Coordinated Multiple Spacecraft Attitude Control with Communication Time Delays and Uncertainties

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Abstract

In this paper, we consider the coordinated attitude control problem of spacecraft formation with communication delays, model and disturbance uncertainties, and propose novel synchronized control schemes. Since the attitude motion is essential in non-Euclidean space, thus, unlike the existing designs which describe the delayed relative attitude via linear algorithm, we treat the attitude error and the local relative attitude on the nonlinear manifold-Lie group, and attempt to obtain coupling attitude information by the natural quaternion multiplication. Our main focus is to address two problems: 1) Propose a coordinated attitude controller to achieve the synchronized attitude maneuver, i.e., synchronize multiple spacecraft attitudes and track a time-varying desired attitude; 2) With known model information, we achieve the synchronized attitude maneuver with disturbances under angular velocity constraints. Especially, if the formation does not have any uncertainties, the designer can simply set the controller via an appropriate choice of control gains to avoid system actuator saturation. Our controllers are proposed based on the Lyapunov-Krasovskii method and simulation of a spacecraft formation is conducted to demonstrate the effectiveness of theoretical results.

Keywords: attitude control; synchronized attitude maneuver; quaternion; communication time delays; uncertainties; angular velocity constraints

1. Introduction

In recent years, distributed cooperative control of multi-agents is a new promising trend due to its important application in mechanical systems such as the robot formation\textsuperscript{[1-2]} and spacecraft formation\textsuperscript{[3-4]}. For the satellite formation mission, the coordinated attitude control has received extensive interests for its application in the space-based interferometer and has led to many significant theoretical developments.

In Ref. \textsuperscript{[5]}, the coordinated attitude control law for each spacecraft, based on the behavioral approach, was involved using the state information of its two adjacent neighboring members. For the angular velocity-free case, a passivity method\textsuperscript{[6]} was adopted. Based on an auxiliary dynamical system, Abdessameud and Tayebji\textsuperscript{[7]} designed a velocity-free attitude tracking and synchronized control scheme under the undirected topologies. Ren\textsuperscript{[8-9]} generalized the attitude synchronization to the case of tracking and for directed topologies, then extended his results using the modified Rodriguez parameters (MRP) parameterization for the attitude representation\textsuperscript{[10]}. The attitude coordination control with the geometrical structure was shown by Ref. \textsuperscript{[11]}.

In applications, affected by various reasons, the information flow sometimes is delayed in many practical situations. Wang and Xie\textsuperscript{[12]} addressed the delayed attitude synchronization on SO(3) by employing geometrical structure, and generalized Ref. \textsuperscript{[11]} to the constant delayed links, but it was hard to deal with the multiple equilibrium points in this space; also, via the
passivity, the final constant attitude was unknown. For the bilateral teleoperation with Lagrange form in task space, Ref. [13] addressed two agents, i.e., the master-slave coordination scheme. However, both did not formulate the synchronization problem when a desired trajectory was explicitly defined. Also, the problem with uncertainties, such as model and disturbance uncertainties and angular velocity constraints, is not considered.

In fact, due to the jet propulsion, the spacecraft is hard to obtain exact model information. Also, the space environment may contain some unknown disturbances. An adaptive controller was designed for model insensitivity in Ref. [14], and a robust adaptive controller appeared in Ref. [15] for disturbance rejection. For multiple spacecraft system, the uncertainty problem also attracted some interests. Ref. [16] adopted an adaptive law to propose a model-insensitive controller, and to minimize the error of relative attitude in the presence of disturbances, optimal control was presented in Ref. [17].

For the delayed coordinated attitude control problem (CACP) with model or and disturbance uncertainties, Jin, et al. proposed a synchronized attitude controller, achieving the robustness for identical communication delays existing in topologies in Ref. [18], and then considered the synchronized attitude maneuver controller in Ref. [19]. Meng, et al. [20] presented model-independent cooperative controllers to achieve the uniformly ultimate boundedness with delayed and switched topologies. In other works, the delayed CACP was described by MRP and then was transferred to Euler-Lagrange form. Ref. [21] investigated the coordinated attitude regulation problem with the adaptive law for uncertainties, and a recent literature Ref. [22] proposed a unified scheme that handled the mentioned problem with/without a desired trajectory.

Note that, in the most existing schemes for the delayed CACP with uncertainties, the local relative attitude information is obtained via making the differences of absolute attitude errors, instead of via the nature quaternion multiplication. This difference manner can provide the convenience for the controller design; however, from theoretical aspect, the attitude motion on Lie group is non-convex, and thus exhibits different property from \( R^3 \). Therefore, we would like to state the relative attitude via the natural nonlinear manifold.

Since the coordinated controller via relative attitude on its natural nonlinear manifold can solve the delayed-free CACP, we attempt to extend this result to the delayed case, and propose a unified scheme that handles the CACP in the presence or absence of time delays with uncertainties. Due to the complexity of attitude motion in non-convex space, this extension is not trivial, and more work should be done, just as pointed out in Ref. [19] and Ref. [22]. In addition, we remove several inequality constraints, required in Ref. [19] for the stabilization.

Due to the complexity of disturbances in space, we try to deal with linearly parameterizable disturbances, just as mentioned in Ref. [23] for single spacecraft. Also, an adaptive law for disturbance rejection is considered, under the angular velocity constraint for each member, with the method inspired by Ref. [24]. In applications, the angular velocity sometimes is restricted due to the property of the physical gyros and the rendezvous mission. Additionally, for the formation without disturbances, the designer can choose the controller gains to avoid system actuator saturation simply, under this constraint.

2. Problem Formulation

2.1. Preliminary

In this paper, undirected topologies [25] are adopted to describe the interaction between the group members. A graph \( G \) consists of a finite nonempty vertex set \( V(G) \) and an edge set \( E(G) \), and an edge of \( G \) is denoted by \((i, j) \in E\). Assume \( a_0 = 1 \) if \((i, j) \in E\), and \( a_0 = 0 \) otherwise. For all \( i \in I \), we define \( a_i = 0 \).

It is worth noting that the agents are interconnected in a non-ideal network with communication delayed links, where \( T_{ij} \) denotes the delayed propagation of state information from \( j \) to \( i \). The time delays are called nonuniform if \( T_{ij} \neq T_{ji} \) for any \((i, j) \in E\).

2.2. Spacecraft attitude dynamics

Throughout this paper, the attitude of the rigid body will be represented in terms of the unit quaternion [26]. A unit quaternion \( \mathbf{q} \in \mathbb{R}^4 \) is defined as \( \mathbf{q} = [\eta \, \mathbf{q}^T]^T \), subjecting to the constraint

\[
\eta^2 + \mathbf{q}^T \mathbf{q} = 1
\]

where \( \eta \in \mathbb{R} \) is the scalar part and \( \mathbf{q} \in \mathbb{R}^3 \) the vector part. The inverse of \( \mathbf{q} \) is \( \mathbf{q}^{-1} = [\eta \, \mathbf{q}^T]^T \). The rotation matrix describing the rotation from the inertial frame to the body frame can be obtained through \( \mathbf{R} \):

\[
\mathbf{R} = I_3 - 2\eta \mathbf{q} \mathbf{q}^T + 2\mathbf{q} \mathbf{q}^T
\]

where \( \cdot^T \) is the skew symmetric matrix of \( \cdot \), and \( I_3 \) the identity matrix. Let us consider a group of \( n \) rigid spacecraft. The dynamical equation of the \( i \)-th body is
\[ J \dot{q} = \tau + I - \omega J \omega + J R_d \dot{\omega}_d + J \dot{R}_d \omega_d \] (3)

and the kinematical equation is

\[ \dot{q}_i = \frac{1}{2} q^T_i \dot{w}_i, \quad \ddot{q}_i = \frac{1}{2} (\eta I + \ddot{q}_i) \omega_i \] (4)

where \( \omega_i \in \mathbb{R}^3 \), \( J \in \mathbb{R}^{3 \times 3} \), \( \tau \in \mathbb{R}^3 \) and \( I \in \mathbb{R}^3 \) denote the angular velocity, constant symmetric positive definite inertial matrix, the control torque and external disturbance respectively, all of which are expressed in its body frame. For tracking control, the desired attitude is described as \( \dot{q} = \dot{q}_d \), which is represented in the desired frame. The absolute attitude error is defined as \( \ddot{q} = [\ddot{q}_i \, \ddot{q}_j] \) with given by

\[ \ddot{q}_i = [\ddot{q}_i \, \ddot{q}_j] \] (5)

where \( \ddot{q}_i \) denotes the natural quaternion multiplication and the angular velocity error is shown by

\[ \dot{\omega}_i = \omega_i - \dot{R}_d \omega_d \] (6)

where \( \dot{R}_d = R \ddot{R} \) can be obtained by Eq. (2), and \( \omega_d \) is desired angular velocity.

The attitude error dynamics is

\[ J \dot{\omega}_i = \tau_i + I - \omega_i J \omega_i - J R_d \dot{\omega}_d + J \dot{R}_d \omega_d + k J \dot{\ddot{q}}_i = \tau_i + I - Y_i (R_d \dot{\omega}_d, \omega_d, \dot{\omega}_d, \ddot{\omega}_d, \dot{\ddot{q}}_i) \theta_i \] (7)

where \( \theta_i = [J_{i,11} J_{i,12} J_{i,13} J_{i,23} J_{i,33}] \in \mathbb{R}^8 \) is a constant vector, and \( Y_i \in \mathbb{R}^{3 \times 6} \) is known, expressed as

\[ Y_i = \omega_i L(\omega_i) + L(R_d \dot{\omega}_d) - L(R_d \dot{\omega}_d, \omega_d) - k L(\dot{\ddot{q}}_i) \] (8)

where \( L : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 6} \) is a linear operator acting on \( a = [a_1, a_2, a_3, a_4] \) by

\[ L(a) = \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 \\ 0 & a_2 & a_3 & 0 & a_1 \\ 0 & a_3 & a_2 & 0 & a_1 \end{bmatrix} \] (9)

\[ L (a) = \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 \\ 0 & a_2 & a_3 & 0 & a_1 \\ 0 & a_3 & a_2 & 0 & a_1 \end{bmatrix} \] (10)

Assumption 1 [23] We assume that the disturbance \( I \) can be linearly parameterized as

\[ \dot{I} = \Psi \dot{\theta}_i \] (11)

where \( \theta_i \) is a constant \( p \)-dimensional vector, \( \Psi \in \mathbb{R}^{3 \times p} \) the matrix of known functions of states, their derivatives and \( t \). \( \Psi \) is uniformly continuous with respect to \( t \).

Let \( \ddot{\theta}_i \) and \( \dddot{\theta}_i \) be estimated states of \( \dot{\theta}_i \) and \( \theta_i \), \( \dot{\theta}_i = \ddot{\theta}_i - \theta_i \) and \( \dddot{\theta}_i = \dddot{\theta}_i - \theta_i \) be estimated errors.

It is worth noting that \( \ddot{q}_i \) and \( \dddot{q}_i \) also meet

\[ \ddot{q}_i = \frac{1}{2} q^T_i \dot{w}_i, \quad \dddot{q}_i = \frac{1}{2} (\ddot{q}_i I + \dddot{q}_i) \omega_i \] (12)

We define the delayed relative attitude information \( \dddot{Q}_i(t) = [\ddot{q}_i \, \dddot{q}_i] \) by

\[ \dddot{Q}_i(t) = \dddot{Q}_i(t - T) \odot \dddot{Q}_i(t) \] (13)

and the delayed relative angular velocity \( \dddot{\omega}_i(t) \) by

\[ \dddot{\omega}_i(t) = \dddot{\omega}_i(t - T_0) \] (14)

where \( \dddot{Q}_i = R(\dddot{Q}_i(t)) \).

According to the definition of \( T_0 \), the delayed states are

\[ \dddot{Q}_i(t - T_0) = \dddot{Q}_i(t - T) \odot \dddot{Q}_i(t - T_0) \] (15)

\[ \dddot{\omega}_i(t - T_0) = \dddot{\omega}_i(t - T) - \dddot{\omega}_i(t - T_0) \] (16)

where \( \dddot{Q}_i = R(\dddot{Q}_i(t - T_0)) \). If \( T_0 \) vanishes, then \( \dddot{Q}_i(t) \) and \( \dddot{\omega}_i(t) \) are denoted by \( \dddot{Q}_i(t) \) and \( \dddot{\omega}_i(t) \), with \( \dddot{R} = R(\dddot{Q}_i(t)) \).

2.3. Problem statement

In this note, for the CACP, we firstly address the synchronized attitude maneuver problem (SAMP), i.e., \( q_i(t) \rightarrow q_i(t) \rightarrow q_i(t) \) and \( \omega_i(t) \rightarrow \omega_i(t) \rightarrow \omega_i(t) \) asymptotically, with delayed links as well as model and disturbance uncertainties. Secondly, with known model information, we consider the delayed SAMP with disturbances and angular velocity constraints.

For delay-free links, without any uncertainties, Ref. [4] proposed a controller, similarly to the form in Ref. [8]:

\[ \tau_i = -k_1 \dddot{q}_i - d_1 \dddot{\omega}_i + \omega_i J \dot{\omega}_i + J R_d \dot{\omega}_d - J \dot{\omega}_d R_d \omega_d + \dddot{A} \] (17)

where \( \dddot{A} = -\sum_{i=1}^{n} \alpha_i k_i \dddot{q}_i - \sum_{i=1}^{n} \alpha_i d_i \dddot{\omega}_i \), while \( k_i, d_i, k_i \) and \( d_i \) are positive gains. Note that \( \dddot{q}_i \) and \( \dddot{\omega}_i \) are obtained by Eqs. (13)-(14) with \( T_0(t) = 0 \).

Considering the delayed communication links, Jin, et al. studied the synchronized attitude regulation problem (SARP), a special case of SAMP with constant \( q_i(t) \) in Ref. [18] and the SAMP in Ref. [19], and Meng, et al. designed the model-independent controllers in Ref. [20] for SAMP. However, for all the mentioned controllers, also in Ref. [16], Refs. [21]-[22], the delayed relative attitude is defined via the differences of absolute attitude errors, as \( q_i(t) - R_i \dddot{q}_i(t - T_0) \) or \( q_i(t) - \dddot{q}_i(t - T_0) \), instead of the natural quaternion multiplication as Eq. (5) or Eq. (13). Their definition manner aims to provide convenience for the controller design, since it deals with this local nonlinear term living on linear manifolds simply. However, from
3. Design of Coordinated Attitude Controllers

3.1. Controller design with model and disturbance uncertainties

First, for the delayed SAMP without any uncertainties, we will show that the controller (17) has the robustness for some constant nonuniform delays.

**Theorem 1** If information exchanged topologies are connected, then the controller (17), designed for the delay-free links, is also effective for the delayed SAMP, if the following constraints hold:

\[ d_i > \frac{1}{2} \sum_{j \in N_i} k_y T_y \]  
\[ k_i > 2 \sum_{j \in N_i} k_y \]  

Note that for the delayed problem, \( \tilde{q}_y \), \( \tilde{\omega}_y \) and \( \tilde{A} \) in Eq. (17) should be replaced by \( \hat{q}_y \), \( \hat{\omega}_y \) and \( \hat{A} \) respectively. Prior to the proof, we state two lemmas first.

**Lemma 1** [27] Let \( f : \mathbb{R} \to \mathbb{R} \) and \( 1 \leq p < \infty \). If \( f \in L_p \) and if \( f \) is uniformly continuous on \( \mathbb{R} (j \in L_p) \), then \( \int f(t) \) → 0 as \( t \to \infty \).

**Lemma 2** [28] For any vector signals \( x \), \( y \) and any \( T, \alpha > 0 \), the following in equality holds:

\[ 2 \int_0^T x^T(\tau) \int_0^T y(\tau - \sigma) d\sigma d\tau \leq \alpha \left\| x^T \right\|_p^2 + \frac{T^2}{\alpha} \left\| y \right\|_p^2 \]

where \( \left\| \cdot \right\|_p \) is the \( L_p \)-norm of the signal \( \cdot \).

**Proof of Theorem 1** The closed-loop system dynamics formed by Eq. (7) and Eq. (17) with delays is as

\[ J_i \dot{\hat{\omega}}_i = -k_i \hat{q}_y - d_i \hat{\omega}_y + \tilde{A} \]

For the Lyapunov function candidate,

\[ V_i = \sum_{j=1}^{n} k_i (2 - 2\eta_j) + \frac{1}{2} \sum_{j \in N_i} \hat{\omega}_y^T J \hat{\omega}_y + \frac{1}{2} \sum_{j \in N_i} k_i \int_{\tau_0}^{T} \hat{\omega}_y^T d\tau \]

(21)

where \( \eta_i \) is the scalar part of \( \tilde{\mathcal{Q}}_i (t) = \tilde{\mathcal{Q}}_{ij} (t) \circ \mathcal{Q}_i (t) \), differentiating \( V_i \) with respect to time, we get

\[ V_i = -\sum_{j=1}^{n} k_i \int_{\tau_0}^{T} \hat{\omega}_y^T \tilde{\mathcal{Q}}_{ij} (t-T_\tau) + \frac{1}{2} \sum_{j \in N_i} k_i \hat{\omega}_y^T \tilde{\mathcal{Q}}_{ij} \]

Then, \( V_i \) can be rewritten as the following form:

\[ V_i = -\sum_{j=1}^{n} k_i d_i \tilde{\mathcal{Q}}_{ij} \hat{\omega}^T \hat{\omega}_y + \frac{1}{2} \sum_{j \in N_i} k_i \hat{\omega}_y^T \tilde{\mathcal{Q}}_{ij} \]

(22)

Switching the order of the summation signs of the term \( \sum_{i=1}^{n} \sum_{j=1}^{n} a_i d_i \tilde{\mathcal{Q}}_{ij} \hat{\omega}_y \), and selecting \( d_i = d_{ij} \) for any \( (i, j) \in E \), then we can obtain

\[ V_i = -\sum_{i=1}^{n} d_i \tilde{\mathcal{Q}}_{ij} \hat{\omega}_y \]

(23)

Let \( k_{ij} = k_{ij} \) and integrate Eq. (23) from 0 to \( T \), then it is not hard to get

\[ V_i (t) - V_i (0) = -\sum_{i=1}^{n} d_i \tilde{\mathcal{Q}}_{ij} \hat{\omega}_y \]

(24)

To overvalue the last term in Eq. (26) so as to continue this proof, we introduce Property 1, whose proof is given in Appendix A based on Lemma 2 and Eq. (12).

**Property 1** For the last term in Eq. (26), it holds

\[ \sum_{i=1}^{n} \sum_{j \in N_i} k_i \int_{\tau_0}^{T} \hat{\omega}_y^T (\tilde{q}_y - \hat{q}_y) d\tau \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} k_i T_y \hat{\omega}_y \]

Using Property 1, we have
\[
V_i(t) - V_i(0) \leq -\frac{1}{2} \sum_{j=1}^{n} d_i \left\| \dot{q}_i \right\|^2 - \frac{1}{2} \sum_{j=1}^{n} \sum_{i,j \neq i} d_{ij} \left\| \dot{\omega}_i \right\|^2 + \frac{1}{2} \sum_{j=1}^{n} \sum_{i,j \neq i} k_{ji} \dot{T}_{ji} \left\| \dot{\omega}_i \right\|^2 
\]

(27)

Obviously, if we select velocity damps as Eq. (18) for \( i \), then \( V_i(t) \) is bounded, thus, via Eq. (27), \( \dot{\omega}_i(t) \in L_\infty \). From Eq. (21), \( \dot{q}_i(t) \in L_\infty \), \( \dot{\omega}_i(t) \in L_\infty \) and then \( \dot{\omega}_i(t) \in L_\infty \). Since Eq. (1) holds, then from Eq. (20), we can get \( \dot{\omega}_i(t) \in L_\infty \). Hence, from Lemma 1, it holds \( \lim_{t \to \infty} \dot{\omega}_i(t) = 0 \) for all \( i \in I \). Then from Eq. (14), we know. Clearly, Eq. (20) can be rewritten as

\[
k_i \dot{q}_i + \sum_{j=1}^{n} a_{ij} k_{ji} \dot{q}_j = 0
\]

(28)

Recall Eq. (12), we have \( \lim_{t \to \infty} \dot{q}_i(t) = 0 \). According to the finite increment theorem, it holds \( \dot{q}_i(t) = \dot{q}_i(t - T_g) \), as \( t \to \infty \). From the definition of \( \dot{q}_i(t) \), one has \( \dot{q}_i(t) = \dot{q}_i(t) \), as \( t \to \infty \). Therefore, Eq. (28) is equal to

\[
k_i \dot{q}_i + \sum_{j=1}^{n} a_{ij} k_{ji} \dot{q}_j = 0
\]

(29)

as \( t \to \infty \). Finally, motivated by the analysis in Refs. [7]-[8], for the connected topologies, if the control parameters meet Eq. (19), then \( \dot{q}_j(t) \to 0 \) and \( \dot{q}_j(t) \to 0 \), as \( t \to 0 \) (see the detailed discussion in Refs. [7]-[8]).

**Remark 1** We achieve the delayed robustness of Eq. (17), and thus, propose a unified scheme that handles the SAMP with/without delays. Obviously, our approach is more realistic in applications, due to the potential existence of delays. In contrast to previous results, we deal with the relative attitude based on the non-convex property via its natural nonlinear manifolds.

The controller (17) is model-dependent and disturbance sensitive. Next, let us consider the delayed SAMP with model and disturbance uncertainties, and propose an adaptive coordinated controller

\[
\tau_i = -k_i \dot{q}_i - d_i \dot{\omega}_i + \dot{\bar{A}} + \dot{Y} \dot{\bar{\Theta}} - \mathbf{\Psi} \dot{\Theta}_i
\]

(30)

where \( \dot{\bar{A}} \) and all gains are the same as the ones in Theorem 1. Design the updated laws as

\[
\dot{\bar{\Theta}} = -\Gamma_i^{-1} \mathbf{Y}_i \dot{s}_i
\]

(31)

\[
\dot{\bar{\Theta}} = \Gamma_i^{-1} \mathbf{\Psi} i \dot{s}_i
\]

(32)

where \( \Gamma_i \in \mathbb{R}^{n \times n} \) and \( \Gamma_i \in \mathbb{R}^{n \times p} \) are positive definite gain matrices, then we have the second result.

**Theorem 2** The controller (30), together with the estimated laws (31) and (32), can solve the delayed SAMP with model and disturbance uncertainties.

**Proof** The closed-loop system formed by Eq. (8) and the controller (30) is as

\[
J_s = -k_i \dot{q}_i - d_i \dot{\omega}_i + \dot{A} + \dot{Y} \dot{\bar{\Theta}} - \mathbf{\Psi} \dot{\Theta}_i
\]

(33)

For the Lyapunov function candidate

\[
V_2 = \frac{1}{2} \sum_{i=1}^{n} \dot{s}_i^T J_i \dot{s}_i + \frac{1}{2} \sum_{i=1}^{n} \left( k_d + k_t \right) \sum_{j=1}^{n} d_{ij} \left\| \dot{\omega}_i \right\|^2 (2 - 2T_i^2) + \frac{1}{2} \sum_{i=1}^{n} k_{ji} \dot{T}_{ji} \left\| \dot{\omega}_i \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} \dot{\Theta}_i^T \Gamma_{\dot{\Theta}} \dot{\Theta}_i + \frac{1}{2} \sum_{i=1}^{n} \dot{\Theta}_i^T \Gamma_{\dot{\Theta}} \dot{\Theta}_i + \frac{1}{2} \sum_{i=1}^{n} k_{ji} \dot{T}_{ji} \left\| \dot{\omega}_i \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} k_{ji} \dot{T}_{ji} \left\| \dot{\omega}_i \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} d_{ij} \left\| \dot{\omega}_i \right\|^2 - \frac{1}{2} \sum_{i=1}^{n} d_{ij} \left\| \dot{\omega}_i \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} d_{ij} \left\| \dot{\omega}_i \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} d_{ij} \left\| \dot{\omega}_i \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} d_{ij} \left\| \dot{\omega}_i \right\|^2
\]

(26)

(34)

and based on Eqs. (31)-(33), we have

\[
\dot{V}_2 = \frac{1}{2} \sum_{i=1}^{n} \left( d_i \dot{\omega}_i + k_{ji} \dot{T}_{ji} \right) (2 - 2T_i^2)
\]

(35)

(36)

Then from Eq. (35), it is easy to get

\[
V_2 \leq -\frac{1}{2} \sum_{i=1}^{n} d_{ij} \dot{\omega}_i \dot{\omega}_i + \frac{1}{2} \sum_{i=1}^{n} d_{ij} \dot{\omega}_i \dot{\omega}_i
\]

(37)

Integrate Eq. (37) from 0 to \( t \), and invoke Property 1, then

\[
V_2(t) - V_2(0) \leq -\frac{1}{2} \sum_{i=1}^{n} d_{ij} \left\| \dot{\omega}_i \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} d_{ij} \left\| \dot{\omega}_i \right\|^2 + \frac{1}{2} \sum_{i=1}^{n} d_{ij} \left\| \dot{\omega}_i \right\|^2
\]

(38)
Obviously, if the following inequalities hold
\[
k_i > 3 \sum_{j \neq i} k_{ij} + \frac{1}{2} \sum_{j \neq i} d_{ij} \tag{39}
\]
\[
d_i > \frac{1}{2} \sum_{j \neq i} k_{ij} T_{ij} + \frac{1}{2} \sum_{j \neq i} d_{ij} \tag{40}
\]
for each member \(i\), then from Eq. (38), \( V_i(t) \) is bounded and thus \( \tilde{\theta}_i(t) \in L_2\), \( \tilde{\phi}_i(t) \in L_2\). Then from Eq. (34), \( \tilde{\theta}_i(t) \in L_\infty\), \( \tilde{\phi}_i(t) \in L_\infty\) and \( \tilde{\theta}_i(t) \in L_\infty\). Via Eq. (12), \( \tilde{\phi}_i(t) \in L_\infty\), then from Eq. (33) and Assumption 1, we can deduce that \( \tilde{\phi}_i(t) \in L_\infty\). Therefore, from Lemma 1, we get \( \tilde{\theta}_i(t) \to 0\). Recall \( \tilde{\phi}_i(t) \in L_\infty\) and \( \tilde{\phi}_i(t) \in L_\infty\), then we also have \( \tilde{\phi}_i(t) \to 0\). Hence, we complete this proof.

Note that the conditions (39) and (40) are more rigorous than conditions (18) and (19) for the system without uncertainties. Fortunately, it is simple to select all gains. Let us firstly select the synchronized tracking gains. Let us consider the Lyapunov function candidate: \( V_i(t) = \frac{1}{2} \sum_{j \neq i} k_{ij} (\theta_j - \theta_i)^2 + \frac{1}{2} \sum_{j \neq i} d_{ij} (\phi_j - \phi_i)^2 \).

**Remark 2** Obviously, for constant \( q_d \), as the delayed SARP in Ref. [18] and Ref. [21], our proposed controller is also effective. Since \( q_d \) is constant, the SAPR can be treated as an extension of regulation problem, which can be solved via a simple model-independent PD controller [29]. Therefore, for the SAPR, model information (or the updated law for \( \tilde{\theta}_i \)) is unnecessary, i.e., the proposed controller
\[
\tau_i = -k_i \tilde{\phi}_i - d_i \tilde{\theta}_i + \hat{A}_i \tag{41}
\]
can solve the delayed SAPR with model uncertainties, while a Lyapunov function, similar to Eq. (34), benefits the stability analysis. However, for the SAMP or the SAPR with external disturbances, the model information (or the undated laws) is needed, just as the design in Ref. [21]; otherwise, one could only obtain the ultimate boundedness [20], rather than the asymptotical stability.

**Remark 3** Compared with Ref. [19], we deal with the delayed relative attitude via Eq. (13), and thus the whole stability analysis is different. Seen from our approach, this extension from the difference to the quaternion multiplication is not trivial but more complex due to its natural non-convex property. Also, besides the benefit stated in Remark 1, our controller admits some other advantages. First, the control gain selection with multiple inequality constraints in Ref. [19] is too strict, which is removed here, and our control gain selection is simple and flexible; second, from our design, the basic information about unknown inertial matrix, such as its singular values, is not needed; third, the second term (32) designed for disturbance rejection is more flexible, unlike Ref. [19], which required a small gain in its controller and thus limited the rejection performance. Also, we consider the general form disturbance, rather than the constant disturbance, and thus our design will be more useful in applications.

Note that, in Ref. [19], the definition of rotation matrix in its controller (23) may be inaccurate. Since the local relative information is obtained from the tracking errors \( \tilde{\phi}_i \) and \( \tilde{\theta}_i \), instead of the absolute states \( \theta_i \) and \( \phi_i \), then the corresponding rotation matrix \( \tilde{R}_i \) should be obtained by \( \tilde{R}_i = R(\tilde{\phi}_i(t), \tilde{\theta}_i(t)) \), where \( \tilde{\phi}_i(t) \) is described by Eq. (13). From Eq. (5) and Eq. (15), it holds \( \tilde{R}_i = R(\tilde{\phi}_i(t), \tilde{\theta}_i(t)) \). Therefore, from the relation adopted in Ref. [19], where it is \( C_y(t-T_y) = R(\tilde{\phi}_i(t), \tilde{\theta}_i(t)) \). As a matter of fact, \( \tilde{R}_i \) describes the delayed rotation matrix from \( \tilde{\theta}_i(t-T_y) \) to \( \tilde{\theta}_i(t) \), while \( C_y(t-T_y) \) describes the delayed rotation matrix from the signal \( \theta_i(t-T_y) \) to \( \theta_i(t) \). If the local relative information is obtained by absolute states rather than tracking errors, then the corresponding controller cannot solve the delayed SAMP. We will discuss this issue elsewhere.

### 3.2. Controller design with disturbances and angular velocity constraints

We will address the delayed SAMP with disturbances, under the angular velocity constraints. Assume that the model information is accurate. First, consider the adaptive controller for disturbance rejection.

The closed-loop system via Eq. (8) and controller
\[
\tau_i = J_i K_i^{-1}(-k_i \tilde{\phi}_i - d_i \tilde{\theta}_i + \hat{A}_i) + Y_i \tilde{\theta}_i - \Psi_i \tilde{\theta}_i \tag{42}
\]
is expressed as
\[
\dot{s}_i = K_i^{-1}(-k_i \tilde{\phi}_i - d_i \tilde{\theta}_i + \hat{A}_i) - J_i \Psi_i \tilde{\theta}_i \tag{43}
\]
where \( K_i \) is the positive gain matrix to be designed.

Let us consider the Lyapunov function candidate:
\[
V_i = \frac{1}{2} \sum_{i=1}^n \ln \left( \frac{\ell^n - s_i^n}{\ell^n - s_i^n} \right) \tag{44}
\]
\[
+ \sum_{i=1}^n k_{ij} d_{ij} (2-2\tilde{n}_i) + \frac{1}{2} \sum_{i=1}^n k_{ij} (2-2\tilde{n}_i) + \frac{1}{2} \sum_{i=1}^n \tilde{n}_i^2 \tilde{\theta}_i^2 \tilde{\phi}_i^2 \tag{45}
\]
\[
+ \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} d_{ij} \int_{T_i}^{T_j} \tilde{n}_j^2 \tilde{\theta}_i \tilde{\phi}_i d\tau + \frac{k}{2} \sum_{i=1}^n \sum_{j \neq i} k_{ij} \int_{T_i}^{T_j} \tilde{n}_j^2 \tilde{\theta}_i \tilde{\phi}_i d\tau \tag{46}
\]
where $s_{i,v}$ is the $v$th entry of the vector $s_i$, and $l$ some positive constant. Then we have

$$
V_1 = \sum_{i=1}^{n} s_{i,v} \mathbf{K_i} s_i + \sum_{i=1}^{n} \left( k d_i + k_i + k \sum_{j=1}^{2} d_j \right) \hat{\omega}_i \hat{\omega}_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=N}^{n} k_{ij} \hat{q}_{ij} \hat{q}_{ij} + \sum_{i=1}^{n} \mathbf{\Theta}_{i} \mathbf{\Gamma}_2 \mathbf{\Theta}_{i} + \frac{1}{2} k \sum_{i=1}^{n} \sum_{j=N}^{n} \left( kd_i + k_i + k \sum_{j=1}^{2} d_j \right) \hat{\omega}_i \hat{\omega}_i - \frac{1}{2} k \sum_{i=1}^{n} \sum_{j=1}^{n} \left( d_i \hat{\omega}_i \right) (t - T_y) \cdot \hat{\omega}_i (t - T_y) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=N}^{n} \left( k \hat{q}_{ij} \hat{q}_{ij} - \frac{1}{2} k \sum_{i=1}^{n} \sum_{j=1}^{n} \left( d_i \hat{\omega}_i \right) (t - T_y) \cdot \hat{\omega}_i (t - T_y) \right)
$$

where $\mathbf{K}_i = \text{diag}(1/(l^2 - s_{i,1}^2), 1/(l^2 - s_{i,2}^2), 1/(l^2 - s_{i,3}^2))$.

Consider Eq. (43) and design the following updated law

$$
\hat{\Theta}_i = \hat{G}_2^{-1} \hat{\Psi}_i^{-1} \hat{J}_i^{-1} \hat{K}_i \hat{s}_i
$$

then we can get

$$
\dot{V}_1 = \sum_{i=1}^{n} s_{i,v} \left( -k \hat{q}_{ij} - d \hat{\omega}_i + \hat{A} \right) + \sum_{i=1}^{n} \left( k d_i + k_i + k \sum_{j=1}^{2} d_j \right) \hat{\omega}_i \hat{\omega}_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=N}^{n} k_{ij} \hat{q}_{ij} \hat{q}_{ij} + \frac{1}{2} k \sum_{i=1}^{n} \sum_{j=1}^{n} \left( d_i \hat{\omega}_i \right) (t - T_y) \cdot \hat{\omega}_i (t - T_y)
$$

which is equal to $\dot{V}_2$ in Eq. (35). Thus, following the procedure steps in Theorem 2, we obtain

$$
V_1(t) - V_1(0) \leq \sum_{i=1}^{n} \sum_{j=N}^{n} \left( k \left\| \hat{q}_{ij} \right\| + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=N}^{n} d_j \left\| \hat{\omega}_i \right\| \right) \frac{1}{2} \frac{1}{2} k \sum_{i=1}^{n} \sum_{j=1}^{n} \left( d_i \hat{\omega}_i \right) (t - T_y) \cdot \hat{\omega}_i (t - T_y)
$$

Hence, constraints (39) and (40) guarantee that $V_1(t)$ is bounded, and thus $\hat{\omega}_i(t) \in L_2, \hat{q}_{ij}(t) \in L_2, \hat{\Theta}_i \in L_1$. From Eq. (44), if $s_{i,v} \rightarrow 1$ for any $v \in \{1, 2, 3\}$, then it holds $V_1(t) \rightarrow 0$. Since $V_1(t)$ is bounded, then we can conclude that for all initial conditions $s_{i,v}(0) < l$, it must hold $s_{i,v}(t) < l, \forall t > 0$, for $v \in \{1, 2, 3\}$. Thus, $\hat{\omega}_i(t) \in L_2, \hat{K}_i^{-1} \hat{q}_{ij} \in L_2, \hat{\Theta}_i \in L_1$. So, from Eq. (43) and Assumption 1, we can deduce $\hat{\omega}_i(t) \in L_2$. Based on Lemma 1, we get $\hat{\omega}_i(t) \rightarrow 0$ and $\hat{q}_{ij}(t) \rightarrow 0$. Then we can get the following result.

**Theorem 3** For the constant $l > 0$, if the initial condition meets $s_{i,v}(0) < l$ for $v \in \{1, 2, 3\}$, then the controller (42), together with the undated law (46), can solve the delayed SAMP with disturbance rejection, while it holds $s_{i,v}(t) < l, \forall t > 0$ and $v \in \{1, 2, 3\}$.

Recall $s_i = \hat{\omega}_i + k \hat{q}_{ij}$, and consider the fact $\left\| \hat{q}_{ij}(t) \right\| \leq 1$, where $\left\| \cdot \right\|$ is the Euclidean norm of "\cdot". Then for some small $k > 0$, we can restrict the error $\hat{\omega}_i$, and thus restrict the actual angular velocity $\omega_i$ to the interested region $\left\| \omega_i \right\| < l + k + \rho$, where $\rho$ denotes the upper bound of $\omega_i$.

As a matter of fact, if disturbances vanish, we can overvalue the bound of the controller simply. Note that for this case we can define $k = 0$. Then, from Eq. (48), if the damp gain meets Eq. (18) for agent $i$, we conclude that $V_1(t)$ is bounded, $\hat{\omega}_i(t) \in L_2$. Similar to the mentioned analysis, $\left\| \hat{\omega}_i(t) \right\| < l$. If $\left\| \hat{\omega}_i(0) \right\| < l$ for $v \in \{1, 2, 3\}$, then via the definition of $\mathbf{K}_i$, we have $\mathbf{K}_i^{-1} = \mathbf{I} - \mathbf{\hat{\omega}}_i$, and thus $\mathbf{K}_i^{-1}$ is positive definite and $\mathbf{K}_i^{-1} \in L_1$. Then from the closed-loop system $\dot{\hat{\omega}}_i = \mathbf{K}_i^{-1}(-k \hat{q}_{ij} - d \hat{\omega}_i + \hat{A})$, we know $\hat{\omega}_i \in L_1$. Hence, from Lemma 1, $\hat{\omega}_i(t) \rightarrow 0$. Consequently, it is simple to get Eq. (29). Then according to the connected topologies, we obtain $\hat{q}_{ij}(t) \rightarrow 0$.

Now, let us estimate the bound of the controller

$$
\tau_e = \mathbf{J}_i \mathbf{K}_i^{-1}(-k \hat{q}_{ij} - d \hat{\omega}_i + \hat{A}) + Y \mathbf{\Theta}_i
$$

with $k = 0$. Invoke Eq. (8) and then this control effort is bounded as

$$
\left\| \tau_e \right\| < \left\| \mathbf{J}_i \mathbf{K}_i^{-1}(-k \hat{q}_{ij} - d \hat{\omega}_i + \hat{A}) \right\| + \left\| Y \mathbf{\Theta}_i \right\| \leq \left\| \mathbf{J}_i \mathbf{K}_i^{-1} \right\| \left( k \hat{q}_{ij} + d \hat{\omega}_i \right) + \left\| Y \mathbf{\Theta}_i \right\| \rho + \left\| \mathbf{J}_i \mathbf{K}_i^{-1} \right\| \rho
$$

where $\rho$ denotes the upper bound of $\hat{\omega}_i$. Hence, via an appropriate choice of control gains, a designer can set the controller to avoid system actuator saturation. Noted that Ref. [30] addressed an adaptive law for single flexible spacecraft with angular velocity constraints to overcome model uncertainty, but it needed the angular acceleration signal, which cannot be meas-
ured generally in applications. In fact, the delayed SAMP with model uncertainties and angular velocity constraints is still open, which is our future work.

4. Simulation Results

In this section, simulation results are presented to illustrate the performance of our proposed schemes. We apply controller (17) to solve the delayed SAMP without uncertainties, controller (30) the delayed SAMP with model and disturbance uncertainties, controller (41) the delayed SARP with model uncertainties, and controller (42), the delayed SAMP with disturbance uncertainties and velocity constraints. For the formation spacecraft, we assume that the interaction information flows among the members can be shown by Fig. 1, with time delays existing in the links.

Fig. 1  Information topology of spacecraft formation.

In our simulations, the desired angular velocity is given by $\omega_d(t) = 0.2 \sin(0.1 \pi t) m, m = [1 1 1]^T$, and the spacecraft specifications are shown in Table 1. The disturbance $\mathbf{\Psi}_i = [\sin(0.01 \pi t) \otimes I_3, 0^T] \in \mathbb{R}^{9\times 1}$, with $\mathbf{\Theta} = [0.3 0.4 0.5 0.5 0.4 0.3 0.5]^T$. For the extended controller (17), we choose the delayed values as follows: $T_{12}, T_{21}, T_{13}, T_{14}, T_{24}, T_{42}$, equaling 1.2, 1.5, 1.6, 1.3, 1.0, 0.8 s respectively. With the control parameters $k_i = 15, d_i = 20, k_g = k_d = 5$ and $d_g = d_d = 8$ and other random parameters, the simulation results are shown in Fig. 2, which illustrates the delayed robustness of controller (17) without any uncertainties.

Let $k_i = 31, k = 0.1, \mathbf{I}_i = 0.05 \mathbf{I}_i$, and $\mathbf{I}_d = 0.005 \mathbf{I}_d$, others are the same as controller (17), then we have the result in Fig. 3 under the controller (30) with updated laws (31) and (32).

For the delayed SARP, the controller controller (41) without the model information or the adaptive updating law is effective. The simulation results are shown in Fig. 4.

Table 1 Spacecraft specifications

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\mathbf{J}_i \text{ (kg m}^2\text{)}$</th>
<th>$\mathbf{\Omega}_i \text{ (0)}$</th>
<th>$\mathbf{\omega}_i \text{ (0)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>diag(30,18,24)</td>
<td>$[0.54 \ 0.41 \ 0.71 \ 0.17]$</td>
<td>$[0.1 \ 0.1 \ 0]$</td>
</tr>
<tr>
<td>2</td>
<td>diag(30,24,30)</td>
<td>$[-0.42 \ 0.77 \ 0.18 \ 0.45]$</td>
<td>$[0 \ 0.2 \ 0.1]$</td>
</tr>
<tr>
<td>3</td>
<td>diag(15,60,60)</td>
<td>$[-0.99 \ 0.01 \ 0.14 \ 0.01]$</td>
<td>$[0.1 \ 0 \ 0.2]$</td>
</tr>
<tr>
<td>4</td>
<td>diag(60,42,54)</td>
<td>$[0.54 \ 0.25 \ 0.62 \ 0.5]$</td>
<td>$[0.2 \ 0 \ 0.1]$</td>
</tr>
</tbody>
</table>

For delayed SAMP with disturbance uncertainties under velocity constraints, let $l = 0.8$, then controller (42) is effective, which can be illustrated in Fig. 5.
5. Conclusions

In this paper, we present a uniform control scheme in the natural non-convex space for delayed-free or delayed CACP with model and disturbance uncertainties, while an explicit desired attitude is considered. Compared with the existing studies, we deal with the delayed relative attitude via nonlinear manifolds, and the control gain selection with multiple inequality constraints is removed. Also, our controllers do not require any basic information about unknown inertial matrix and can obtain rejection for linear parameterized disturbances. If the model information is accurate, under angular velocity constraints, an adaptive law is proposed to overcome this kind of disturbances. When disturbances vanish, the restricted feature allows the designer to set the controller gains simply to avoid system actuator saturation.

However, the extension of the present work to velocity constraints problem with model uncertainties, or the synchronization under directed topologies, is challenging topics.

References


Biography:

LI Guiming received his B. S. and M. S. degrees from Harbin Institute of Technology in 2006 and 2008 respectively. Since 2008, he has been with Beijing Institute of Control Engineering to pursue his Ph.D. study. His research interests focus on the attitude control of spacecraft and the consensus theory.

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Appendix A

Property 1 is inspired by the structural method, cited in Ref. [28], Refs. [12]-[13], rooted in Ref. [31]. Prior to this proof, we would like to present Property 2.

Property 2 For the attitude error kinematics Eq. (12), and the constraint $\dot{\eta}_{ij} + 4\dot{\eta}_{ij}^2 = 1$, it is easy to verify

$$\dot{\eta}_{ij} + 4\dot{\eta}_{ij}^2 = I_3$$

then from Eq. (A1) and Eq. (12), it holds

$$\dot{\eta}_{ij} + 4\dot{\eta}_{ij}^2 = 1.$$
Now, let us consider Property 1.

**Proof of Property 1** According to the quaternion multiplication, we have \( \vec{q}_y = \vec{\eta}_r \hat{\vec{q}}_i - \vec{\eta}_a \hat{\vec{q}}_j - \hat{\vec{q}}_i \) and \( \vec{q}_y = \vec{\eta}_r (t - T_q) \hat{\vec{q}}_i - \vec{\eta}_a (t - T_q) - \hat{\vec{q}}_i \). Then, \( \vec{q}_y - \vec{q}_y = (\vec{\eta}_r - \vec{\eta}_r (t - T_q) \hat{\vec{q}}_i - (\vec{\eta}_r - \vec{\eta}_r (t - T_q)) \hat{\vec{q}}_j - \hat{\vec{q}}_i \).

By the Newton-Leibnitz formula, we get

\[
\vec{q}_y - \vec{q}_y = -\int_0^{T_q} \vec{\eta}_r (t - \sigma) d\sigma \hat{\vec{q}}_i + (\vec{\eta}_r \hat{\vec{q}}_i - \vec{\eta}_r) \int_0^{T_q} \hat{\vec{q}}_j (t - \sigma) d\sigma
\]

Then, integrating \( \vec{\omega}_y (\vec{q}_y - \vec{q}_y) \) from 0 to \( t \), we have

\[
\int_0^t \vec{\omega}_y (\vec{q}_y - \vec{q}_y) d\tau = -\int_0^t \vec{\omega}_y (\vec{\eta}_r (t - \sigma) d\sigma \hat{\vec{q}}_i + \int_0^t \vec{\omega}_y (\vec{\eta}_r \hat{\vec{q}}_i - \vec{\eta}_r) \int_0^{T_q} \hat{\vec{q}}_j (t - \sigma) d\sigma d\tau
\]

(A3)

For the first term on the right side of Eq. (A3), invoke the Lemma 2, then we have

\[
-2 \int_0^t \vec{\omega}_y \hat{\vec{q}}_i (\vec{\eta}_r (t - \sigma) d\sigma \hat{\vec{q}}_i) d\tau \leq \alpha_y \int_0^t \vec{\omega}_y \hat{\vec{q}}_i \hat{\vec{q}}_j \hat{\vec{q}}_j + \frac{T_y^2}{\alpha_y} \int_0^t \hat{\vec{q}}_i \hat{\vec{q}}_i d\tau
\]

(A4)

where \( \alpha_y > 0 \). Similarly, for the second one on the right side of Eq. (A3), we get

\[
2 \int_0^t \vec{\omega}_y \hat{\vec{q}}_i (\vec{\eta}_r = \vec{\eta}_r) \int_0^{T_q} \hat{\vec{q}}_j (t - \sigma) d\sigma \int_0^t d\tau \leq \alpha_y \int_0^t \vec{\omega}_y \hat{\vec{q}}_i \hat{\vec{q}}_j \hat{\vec{q}}_j + \frac{T_y^2}{\alpha_y} \int_0^t \hat{\vec{q}}_i \hat{\vec{q}}_i d\tau
\]

(A5)

Recall the relations in Property 2, we get

\[
\int_0^t \vec{\omega}_y (\vec{q}_y - \vec{q}_y) d\tau \leq \alpha_y \int_0^t \vec{\omega}_y \hat{\vec{q}}_i \hat{\vec{q}}_i d\tau + \frac{T_y^2}{8\alpha_y} \int_0^t \vec{\omega}_y \hat{\vec{q}}_i d\tau
\]

(A6)

Hence, it is simple to obtain

\[
\sum_{i=1}^n \sum_{j=1}^m k_{ij} \int_0^t \vec{\omega}_y (\vec{q}_y - \vec{q}_y) d\tau \leq \sum_{i=1}^n \sum_{j=1}^m k_{ij} \left( \frac{\alpha_y}{2} + \frac{T_y^2}{8\alpha_y} \right) \int_0^t \vec{\omega}_y \hat{\vec{q}}_i \hat{\vec{q}}_i d\tau
\]

(A7)

Since Lemma 2 holds for any \( \alpha_y > 0 \), then based on the inequality \( a^2 + b^2 \geq 2ab \), let \( \alpha_y = T_y / 2 \), it holds

\[
\sum_{i=1}^n \sum_{j=1}^m k_{ij} \int_0^t \vec{\omega}_y (\vec{q}_y = \vec{q}_y) d\tau \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m k_{ij} T_y \int_0^t \vec{\omega}_y \hat{\vec{q}}_i \hat{\vec{q}}_i d\tau
\]

(A8)

Then, Property 1 holds, clearly.