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TWO COUNTERFEIT COINS

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We consider the problem of ascertaining the minimum number of weighings which suffice to determine the counterfeit (heavier) coins in a set of n coins of the same appearance, given a balance scale and the information that there are exactly two heavier coins present. An optimal procedure is constructed for infinitely many n's, and for all other n's a lower bound and an upper bound for the maximum number of steps of an optimal procedure are determined which differ by just one unit. Some results of Cairns are improved, and his conjecture at the end of [3] is proved in a slightly modified form.

1. Introduction

Consider the following problem. Let $X=\{c_1, c_2, \ldots, c_n\}$ be a set of n coins, indistinguishable except that exactly two of them are slightly heavier than the rest (in the sense specified below). Given a balance scale, we want to find an optimal weighing procedure, i.e., a procedure which minimizes the maximum number of steps (weighings) which are required to identify both heavier coins.

We suppose that both heavier coins are of equal weight, and so are all light coins. If λ is the weight of a light (good) coin, then the weight of a heavy (defective) coin is less than $\frac{3}{2}\lambda$, so that the larger of two numerically unequal subsets of X is always the heavier. It means that no information is gained by balancing two numerically unequal sets. We also suppose that the scale reveals which, if either, of two subsets of X is heavier but not by how much.

Consider a pair of numerically equal disjoint subsets (A, B) of X. Step (A, B) will mean the balancing of A against B. The following outcomes are possible:

(a) The sets balance, symbolized by $A=B$.

(b) The sets do not balance, symbolized by $A\neq B$. We use the notation, if necessary, $A>B$, $B>A$, where $>$ between two sets means 'is heavier than'.

Let $P_n^2(l)$ denote any procedure which enables us to identify both heavier coins in the set X of n coins, l being the maximum number of weighings to be required. Similarly, $P_n^1(r)$ denotes any procedure which enables us to identify the heavier coin, if there is exactly one in the set of n coins, r being the maximum number of weighings to be required. It is well known that $P_n^1(r)$ is optimal if and only if $3^{r-1}<n\leq 3^r$.

We write $\mu_2(n)=l$ if $P_n^2(l)$ is optimal. It follows by information-theoretical reasonings that

$$\mu_2(n) \geq \left\lceil \log_3 \binom{n}{2} \right\rceil \tag{1}$$

where $\lceil x \rceil$ denotes the least integer $\geq x$.

In this paper we determine an infinite set of n's for which this lower bound is reached, and the upper bound which differs by just one unit from the lower bound. The corresponding procedures are constructed inductively.

The results

Theorem 1.

$$\left\lceil \log_3 \binom{n}{2} \right\rceil \leq \mu_2(n) \leq \left\lceil \log_3 \binom{n}{2} \right\rceil + 1. \tag{2}$$

Proof. It is easy to check that the following statements hold:

$$n > 3^k \Rightarrow \binom{n}{2} > 3^{2k-1}, \tag{3a}$$

$$n > 2 \cdot 3^k \Rightarrow \binom{n}{2} > 3^{2k}, \tag{3b}$$

i.e.,

$$n > 3^k \Rightarrow \left\lceil \log_3 \binom{n}{2} \right\rceil \geq 2k, \tag{4a}$$

$$n > 2 \cdot 3^k \Rightarrow \left\lceil \log_3 \binom{n}{2} \right\rceil \geq 2k+1. \tag{4b}$$

From (4a), (4b) and (1), we have

$$n > 3^k \Rightarrow \mu_2(n) \geq 2k, \tag{5a}$$

$$n > 2 \cdot 3^k \Rightarrow \mu_2(n) \geq 2k+1. \tag{5b}$$

Now, (2) will be proved if we prove the following two statements:

$$n \leq 2 \cdot 3^k \Rightarrow \mu_2(n) \leq 2k+1 \qquad (k = 0, 1, 2, \ldots), \tag{6a}$$

$$n \leq 3^{k+1} \Rightarrow \mu_2(n) \leq 2k+2 \qquad (k = 0, 1, 2, \ldots). \tag{6b}$$

The proof of (6a) and (6b) uses mathematical induction. The statements are obviously true for $k = 0$. In fact, we have $\mu_2(2) = 0$ and $\mu_2(3) = 1$. Suppose now that $k \geq 1$ and (6a) and (6b) are true for all $k \leq m-1$ and that the corresponding procedures are constructed. Then the procedures $P_n^2(2m+1)$ for $3^m < n \leq 2 \cdot 3^m$ and $P_n^2(2m+2)$ for $2 \cdot 3^m < n \leq 3^{m+1}$ can be constructed according to the following schemes:

(a) *Construction of* $P_n^2(2m+1)$ *for* $n \leq 2\cdot 3^m$. The first step is (A, B), where $A=\{c_1, c_2, \ldots, c_{\lfloor n/2\rfloor}\}$ and $B=\{c_{\lfloor n/2\rfloor+1}, \ldots, c_{2\lfloor n/2\rfloor}\}$. (Here, $\lfloor x\rfloor$ denotes the greatest integer $\leq x$.) If $A=B$, then each of the sets A and B contains exactly one heavier coin. Continue now by successively applying the two independent procedures $P^1_{|A|}(m)$ and $P^1_{|B|}(m)$. These procedures exist because $\lfloor \frac{1}{2}n\rfloor=|A|=|B|\leq 3^m$.

If $A\neq B$ we may suppose $A>B$ by symmetry. $A>B$ implies that both heavier coins are in $X\backslash B$, where $|X\backslash B|=\lceil\frac{1}{2}n\rceil\leq\lceil 2\cdot 3^m/2\rceil=3^m$. Continue now the procedure by applying $P^2_{|X\backslash B|}(2m)$. It can be constructed by the induction hypothesis.

A procedure $P_n^2(2m+1)$ for $n\leq 2\cdot 3^m$ is constructed.

(b) *Construction of* $P_n^2(2m+2)$ *for* $n\leq 3^{m+1}$. Let us define the sets $A, B, C, \tilde{B}$ and $\tilde{C}$ in the following way:

$$A=\{c_1, c_2, \ldots, c_{\lfloor n/3\rfloor}\}, \qquad B=\{c_{\lfloor n/3\rfloor+1}, \ldots, c_{2\lfloor n/3\rfloor}\},$$

$$C=\{c_{2\lfloor n/3\rfloor+1}, \ldots, c_{3\lfloor n/3\rfloor}\}.$$

Put $\tilde{B}=B$, $\tilde{C}=C$ if $n\neq 3^{m+1}-1$ and $\tilde{B}=B\cup\{c_{3^{m+1}-2}\}$, $\tilde{C}=C\cup\{c_{3^{m+1}-1}\}$ if $n=3^{m+1}-1$.

Step (b1). (A, B).

Step (b2). $(\tilde{B}, \tilde{C})$.

Several cases are distinguished according to the results of the above steps.

If $A<B$ and $\tilde{B}=\tilde{C}$, then each of the sets $\tilde{B}$ and $\tilde{C}$ contains exactly one heavier coin. Continue now by successive application of the two independent procedures $P^1_{|\tilde{B}|}(m)$ and $P^1_{|\tilde{C}|}(m)$ to the sets $\tilde{B}$ and $\tilde{C}$, respectively. These procedures exist since $|\tilde{B}|=|\tilde{C}|\leq 3^m$.

If $A<B$ and $\tilde{B}>\tilde{C}$, then both heavier coins are in $X\backslash(A\cup\tilde{C})$, where $|X\backslash(A\cup\tilde{C})|\leq 3^m$. Continue now by applying $P^2_{|X\backslash(A\cup\tilde{C})|}(2m)$ which can be constructed by the induction hypothesis.

The case $A<B$, $\tilde{B}<\tilde{C}$ is impossible.

If $B<A$ and $\tilde{B}=\tilde{C}$, then both heavier coins are in $X-(\tilde{B}\cup\tilde{C})$ where $|X-(\tilde{B}\cup\tilde{C})|\leq 3^m$. Continue now by applying $P^2_{|X-(\tilde{B}\cup\tilde{C})|}(2m)$ which exists by the induction hypothesis.

If $B<A$ and $\tilde{B}<\tilde{C}$, then each of the sets A and $\tilde{C}$ contains exactly one heavier coin. As $|A|\leq|\tilde{C}|\leq 3^m$ we may apply the procedures $P^1_{|A|}(m)$ and $P^1_{|\tilde{C}|}(m)$.

If $B<A$ and $\tilde{B}>\tilde{C}$, then $\tilde{B}$ and B must be different and $c_{3^{m+1}-2}$ is one of the heavier coins. The other one is in A. Here $|A|\leq 3^m$, therefore $P^1_{|A|}(m)$ completes our procedure.

If $A=B$ and $\tilde{B}<\tilde{C}$, then both heavier coins are in $X\backslash(A\cup\tilde{B})$, where $|X\backslash(A\cup\tilde{B})|\leq 3^m$. Continue now by applying $P^2_{|X\backslash(A\cup\tilde{B})|}(2m)$ which can be constructed by the induction hypothesis.

If $A=B$ and $\tilde{B}=\tilde{C}$, this is possible only if $n-3\lfloor\frac{1}{3}n\rfloor=2$. If $n<3^{m+1}-1$, then the heavier coins are $c_{3\lfloor n/3\rfloor+1}$ and $c_{3\lfloor n/3\rfloor+2}$; if $n=3^{m+1}-1$, then one heavier coin is in $\tilde{C}$, the other is $c_{3^{m+1}-2}$, apply $P^1_{|\tilde{C}|}(m)$.

If $A = B$ and $\tilde{B} > \tilde{C}$, then each of the sets A and $\tilde{B}$ contains exactly one heavier coin. Continue now by successive application of the two independent procedures $P^1_{|A|}(m)$ and $P^1_{|\tilde{B}|}(m)$. These procedures exist because $|A| \leq |\tilde{B}| \leq 3^m$.

A procedure $P^2_n(2m+2)$ for $n \leq 3^{m+1}$ is constructed. The theorem is proved.

Theorem 2. *There are infinitely many n's for which the procedure described in the proof of Theorem* 1 *is optimal. More precisely, the procedure is optimal at least for all n's belonging to the set*

$$\bigcup_{k=1} ([\lceil 3^k\sqrt{2}+1\rceil, 2\cdot 3^k] \cup [\lceil 3^k\sqrt{6}+1\rceil, 3^{k+1}]),$$

where $[p, q]$ *denotes the set of all integers n such that* $p \leq n \leq q$.

Proof. It is easy to check the following inequalities:

$$\binom{\lceil 3^k\sqrt{2}+1\rceil}{2} > 3^{2k}, \qquad \binom{\lceil 3^k\sqrt{6}+1\rceil}{2} > 3^{2k+1}. \tag{7a,b}$$

It follows from (7a) and (7b) that

$$\binom{n}{2} > 3^{2k} \quad \text{for all } n \in [\lceil 3^k\sqrt{2}+1\rceil, 2\cdot 3^k]$$

and

$$\binom{n}{2} > 3^{2k+1} \quad \text{for all } n \in [\lceil 3^k\sqrt{6}+1\rceil, 3^{k+1}],$$

which together with (2) implies the statement.

E.g. our procedure is optimal for all n's from the intervals [9, 9], [14, 18], [24, 27], [40, 54], [68, 81], [116, 162] etc.

Our results are stronger than those of Cairns [3, Corollary of Lemma 8.3 and Theorem $M(n, 2)$]. Also, his conjecture that $\mu_2(n)$ has one of the values $2k-1$, $2k$, $2k+1$, depending on n, for $3^{k-1} < n \leq 3^k$, is shown to be slightly incorrect. It follows from Theorem 1 and 2 that $\mu_2(n)$ has one of the values $2k-2$, $2k-1$, $2k$, for $3^{k-1} < n \leq 3^k$, and the corresponding procedures are constructed.

References

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