:

(i) $\rho(H) \leqslant \rho(\widetilde{H}) < 1$;

(ii) $\rho(H) = \rho(\widetilde{H}) = 1$;

(iii) $\rho(H) \geqslant \rho(\widetilde{H}) > 1$.

Proof. Follows immediately from Theorem 3.1. □

Remark 3.6. From Lemmas 3.3, Corollary 3.4 of [9] and Theorem 3.1, it is easy to see that Corollary 3.2 is still true under the assumption that (M_l, N_l, E_l) is a block upper triangular nonnegative multisplitting. Theorem 3.1 gives a Stein–Rosenberg type theorem for a general parallel multisplitting method and a Jacobi-type multisplitting method.

4. Parallel multisplitting AOR methods

In this section, we present a general comparison theorem and applications to some Jacobi-type multisplittings. Here we consider some more classical parallel multisplittings.

AOR iterative methods were first introduced by Hadjidimos [8] to solve nonsymmetric systems. The iteration matrix can be written in general

$$(D - \gamma L)^{-1}[(1 - \omega)D + (\omega - \gamma)L + \omega U]$$

with two parameters γ and ω, where $A = D - L - U$, D is nonsingular and diagonal, and L and U are strictly lower triangular and strictly upper triangular, respectively. Parallel multisplitting AOR methods were proposed by Wang [19] and later, studied by many authors. For example, a generalized AOR method (GAOR) was considered in [18] and the iteration matrix is given by

$$L(\gamma, \omega) = \sum_{l=1}^{k} E_l(D_l - \gamma_l L_l)^{-1}[(1 - \omega_l)D_l + (\omega_l - \gamma_l)L_l + \omega_l U_l],$$

where $A = D_l - L_l - U_l$, $D_l = \operatorname{diag}(d_1^{(l)}, \ldots, d_n^{(l)})$ is nonsingular and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_k)$ are parameters. In [20] the author only considered the case that $D_l = I$ for any l. Let

$$d = \min_{1 \leqslant l \leqslant k, 1 \leqslant i \leqslant n} \{d_i^{(l)}\}.$$

By using Theorem 3.2, we can obtain a comparison result for the spectral radii of iteration matrices of parallel multisplitting methods.

Theorem 4.1. *Let $A = D_l - L_l - U_l$, where $D_l = \operatorname{diag}(d_1^{(l)}, \ldots, d_n^{(l)})$ is a positive diagonal matrix, L_l and U_l are nonnegative, $l = 1, \ldots, k$. Let $\rho(L_l) < d$ and $0 \leqslant \gamma_l \leqslant \omega_l \leqslant 1, \omega_l \neq 0,\ l = 1, \ldots, k$. Then for any given weighting E_l we have:*

(i) $\rho(L(\gamma, \omega)) < 1$ *if and only if* $\rho(L(\mathbf{0}, \omega)) < 1$, *in which case,* $\rho(L(\gamma, \omega)) \leqslant \rho(L(\mathbf{0}, \omega)) < 1$,

(ii) $\rho(L(\gamma, \omega)) = 1$ *if and only if* $\rho(L(\mathbf{0}, \omega)) = 1$,

(iii) $\rho(L(\gamma, \omega)) > 1$ *if and only if* $\rho(L(\mathbf{0}, \omega)) > 1$, *in which case* $\rho(L(\gamma, \omega)) \geqslant \rho(L(\mathbf{0}, \omega)) > 1$,

where $\mathbf{0}$ *denotes the zero vector.*

Proof. Since $\rho(L_l) < d$, $D_l - L_l$ is a nonsingular M-matrix and $U_l \geqslant 0$, which implies that $A = (D_l - L_l) - U_l$ is an M-splitting, and hence is block upper triangular regular (see Section 2). Let

$$M_l = \frac{1}{\omega_l}(D_l - \gamma_l L_l), \quad N_l = \frac{1}{\omega_l}[(1-\omega_l)D_l + (\omega_l - \gamma_l)L_l + \omega_l U_l],$$
$$\widetilde{M}_l = \frac{1}{\omega_l}D_l \quad \widetilde{N}_l = \frac{1}{\omega_l}[(1-\omega_l)D_l + \omega_l L_l + \omega_l U_l].$$

Then $\widetilde{M}_l$ is diagonal and $M_l \leqslant \widetilde{M}_l, l = 1, \ldots, k$. Clearly, $A = M_l - N_l = \widetilde{M}_l - \widetilde{N}_l$ are all M-splittings since $0 \leqslant \gamma_l \leqslant \omega_l \leqslant 1$. Because

$$H = L(\gamma, \omega) \quad \text{and} \quad \widetilde{H} = \sum_{l=1}^{k} E_l \widetilde{M}_l^{-1} \widetilde{N}_l = L(\mathbf{0}, \omega),$$

the result follows from Theorem 3.1. □

If the conditions that $\gamma_l \leqslant \omega_l, \; l = 1, \ldots, k,$ are omitted, then we have:

Theorem 4.2. *Let $A = D_l - L_l - U_l$, where $D_l = \operatorname{diag}(d_1^{(l)}, \ldots, d_n^{(l)})$ is a positive diagonal matrix, L_l and U_l are nonnegative, $l = 1, \ldots, k$. Let $\rho(L_l) < d$ and $0 \leqslant \gamma_l \leqslant 0, 0 < \omega_l \leqslant 1, l = 1, \ldots, k$. Then one and only one of the following mutually exclusive results holds for any given weighting E_l:*
(i) $\rho(L(\gamma, \omega)) \leqslant \rho(L(\mathbf{0}, \omega)) < 1$,
(ii) $\rho(L(\gamma, \omega)) = \rho(L(\mathbf{0}, \omega)) = 1$,
(iii) $\rho(L(\gamma, \omega)) \geqslant \rho(L(\mathbf{0}, \omega)) > 1$.

Proof. Let M_l, N_l, $\widetilde{M}_l$ and $\widetilde{N}_l$ be as in the proof of Theorem 4.1. Then $\widetilde{M}_l$ is a diagonal and $M_l \leqslant \widetilde{M}_l, l = 1, \ldots, k$. Clearly, $A = M_l - N_l = \widetilde{M}_l - \widetilde{N}_l$ are block upper triangular, and the second splitting is an M-splitting. Now we need to show that the first splitting is nonnegative. It is easy to check that M_l is a nonsingular M-matrix and

$$M_l^{-1} N_l = (1-\omega_l)I + \omega_l (D_l - \gamma_l L_l)^{-1}[(1-\gamma_l)L_l + U_l],$$
$$N_l M_l^{-1} = (1-\omega_l)I + \omega_l [(1-\gamma_l)L_l + U_l](D_l - \gamma_l L_l)^{-1}.$$

Since $0 \leqslant \gamma_l, \omega_l \leqslant 1$, $M_l^{-1} N_l$ and $N_l M_l^{-1}$ are nonnegative, which proves our assertion. Because $H = L(\gamma, \omega)$ and $\widetilde{H} = L(\mathbf{0}, \omega)$, the result follows from Remark 3.6. □

When $\gamma = 0$, the parallel GAOR method $(L(0, \omega))$ corresponds to the extrapolated Jacobi-type multisplitting and the iterative matrix is $L(\mathbf{0}, \omega)$. When $\gamma = \omega = \mathbf{1} := (1, 1, \ldots, 1)^{\mathrm{T}}$, $L(\mathbf{1}, \mathbf{1})$ denotes Gauss–Seidel-type multisplitting. Applying the

Theorem 4.1 for this case gives a Stein–Rosenberg type theorem for Gauss–Seidel-type multisplittings and Jacobi-type multisplittings for any weighting E_l:

(i) $\rho(L(\mathbf{1},\mathbf{1})) < 1$ if and only if $\rho(L(\mathbf{0},\mathbf{1})) < 1$, in which case, $\rho(L(\mathbf{1},\mathbf{1})) \leqslant \rho(L(\mathbf{0},\mathbf{1})) < 1$,

(ii) $\rho(L(\mathbf{1},\mathbf{1})) = 1$ if and only if $\rho(L(\mathbf{0},\mathbf{1})) = 1$,

(iii) $\rho(L(\mathbf{1},\mathbf{1})) > 1$ if and only if $\rho(L(\mathbf{0},\mathbf{1})) > 1$, in which case, $\rho(L(\mathbf{1},\mathbf{1})) \geqslant \rho(L(\mathbf{0},\mathbf{1})) > 1$.

Remark 4.1. Theorems 4.1 and 4.2 give a general comparison theorem between a GAOR multisplitting and an extrapolated Jacobi-type multisplitting, from which many previous results can be obtained.

- Applying Theorem 4.2 for the case $D_l = I$ gives the Theorem 2.1 of [20].
- If we take $k = 1$, we obtain the Stein–Rosenberg type theorem for generalized single Jacobi and generalized single Gauss–Seidel iterations (also see [21, Theorem 3.1] or [11, Corollary 2.1]).
- If we assume that A is a nonsingular M-matrix, (M_l, N_l, E_l) and $(\widetilde{M}_l, \widetilde{N}_l, \widetilde{E}_l)$, $l = 1, \ldots, k$, are overlapping Jacobi multisplitting and nonoverlapping Jacobi multisplitting, respectively, then (1.6) follows immediately from Theorem 3.1, which is Theorem 2.2 of [7].
- Since SOR iteration is a special case of AOR iterations with $\gamma = \omega$, applying Theorem 4.1 and letting $\gamma = \omega$ give the comparison result for the SOR-type multisplitting and extrapolated Jacobi multisplitting.

It is also noted that Theorem 4.1 can be extended to block Jacobi-type multisplittings. In the following corollary, we obtain a generalization of [18, Theorem 4.1].

Corollary 4.1. *Let A be a nonsingular M-matrix and $A = D - L_l - U_l$, where D is a nonsingular Z-matrix, L_l and U_l are nonnegative, $l = 1, \ldots, k$. Let*

$$0 \leqslant \underline{L} \leqslant L_l \leqslant \overline{L} \leqslant D - A.$$

and $A = D - \underline{L} - \underline{U} = D - \overline{L} - \overline{U}$. Then

$$\rho(L(\overline{\gamma}, \overline{\omega})) \leqslant \rho(L(\gamma, \omega)) \leqslant \rho(L(\underline{\gamma}, \underline{\omega})) < 1, \tag{4.1}$$

where $\underline{\gamma} = \min\{\gamma_l\}$, $\underline{\omega} = \min\{\omega_l\}$, $\overline{\gamma} = \max\{\gamma_l\}$ and $\overline{\omega} = \max\{\omega_l\}$, and $L(\overline{\gamma}, \overline{\omega})$ and $\rho(L(\underline{\gamma}, \underline{\omega}))$ denote the classical AOR methods with the parameters $(\overline{\gamma}, \overline{\omega})$ and $(\underline{\gamma}, \underline{\omega})$, respectively.

Proof. Taking

$$M_l = \frac{1}{\omega_l}(D - \gamma_l L_l), \quad N_l = \frac{1}{\omega_l}[(1 - \omega_l)D + (\omega_l - \gamma_l)L_l + \omega_l U_l],$$

$$\overline{M} = \frac{1}{\overline{\omega}}(D - \overline{\gamma}\overline{L}), \quad \overline{N} = \frac{1}{\overline{\omega}}[(1 - \overline{\omega})D + (\overline{\omega} - \overline{\gamma})\overline{L} + \overline{\omega}\overline{U}],$$

and

$$\underline{M} = \frac{1}{\underline{\omega}}(D - \underline{\gamma}\underline{L}), \quad \underline{N} = \frac{1}{\underline{\omega}}[(1 - \underline{\omega})D + (\underline{\omega} - \underline{\gamma})\underline{L} + \underline{\omega U}],$$

then $A = \underline{M} - \underline{N} = \overline{M} - \overline{N}$ with $\overline{M} \leqslant M_l \leqslant \underline{M},\ l = 1, \ldots, k$. Since $D \geqslant D - \underline{L} \geqslant D - L_l \geqslant D - \overline{L} \geqslant A$ and A is a nonsingular M-matrix, D, $D - \underline{L}$, $D - L_l$ and $D - \overline{L}$ are all nonsingular M-matrices, which implies that $\underline{M}^{-1} \geqslant 0$ and $\overline{M}^{-1} \geqslant 0$. By taking the approach similar to the proof of Theorem 4.1 it is easy to show that $\overline{M}^{-1}\overline{N} \geqslant 0$ and $\overline{NM}^{-1} \geqslant 0$. Hence $A = \overline{M} - \overline{N}$ is a nonnegative splitting. $A = \underline{M} - \underline{N}$ is also a nonnegative splitting. Then (4.1) follows from Theorem 3.4 of [5] immediately. □

Remark 4.2. In [18], the author proved (4.1) only for $\underline{\gamma} \leqslant \underline{\omega}$, $\overline{\gamma} \leqslant \overline{\omega}$ and under the assumption that D is an M-matrix. These conditions are not necessary in Corollary 4.1.

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References

[1] G. Alefeld, I. Lenhardt, G. Mayer, On multisplitting methods for band matrices, Numer. Math. 75 (1997) 267–292.
[2] Z. Bai, On the monotone convergence of matrix multisplitting relaxation methods for the linear complementarity problem, IMA J. Numer. Anal. 18 (1998) 509–518.
[3] A. Berman, R. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, PA, 1994.
[4] R. Bru, R. Canto, J. Climent, On M-multisplittings of singular M-matrices with application to Markov chains, Numer. Linear Algebra Appl. 5 (1998) 299–311.
[5] J.J. Climent, C. Perea, Convergence and comparison theorems for multisplittings, Numer. Linear Algebra Appl. 6 (1999) 93–107.
[6] L. Elsner, Comparisons of weak regular splitting and multisplitting methods, Numer. Math. 56 (1989) 283–289.
[7] A. Frommer, B. Pohl, A comparison result for multisplittings and waveform relaxation methods, Numer. Linear Algebra Appl. 2 (1995) 335–346.
[8] A. Hadjidimos, Accelerated overrelaxation method, Math. Comput. 32 (1978) 149–157.
[9] J.P. Kavanagh, M. Neumann, Consistency and convergence of the parallel multisplitting method for singular M-matrices, SIAM J. Matrix Anal. Appl. 10 (1989) 210–218.
[10] W. Li, M. Zhang, On upper triangular block weak regular splittings of a singular M-matrix, Linear Algebra Appl. 233 (1996) 175–187.
[11] W. Li, Z. You, The multi-parameters overrelaxation method, J. Comput. Math. 16 (1998) 231–238.
[12] W. Li, G. Lei, Convergence theorems of parallel multisplitting AOR iterative methods, Acta Math. Appl. Sinica 22 (1999) 278–283.

[13] R. Nabben, A note on comparison theorems for splittings and multisplittings of Hermitian positive definite matrices, Linear Algebra Appl. 233 (1996) 67–80.
[14] M. Neumann, R.J. Plemmons, Convergence of parallel multisplitting iterative methods for *M*-matrices, Linear Algebra Appl. 88/89 (1987) 559–573.
[15] D. O'Leary, R. White, Multisplittings of matrices and parallel solutions of linear systems, SIAM J. Algebra Dis. Math. 6 (1985) 630–640.
[16] H. Schneider, Theorems on *M*-splittings of a singular *M*-matrix which depend on graph structure, Linear Algebra Appl. 58 (1984) 407–427.
[17] D.B. Szyld, M.T. Jones, Two-stage and multisplitting methods for the parallel solution of linear systems, SIAM J. Matrix Anal. Appl. 13 (1992) 671–679.
[18] Y. Song, Convergence of parallel multisplitting methods for *H*-matrices, Int. J. Comput. Math. 50 (1994) 213–232.
[19] D. Wang, On the convergence of the parallel multisplitting AOR algorithm, Linear Algebra Appl. 154–156 (1991) 473–486.
[20] X. Wang, Comparison theorems for a class of parallel multisplitting AOR type iterative methods, Linear Algebra Appl. 269 (1998) 1–16.
[21] X. Wang, Generalized Stein–Rosenberg type theorems for the regular splittings and convergence of some generalized iterative methods, Linear Algebra Appl. 184 (1993) 207–234.
[22] X. Wang, Convergence of parallel AOR and GAOR methods applied to *H*-matrices, IMA J. Numer. Anal. 16 (1996) 485–499.
[23] I. Woznicki, Nonnegative splitting theory, Jpn J. Indust. Appl. Math. 11 (1994) 289–342.