The periodic boundary value problem for semilinear elastic beam equations: The resonance case

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Abstract

This paper discusses the existence of generalized solutions to periodic boundary value problems for semilinear elastic beam equations under a resonance condition. The argument presented makes use of the global inverse theorem and Galerkin’s method.

Keywords: Elastic beam equation; Periodic boundary value problem; Resonance; Generalized solution; Unique existence

1. Introduction

Let \( \Omega = (0, 2\pi) \times (0, \pi) \subset \mathbb{R}^2 \) and \( H = [L^2(\Omega)]^n \), with integer \( n \geq 1 \). Now \( H \) is a real Hilbert space with inner product

\[
\langle u, v \rangle = \int_0^{2\pi} \int_0^\pi (u(t, x), v(t, x))dxdt.
\] (1.1)

Note that \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product in \( \mathbb{R}^n \). The norms induced by \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) are denoted by \( \| \cdot \| \) and \( | \cdot | \), respectively.

We consider the system of semilinear elastic beam equations described by the following partial differential equation:

\[
u_{tt} + u_{xxxx} - f(t, x, u) = h(t, x), \quad \forall (t, x) \in J,
\] (1.2)

with the boundary value conditions,

\[
u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0,
\] (1.3)

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where \( J = [0, 2\pi] \times [0, \pi], f : J \times \mathbb{R}^n \to \mathbb{R}^n \) satisfies the Carathéodory conditions and \( h : J \to \mathbb{R}^n \). We will look for periodic solutions so we will need

\[
    u(0, x) = u(2\pi, x), \quad u_t(0, x) = u_t(2\pi, x), \quad \forall x \in [0, \pi]. \tag{1.4}
\]

Much work has been done on the existence of periodic solutions for the beam equations (1.2)–(1.4); we refer the reader to [1–5] and the references therein. In [3], the nonlinear term \( f(t, x, u) \) is a power approximately; and in [4] more general nonlinearities are considered. Besides other assumptions, the function \( f \) is required to be symmetric in the following sense: there exist constants \( \alpha, \beta, M > 0 \) such that

\[
    \alpha f(t, x, u) \leq f(t, x, -u) \leq \beta f(t, x, u), \quad \text{for } u \geq M. \tag{1.5}
\]

Therefore, functions with different powers, say

\[
    g(t, x, u) = \begin{cases} 
        |u|^{p-2}u, & u \geq 0; \\
        |u|^{q-2}u, & u < 0;
    \end{cases} \tag{1.6}
\]

where \( p, q > 2 \) and \( p \neq q \), are excluded. In [1,2], by using Galerkin’s method, and working in finite dimensional spaces and passing to the limit, that case is discussed. In [7,8], Galerkin’s type arguments are also adopted to study the semilinear wave equation with periodic-Dirichlet boundary conditions.

In this paper, motivated by [6,7,9,11], with the use of a version of the global inverse function theorem and Galerkin’s method, we present a result on the existence and uniqueness of generalized solutions to the periodic-Dirichlet problem (GPDS, for short) for the semilinear elastic beam equation under a resonance condition. Our result avoids the above restrictions on \( f(t, x, u) \). In fact we allow the nonlinearity \( f'(t, x, u) \) when \( \langle u, v \rangle \to \infty \) to interact with points of resonance.

Here, by a GPDS for (1.2) we mean a function \( u \in H \) such that

\[
    \langle u, v_{tt} + v_{xxxx} \rangle - \langle f(t, x, u), v \rangle = \langle h(t, x), v \rangle, \tag{1.7}
\]

holds for all \( \forall v \in C^2(\Omega, \mathbb{R}^n) \) and which satisfies the following conditions:

\[
    v(t, 0) = v(t, \pi) = v_{xx}(t, 0) = v_{xx}(t, \pi) = 0, \quad \forall t \in [0, 2\pi]; \tag{1.8}
\]

\[
    v(0, x) = v(2\pi, x), \quad v_t(0, x) = v_t(2\pi, x), \quad \forall x \in [0, \pi]. \tag{1.9}
\]

The set \( \sigma(L) = \{m^4 - l^2|m \in Z, l \in N^*\} \) is called the set of points of resonance of (1.2)–(1.4), where \( Z \) is the set of all integers and \( N^* \) is the set of all positive integers and zero. The periodic boundary value problem (1.2)–(1.4) is said to be at resonance if the following conditions hold:

There exist two constant symmetric \( n \times n \) matrices \( A \) and \( B \) such that

\[
    A \leq f''_u(t, x, u) \leq B
\]

and, if \( \lambda_{11} \leq \lambda_{12} \leq \cdots \leq \lambda_{1n} \) and \( \lambda_{21} \leq \lambda_{22} \leq \cdots \leq \lambda_{2n} \) are the eigenvalues of \( A \) and \( B \), \( \lambda_{1i} \leq \lambda_{2i} \), for \( i = 1, \ldots, n \) respectively, then

\[
    \bigcup_{j=1}^n [\lambda_{1j}, \lambda_{2j}] \cap \sigma(L) \neq \emptyset;
\]

note that the relation \( A \leq B \) means that \( B - A \) is positive semi-definite.

To be more precise, we will prove in this paper that if \( f \) is continuously differentiable, \( f'(t, x, u) \) is symmetric and satisfies

\[
    A + \alpha(\|u\|)I \leq f'(t, x, u) \leq B - \beta(\|u\|)I \tag{1.10}
\]

and

\[
    \int_0^{+\infty} \min\{\alpha(s), \beta(s)\}ds = +\infty, \tag{1.11}
\]
then there exists a unique GPDS to (1.2); here \( I \) is an \( n \times n \) identity matrix, \( A \) and \( B \) are two real symmetric matrices (and we assume that \( \lambda_{1i}, \lambda_{2j} \in \sigma(L) \) are consecutive, for \( i = 1, \ldots, n \), where \( \lambda_{1i}, \lambda_{2j} \) are the eigenvalues of \( A \) and \( B \) respectively) and \( \alpha(s), \beta(s) \) are two continuous and nonincreasing functions from \( [0, \infty) \) to \( (0, \infty) \).

We note that \( \lambda_{1i}, \lambda_{2j} \in \sigma(L) \) are consecutive, for \( i = 1, \ldots, n \), if

\[
\bigcup_{j=1}^{n} (\lambda_{1j}, \lambda_{2j}) \cap \sigma(L) = \emptyset,
\]

and \( \lambda_{11} \leq \lambda_{12} \leq \cdots \leq \lambda_{1n}, \lambda_{21} \leq \lambda_{22} \leq \cdots \leq \lambda_{2n}, \lambda_{1i} < \lambda_{2i} \) for each \( i \).

2. Existence and uniqueness

Denote by \( \Theta \) the set of all continuous and nondecreasing mapping \( \omega \) that satisfy

\[
\omega : R_{+} \rightarrow R_{+}, \quad w(t) > 0, \quad t > 0, \quad \int_{0}^{\infty} \frac{dt}{\omega(t)} = \infty. \tag{2.1}
\]

We first employ the following lemmas.

**Lemma 1** ([9]). Assume that \( H \) is a Hilbert space. Let \( T \in C^{1}(H, H) \), and we assume \( T'(u) \) is everywhere invertible, \( \forall u \in H \). Then \( T \) is a global diffeomorphism onto \( H \) if there exists \( \omega \in \Theta \) satisfying \( \|T'(u)^{-1}\| \leq \omega(\|u\|) \).

**Lemma 2** ([10]). Let \( H \) be a vector space such that for subspaces \( Y \) and \( Z \), \( H = Z \bigoplus Y \). If \( Z \) is finite dimensional and \( X \) is a subspace of \( H \) such that \( X \cap Y = \{0\} \) and dimension \( X = \) dimension \( Z \), then \( H = X \bigoplus Y \).

We will follow the setup in Mawhin [7]. Set \( \varphi_{lm}(t, x) = \exp(ilt) \sin(mx), l \in Z, m \in N^{*} \), and let \( \{e_{1}, e_{2}, \ldots, e_{n}\} \) denote an orthonormal basis in \( R^{n} \). Then the set \( \{\varphi_{lm}e_{k} \mid l \in Z, m \in N^{*}, 1 \leq k \leq n\} \) constitutes a complete orthonormal basis in \( H = [L^{2}(\Omega)]^{n} \). Therefore, for any \( u \in H \) we may write \( u \) in terms of its Fourier series as

\[
u = \sum_{l,m,k} a_{lmk} \varphi_{lm} e_{k}, \tag{2.2}\]

where \( a_{lmk} = \langle u, \varphi_{lm} e_{k} \rangle = \int_{\Omega} (u, \varphi_{lm} e_{k}) dx \ dt \).

Define \( L : \text{Dom } L \subset H \rightarrow H \), the abstract realization of the \( u_{tt} + u_{xxxx} \), by

\[
\text{Dom } L = \left\{ u \in H \mid \sum_{l,m,k} (m^{4} - l^{2})^{2} |a_{lmk}|^{2} < \infty \right\},
\]

\[
Lu = \sum_{l,m,k} (m^{4} - l^{2}) a_{lmk} \varphi_{lm} e_{k}, \tag{2.4}\]

and it is easy to check that \( L \) is a linear, closed densely defined self-adjoint operator such that

\[
\text{Ker } L = \text{span} \{\cos(mt) \sin(mx), \sin(mt) \sin(mx) : m \in N^{*}\}, \quad \text{Im } L = (\text{Ker } L)^{\perp}.
\]

Moreover, for every \( h \in H, u \) is a GPDS on \( J \) of the equation

\[
u_{tt} + u_{xxxx} = h(t, x),
\]

if and only if \( u \in \text{Dom } L \) and \( Lu = h \). Therefore, if we assume the existence of a constant \( \delta \geq 0 \) such that, for all \( u \in R^{n} \), one has

\[
\|f'(u)(t, x)\| \leq \delta, \quad \text{a.e. on } J \tag{2.5}
\]

it is well known [7] that the mapping \( N \) defined on \( H \) by

\[
(Nu)(t, x) = f(t, x, u(t, x)), \quad \text{a.e. on } J
\]
maps continuously $H$ into itself, and then the existence of GPDS on $J$ for (1.2)–(1.4) is equivalent to the existence of a solution $u \in \text{Dom } L$ for the equation in $H$

\[ Lu - Nu = h. \]  

(2.6)

We shall now construct Galerkin’s approximate equations for (2.6) in a way similar to that in [7]. Let $\{a_k : 1 \leq k \leq n\}$ and $\{b_k : 1 \leq k \leq n\}$ be orthonormal bases in $R^n$ such that

\[ Aa_k = \lambda_1 a_k, \quad Bb_k = \lambda_2 b_k. \]

For every $j \in N^*$, define the subspace $H_j$ of $H$ by

\[ H_j = \left\{ \sum_{l,m,k} a_{lmk} \varphi_{lm} b_k : a_{lmk} \in R, a_{lmk} = \bar{a}_{lmk}, (l,m) \in (Z \times N^*)_j, 1 \leq k \leq n \right\}, \]

(2.7)

where $(Z \times N^*)_j = \{(l,m) \in Z \times N^* : |m^4 - l^2| \leq j, m^4 \leq j\}$. Notice from this construction that the restriction of $L$ to $\text{Dom } L \cap H_j$ has, in contrast with $L$, a spectrum bounded below and above and made of eigenvalues having finite multiplicity. Moreover, $\bigcup_{j \in N} H_j$ is dense in $H$ and if we denote by $P_j : H \rightarrow H$ the orthogonal projector onto $H_j(j \in N^*)$, Galerkin’s approximate equations for (2.6) will be

\[ Lu_j - P_j Nu_j = P_j h, \quad u_j \in \text{Dom } L \cap H_j = H_j, \quad j \in N. \]

(2.8)

We now prove the existence of Galerkin’s approximate solutions using a global inverse function theorem.

Let $j \in N^*$ be fixed. We have the following direct sum decomposition of $H_j$:

\[ X_j = \left\{ x \in H_j | x = \sum_{l,m,k} a_{lmk} \varphi_{lm} b_k, a_{lmk} = \bar{a}_{lmk}, m^4 - l^2 \geq \lambda_2 \right\}, \]

\[ Y_j = \left\{ y \in H_j | y = \sum_{l,m,k} a_{lmk} \varphi_{lm} b_k, a_{lmk} = \bar{a}_{lmk}, m^4 - l^2 < \lambda_2 \right\}, \]

\[ Z_j = \left\{ z \in H_j | z = \sum_{l,m,k} a_{lmk} \varphi_{lm} b_k, a_{lmk} = \bar{a}_{lmk}, m^4 - l^2 < \lambda_1 \right\}. \]

Clearly, $H_j = X_j \oplus Y_j$ (orthogonal direct sum) and because $\lambda_{1i}, \lambda_{2i} \in \sigma(L)$, are consecutive, then $\text{dim } Y_j = \text{dim } Z_j = \infty$.

Now we establish an existence uniqueness result. We note that some of the ideas in the proof are modelled off [7, Lemma 2], [11, Lemma 1.2].

**Lemma 3.** If the conditions (1.10) and (1.11) hold for $\forall (t, x) \in \Omega, \forall u \in H$, Eq. (2.8) has, for each $j \in N$ and a fixed $h \in H$, a unique solution $u_j \in \text{Dom } L \cap H_j$, and there exists a constant $\delta$, which depends on $h$ only, such that $\|u_j\| \leq \delta(\|h\|)$, for all $j \in N$.

**Proof.** We shall show that the mapping $F_j : H_j \rightarrow H_j$ defined by

\[ F_j u_j = Lu_j - P_j Nu_j \]

for every $u_j \in H_j$ satisfies all the conditions of Lemma 1.

The continuous Fréchet differentiability of $F_j$ is trivial. Now if $u_j \in H_j$ and $x_j \in \text{Dom } L \cap X_j = X_j$, with

\[ x_j = \sum_{k=1}^{n} \sum_{m^4 - l^2 \geq \lambda_2} (m^4 - l^2) a_{lmk} \varphi_{lm} b_k, \]

we have
\[ (Lx_j - P_j N'(u_j)x_j, x_j) = (Lx_j, x_j) - (P_j N'(u_j)x_j, x_j) \]
\[ \geq \sum_{k=1}^{\infty} \sum_{m^2 - l^2 \geq \lambda_{2k}} (m^4 - l^2) |a_{lmk}| \gamma \]
\[ = \sum_{k=1}^{\infty} \sum_{m^2 - l^2 \geq \lambda_{2k}} \left( m^4 - l^2 - [\lambda_{2k} - \beta(\|u_j\|)] \right) |a_{lmk}| \]
\[ \geq \beta(\|u_j\|) \|x_j\|^2. \] (2.9)

Similarly, if \( z_j \in \text{Dom } L \cap Z_j = Z_j \), we obtain
\[ (Lz_j - P_j N'(u_j)z_j, z_j) \leq -\alpha(\|u_j\|) \|z_j\|^2. \] (2.10)

Inequalities (2.9) and (2.10) imply that \( X_j \cap Z_j = \{0\} \) which, combined with the above condition \( \dim Y_j = \dim Z_j < \infty \) and Lemma 2 imply that \( H_j = X_j \oplus Z_j \) algebraically and hence topologically.

Consequently, if \( u_j \in H_j, \forall j \in H_j, v_j \) with \( x_j \in X_j \) and \( z_j \in Z_j \), and we obtain, using (2.9) and (2.10),
\[ \langle F'(u_j)v_j, x_j - z_j \rangle = \langle F'(u_j)x_j, x_j \rangle - \langle F'(u_j)z_j, z_j \rangle \]
\[ \geq \beta(\|u_j\|) \|x_j\|^2 + \alpha(\|u_j\|) \|z_j\|^2 \geq \gamma(\|u_j\|) \|v_j\|^2, \]
where \( \gamma(s) = \min\{\alpha(s), \beta(s)\} \). Furthermore, we have
\[ \gamma(\|u_j\|) \|v_j\|^2 \leq \|F'(u_j)v_j\|(\|x_j\| + \|z_j\|), \]
and since \( \sqrt{a^2 + b^2} \leq a + b \) and \( (a + b)^2 \leq 2(a^2 + b^2) \) for \( a \geq 0, b \geq 0 \) we have
\[ \gamma(\|u_j\|) \|v_j\| \leq \gamma(\|u_j\|)(\|x_j\| + \|z_j\|) \leq 2 \|F'(u_j)v_j\|. \] (2.11)

Now we will prove that the conditions of Lemma 1 are satisfied. First we show that \( F'_j(u_j) \) is a one to one mapping. If not, suppose \( w_1 \neq w_2 \) (\( w_1, w_2 \in H_j \)), such that \( F'_j(u_j)w_1 = F'_j(u_j)w_2 \). Then from (2.11) we have
\[ 0 = \|F'_j(u_j)w_1 - F'_j(u_j)w_2\| = \|F'_j(u_j)(w_1 - w_2)\| \geq \frac{\gamma(\|u_j\|)}{2} \|w_1 - w_2\| > 0, \]
which is a contradiction. Also \( F'_j(u_j)H_j \) is a closed subspace of \( H_j \). In fact, let \( \{z_m\} \subseteq F'_j(u_j)H_j \) and \( z_m \rightarrow z \), as \( m \rightarrow \infty \). Then there exists \( w_m \) such that
\[ \|z_m - w_m\| = \|F'_j(u_j)w_m - F'_j(u_j)w_m\| = \|F'_j(u_j)(w_m - w_m)\| \geq \frac{\gamma(\|u_j\|)}{2} \|w_m - w_m\|, \]
and we also have \( \|z_m - w_m\| \rightarrow 0 \), as \( n, m \rightarrow \infty \). This implies that \( \{w_m\} \) is a Cauchy sequence and, consequently, converges in \( H_j \), and thus there exists \( w \in H_j \) satisfying \( w_m \rightarrow w \). By the continuity of \( F'_j(u_j) \), we have
\[ F'_j(u_j)w_m \rightarrow F'_j(u_j)w, \] as \( w_m \rightarrow w \).

Hence \( z = F'_j(u_j)w \in F'_j(u_j)H_j \). This proves that \( F'_j(u_j)H_j \) is a closed subspace of \( H_j \).

Next, we prove that \( F'_j(u_j)H_j = H_j \). For this let us assume that there exists a \( v \in \{F'_j(u_j)H_j\}^\perp \) and \( v \neq 0 \). Then \( \langle v, F'_j(u_j)w \rangle = 0, \forall w \in H_j \). Let \( v = v_X + v_Z, v_X \in X, v_Z \in Z \) and set \( w = v_Z - v_X \). Then
\[ 0 = \langle v, F'_j(u_j)w \rangle \geq \beta(\|u_j\|) \|v_X\|^2 + \alpha(\|u_j\|) \|v_Z\|^2 \geq \frac{\gamma(\|u_j\|)}{2} \|w\|^2. \]
This is a contradiction since \( X_j \cap Z_j = \{0\} \). Hence \( F'_j(u_j)H_j = H_j \). Notice that
\[ \|F'_j(u_j)^{-1}\| \leq \frac{2}{\gamma(\|u_j\|)}. \] (2.12)
Now (note (1.11)) Lemma 1 guarantees that $F_j : H_j \rightarrow H_j$ is a homeomorphism. To obtain the estimate for the unique solution $u_j$ in $H_j$ to Eq. (2.8), with $h \in H$,

$$F_j(u_j) = P_jh,$$

we notice that

$$u_j = F_j^{-1}(P_jh) - F_j^{-1}(F_j(0));$$

hence, using the integral mean value theorem, we get

$$\|u_j\| = \|F_j^{-1}(P_jh) - F_j^{-1}(F_j(0))\|$$

$$\leq \int_0^1 \|(F_j')^{-1}(F_j(0)) + \theta(P_jh - F_j(0))\|d\theta \cdot \|P_jh - F_j(0)\|.$$ (2.13)

Now since

$$\|F_j(0) + \theta(P_jh - F_j(0))\| = \|\theta P_jh + (1 - \theta)F_j(0)\| \leq \|P_jh\| + \|F_j(0)\| \leq \|h\| + \|N(0)\|,$$

we have from (2.11) and (2.13) that

$$\|u_j\| \leq \frac{2}{\gamma(\|h\| + \|N(0)\|)}\|h\| + \|N(0)\|,$$ (2.14)

with a right-hand member independent of $j$. The proof of the lemma is complete. \(\square\)

The convergence result for Galerkin’s method associated to nonlinear perturbations of $L$ can be found from the following lemma (see [7, Lemma 3]).

Let $\tilde{L} : \text{Dom } \tilde{L} \subset H \rightarrow H$ be a linear, closed, densely defined operator such that $\text{Im } \tilde{L} = (\text{Ker } L)^{-1}$ and whose right inverse on $\text{Im } \tilde{L}$ defined by $\tilde{K} = (\tilde{L}|_{\text{Dom } \tilde{L}} \cap \text{Im } \tilde{L})^{-1}$ is compact. Denoting by $P : H \rightarrow H$ the orthogonal projector onto $\text{Ker } \tilde{L}$, we shall say that the sequence $\{v_k\}$ in $\text{Dom } \tilde{L}$ is $P$-convergent to $v \in H$, and we shall write

$$v_k \overset{P}{\rightarrow} v,$$

if

$$Pv_k \rightarrow Pv \quad \text{and} \quad (I - P)v_k \rightarrow (I - P)v$$

for $k \rightarrow \infty$, where $\rightarrow$ denotes the weak convergence in $H$.

**Lemma 4.** Assume that there exists a sequence $\{H_j\}$ of finite dimensional vector subspaces of $H$ such that

$$H_j \subset H_{j+1}, \quad \tilde{L}(\text{Dom } \tilde{L} \cap H_j) \subset H_j \quad (j \in N), \quad H = \bigcup_{j \in N} H_j,$$ (2.15)

and let $P_j : H \rightarrow H$ be the orthogonal projector onto $H_j$ ($j \in N^\ast$). Let $\tilde{N} : H \rightarrow H$ be a continuous monotone mapping which takes bounded sets into bounded sets. Assume that for some $h \in H$ and some $r > 0$ the equation

$$\tilde{L}u_j - P_j\tilde{N}u_j = P_jh,$$ (2.16)

has a solution $u_j \in \text{Dom } \tilde{L} \cap H_j$ such that $\|u_j\| \leq r \quad (j \in N)$. Then,

$$\tilde{L}u - P_j\tilde{N}u = P_jh$$ (2.17)

has at least one solution $u \in \text{Dom } \tilde{L}$ such that $\|u\| \leq r$.

Now we present our main theorem. We follow the argument in Mawhin [7].

**Theorem 1.** If the conditions (1.10) and (1.11) hold for $\forall(t,x) \in \Omega$, $\forall u \in H$, then the periodic boundary value problem (1.2)–(1.4) has a unique generalized solution.
Proof. As $\lambda_{1k}, \lambda_{2k} \in \sigma(L)$, for $k = 1, \ldots, n$, are consecutive (note also $0 \in \sigma(L)$), it follows that there exists an integer $0 \leq p \leq n$ such that

$$\lambda_{2k} \geq \lambda_{1k} \geq 0, \quad 1 \leq k \leq p, \quad \lambda_{1k} \leq \lambda_{2k} \leq 0, \quad p + 1 \leq k \leq n.$$ 

Now, define the operators $S_+$ and $S_-$ on $R^n$ as follows:

$$S_+ x = \sum_{k=1}^{p} \xi_k a_k, \quad S_- x = \sum_{k=p+1}^{n} \xi_k b_k,$$

for every $x = \sum_{k=1}^{n} \xi_k a_k$ in $R^n$. Using Lemma 2, we have

$$R^n = \text{Im } S_+ \bigoplus \text{Im } S_-$$

if we show that $\text{Im } S_+ \cap \text{Im } S_- = \{0\}$. In fact, for $x \in \text{Im } S_+$, we have

$$(B - \beta(||u||))x, x) \geq ((A + \alpha(||u||))x, x) = \sum_{k=1}^{p} (\lambda_{1k} + \alpha(||u||)) \xi_k^2 \geq \min_{1 \leq k \leq p} (\lambda_{1k} + \alpha(||u||)) \left( \sum_{k=1}^{p} \xi_k^2 \right),$$

and for $x \in \text{Im } S_-$, we have

$$(A + \alpha(||u||))x, x) \leq ((B - \beta(||u||))x, x) = \sum_{k=p+1}^{n} (\lambda_{2k} - \beta(||u||)) \xi_k^2 \leq \max_{p+1 \leq k \leq n} (\lambda_{2k} - \beta(||u||)) \left( \sum_{k=p+1}^{n} \xi_k^2 \right),$$

and as $\min_{1 \leq k \leq p} (\lambda_{1k} + \alpha(||u||)) > 0$ and $\max_{p+1 \leq k \leq n} (\lambda_{2k} - \beta(||u||)) < 0$, we have $\text{Im } S_+ \cap \text{Im } S_- = \{0\}$.

As is pointed in [7], if we now define the operators $\tilde{S}_+$ and $\tilde{S}_-$ on $H$ by

$$(\tilde{S}_\pm)(t, x) = S_\pm(u(t, x)) \quad \text{a.e. on } J,$$

then $H = \text{Im } \tilde{S}_+ \bigoplus \text{Im } \tilde{S}_-$ (topologically), $\tilde{S}_\pm(\text{dom } L) \subset \text{dom } L$, $\tilde{S}_+ - \tilde{S}_-$ is a linear homeomorphism on $H$ with $(\tilde{S}_+ - \tilde{S}_-)^{-1} = \tilde{S}_+ - \tilde{S}_-$, and, on $\text{dom } L$, we have

$$L \tilde{S}_\pm = \tilde{S}_\pm L.$$

Consequently, if we set in Eq. (2.6)

$$u = (\tilde{S}_+ - \tilde{S}_-)v, \quad \text{so that } v = (\tilde{S}_+ - \tilde{S}_-)u,$$

we obtain the equivalent equation

$$L(\tilde{S}_+ - \tilde{S}_-)v - N((\tilde{S}_+ - \tilde{S}_-)v) = h.$$

Moreover, $u_j \in H_j$ will be a solution of (2.8) if and only if $v_j = (\tilde{S}_+ - \tilde{S}_-)u_j \in H_j$ satisfies the equation

$$\tilde{L} v_j - P_j \tilde{N} v_j = P_j h,$$

where $\tilde{L} = L(\tilde{S}_+ - \tilde{S}_-), \tilde{N} = N \circ (\tilde{S}_+ - \tilde{S}_-)$. Now $\tilde{L}$ has the same domain, kernel, range and spectrum as $L$. For every $w \in H$, $\tilde{N}$ is also of class $C^1$ at $w$, and

$$\tilde{N}'(w) = N'((\tilde{S}_+ - \tilde{S}_-)w) \circ (\tilde{S}_+ - \tilde{S}_-).$$

Next we will show that $\tilde{N}$ is Lipschitzian and monotonic. From condition (1.10) and the above property (i.e., $(\tilde{S}_+ - \tilde{S}_-)^{-1} = \tilde{S}_+ - \tilde{S}_-$) of $\tilde{S}_+ - \tilde{S}_-$, we have (note $\|A\| = \rho(A)$ (the spectral radius of $A$) and $\|B\| = \rho(B)$) that

$$\|\tilde{N}(w)\| = \|N'((\tilde{S}_+ - \tilde{S}_-)w) \circ (\tilde{S}_+ - \tilde{S}_-)||$$

$$\leq \|N'((\tilde{S}_+ - \tilde{S}_-)w)||$$

$$= \|f_\alpha'(t, x, (\tilde{S}_+ - \tilde{S}_-)w)||$$

$$\leq \max_{1 \leq k \leq n} \{ |\lambda_{1k} |, |\lambda_{2k} | \}.$$ 

This implies that $\tilde{N}$ is Lipschitzian.
For every \( w \) and \( v \) in \( H \), we obtain, using the symmetry of \( N'(u) \),
\[
(\hat{N}'(w), v) = (N'((\hat{S}_+ - \hat{S}_-)w)\hat{S}_+ v, \hat{S}_- v) - (N'((\hat{S}_- - \hat{S}_+)w)\hat{S}_+ v) \\
= (N'((\hat{S}_+ - \hat{S}_-)w)\hat{S}_+ v, \hat{S}_+ v) - (N'((\hat{S}_- - \hat{S}_+)w)\hat{S}_+ v, \hat{S}_- v) \\
\leq ((A + \alpha(\|w\|))\hat{S}_+ v, \hat{S}_+ v) - ((B - \beta(\|w\|))\hat{S}_+ v, \hat{S}_- v) \geq 0.
\]
This implies that \( \hat{N} \) is monotone, and takes bounded sets into bounded sets. As \( \sigma(\hat{L}) \setminus \{0\} \) is made of eigenvalues with finite multiplicity with no finite accumulation point, its right inverse \( \hat{K} = (\hat{L}|_{\text{Dom} \hat{L}_\epsilon \setminus \text{Im} \hat{L}})^{-1} \) will be compact and we can apply Lemmas 3 and 4 to obtain the existence of \( v \in \text{Dom} \hat{L} \) such that \( \hat{L}v - \hat{N}v = h \), and hence the existence of the solution \( u = (\hat{S}_+ - \hat{S}_-)v \) for (2.6), i.e., the existence of the solution \( u \) to (1.2)–(1.4).

For the uniqueness, let \( u^1 \) and \( u^2 \) be two periodic solutions to (1.2)–(1.4) and set
\[
u_j^1 = P_ju^1, \quad \nu_j^2 = x_j^1 + z_j^1 \quad (x_j^1 \in X_j, z_j^1 \in Z_j), \quad v_j = x_j^1 - x_j^2, \quad w_j = z_j^1 - z_j^2,
\]
so that \( u_j^1 - u_j^2 = v_j + w_j \) (\( i = 1, 2; j \in \mathbb{N} \)). Therefore, using the notations of Lemma 3, we have
\[
0 = (L(u^1 - u^2) - N(u^1 - u^2), v_j - w_j) \\
= (L(u^1_j - u^2_j), v_j - w_j) - \left( \int_0^1 N'(u^1_j + s(u^1_j - u^2_j))(u^1_j - u^2_j)ds, v_j - w_j \right) \\
= (L(v_j + w_j), v_j - w_j) - \left( \int_0^1 N'(u^2_j + s(u^1_j - u^2_j))(v_j + w_j)ds, v_j - w_j \right) \\
- \left( \int_0^1 N'(u^2_j + s(u^1_j - u^2_j))(u^1_j - u_j^2 + u_j^2 - u_j^1)ds, v_j - w_j \right) \\
= (Lv_j, v_j) - \left( \int_0^1 N'(u^1_j + s(u^1_j - u^2_j))v_j ds, v_j \right) \\
- (Lw_j, v_j) + \left( \int_0^1 N'(u^2_j + s(u^1_j - u^2_j))w_j ds, w_j \right) \\
- \left( \int_0^1 N'(u^2_j + s(u^1_j - u^2_j))(u^1_j - u_j^2 + u_j^2 - u_j^1)ds, v_j - w_j \right) \\
\geq \int_0^1 \beta(\|u^2_j + s(u^1_j - u^2_j)\|)ds \|v_j\|^2 + \int_0^1 \alpha(\|u^2_j + s(u^1_j - u^2_j)\|)ds \|w_j\|^2 - \delta(\|u^1_j\| + \|u^2_j\|^2),
\]
where the first two terms of the last inequality are due to (2.18) and (2.19) and Lemma 3, the last term is due to (2.20) and condition (1.10), and \( \delta > 0 \) is some constant depending only on \( |\lambda_{1k}|, |\lambda_{2k}| \) (\( 1 \leq k \leq n \)) and \( r \) (see Lemma 4). Consequently, \( v_j \to 0, w_j \to 0 \) as \( j \to \infty \), so that
\[
u_j^1 - u_j^2 = \lim_{j \to \infty} (u_j^1 - u_j^2) = \lim_{j \to \infty} (v_j + w_j) = 0,
\]
and the proof of the theorem is now complete. \( \square \)

3. An example

We illustrate our theory with an example.

Example. Consider the following semilinear elastic beam equations in one dimension space,
\[
u_{tt} + \nu_{xxxx} + f(t, x, u) = h(t, x), \quad \forall(x, t) \in J,
\]
(3.1)
with the boundary value conditions,
\begin{align}
    u(t, 0) &= u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0, \\
    u(0, x) &= u(2\pi, x), \quad u_t(0, x) = u_t(2\pi, x),
\end{align}
(3.2)
where \( J = [0, 2\pi] \times [0, \pi] \), and assume \( f(t, x, u) = mu - \frac{1}{2} \arctan(u), m \in \mathbb{Z} \) and \( h : J \to \mathbb{R} \).

Now Theorem 1 guarantees a unique solution to (3.1)–(3.3), for arbitrary \( h(t, x) \), since
\[
    m - 1 + \frac{1}{2} \leq f''_u(t, x, u) = m - \frac{1}{2(1 + u^2)} < m,
\]
\[
    \lim_{\|u\| \to \infty} \|f''_u(t, x, u) - m\| = 0.
\]

References