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J. Math. Anal. Appl. 302 (2005) 97–128

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Quasiasymptotic analysis in Colombeau algebra

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Received 20 June 2002

Submitted by S. Kaijser

Abstract

We consider the main notions of $\mathcal{G}$-quasiasymptotic ($\mathcal{G}$-q.a.) analysis: $\mathcal{G}$-q.a. behavior at zero, $\mathcal{G}$-q.a. expansion at infinity, *Sc*-asymptotic and boundedness, *Sc*-expansion at infinity and their applications to the asymptotic behavior of the solution to some types of nonlinear PDEs in Colombeau algebra of generalized functions $\mathcal{G}$.

Keywords: Colombeau algebra of generalized functions; $\mathcal{G}$-q.a. behavior at zero; Semilinear hyperbolic system; $\mathcal{G}$-q.a. expansion at infinity; Wave equation; *Sc*-asymptotic; *Sc*-boundedness; Semilinear parabolic equations; *Sc*-asymptotic expansion

1. Introduction

The q.a. behavior [27] and S-asymptotic [24] play important role in the investigations of the asymptotic behavior of the generalized functions' integral transforms. Also, asymptotic behaviors of distributions were used in investigations of analytical properties of quantum field elements. These results pushed forward the study of analytical properties of distributions and so the mathematical tool was developed (cf. [5,11,25,27]). Notions of q.a. and

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.07.044

S-asymptotic behavior and expansions of Schwartz distributions are analyzed in [25,27] (cf. references of these monographs). Systematic and simplified approach to asymptotic expansions by the use of distributions is given in [8].

Distributions in the framework of asymptotic analysis allow preformation of many analytical operations and assign values of divergent integrals. These notions are extended to Colombeau algebra by the authors in [18,21,22]. Definition of $\mathcal{G}$-q.a. expansion at zero, basic properties and evaluation of some generalized functions in $\mathcal{G} \setminus \mathcal{D}'$ are given in [21] and [22]. We refer to [19] for related questions via $\mathcal{G}$-q.a. behavior at zero.

Asymptotic analysis in Colombeau algebra of generalized functions $\mathcal{G}$ which is the larger space than the space of distributions $\mathcal{D}'$ allowing their multiplications, has a lot of advantages. We prove that the properties of S-asymptotic and boundedness, q.a. behavior and expansion in $\mathcal{D}'$ are preserved in $\mathcal{G}$, but the space $\mathcal{G}$ is a source of new properties, useful for application. It preserves the classical properties but it gives new ones not available in classical approach. We introduced and investigated in [17] and [20] the $\mathcal{G}$-q.a. behavior at zero in $\mathcal{G}$. We give a new way of q.a. analysis and expansion in Colombeau setting outside the classical approach since we consider elements from $\mathcal{G} \setminus \mathcal{D}'$.

In this paper we consider the three main notions in q.a. analysis: q.a. behavior at zero, q.a. expansion and S-asymptotic at infinity in $\mathcal{G}$. We give the systematical extension of the q.a. method from Schwartz theory of distributions to Colombeau algebra $\mathcal{G}$. The applications of $\mathcal{G}$-q.a. behavior and *Sc*-asymptotic and boundedness to PDEs are displayed.

We show that the Cauchy problem for a semilinear strictly hyperbolic $(n \times n)$-system (2) in two independent variables, has a solution whose $\mathcal{G}$-q.a. behavior at zero is determined by the $\mathcal{G}$-q.a. behavior at zero of the initial data.

We give the definition of $\mathcal{G}$-q.a. expansion at infinity in Colombeau space $\mathcal{G}_t$ and describe its basic properties. We evaluate powers of δ^n, $n \in \mathbf{N}$. We expanded quasiasymptotically the $\delta^2(x,t)$ with respect (w.r.) to the corresponding scale and apply it in the analysis, of the singular part of wave equation of the form (20) below. We obtain the $\mathcal{G}$-q.a. expansion of the solution along the characteristic lines which is a consequence of $\mathcal{G}$-q.a. expansion of the initial data which are (δ^2, δ^2).

In spite of the fact that the embedding of distributions is not canonical in simplified version of Colombeau algebra that version is in the line with other Colombeau algebras. The nonuniqueness of the embedding is not obstacle for the application of the theory.

We give the extension of the notion of S-asymptotic and boundedness from $\mathcal{S}'$ to $\mathcal{G}_t$, as the notions of *Sc*-asymptotic, respectively *Sc*-boundedness at infinity in Colombeau algebra of tempered generalized functions as the analogous to the corresponding ones in classical theory. We give the main characterizations of these notions and application of them in solving semilinear parabolic equations with singular initial data. Results concerning this subject are given mostly in one space dimension. In many dimensional case instead of interval $[0,\infty)$, a cone $\mathbf{R}_+^n$ is used, and in the case of general cone Γ, the assertions are slightly complicated (cf. [25]). We analyze a semilinear parabolic equation with conservative nonlinear term (or without it) and singular initial data through the *Sc*-asymptotic, respectively *Sc*-boundedness of the initial data. In appropriate cases the *Sc*-asymptotic, respectively *Sc*-boundedness of the solution is obtained.

Finally, we introduce the *Sc*-asymptotic expansion at infinity in Colombeau algebra and expand as an example element $\delta^2 \in \mathcal{G}_t \setminus \mathcal{S}'$. We give only the sketch of the *Sc*-asymptotic expansion at infinity and the possibilities for expansion of elements in $\mathcal{G}_t$ outside the $\mathcal{S}'$.

2. Preliminaries

In [1,6,7,10,14,16] is described Colombeau algebra of generalized functions as a useful tool for solving nonlinear problems due to the fact that it allows the multiplication of generalized functions. The essences of the framework of Colombeau generalized function algebras are: equality in Gd sense and association, possibility of multiplication of generalized functions (and embedded distributions) and unrestricted differentiation in $\mathcal{G}$. Recall the simplified version of that theory.

Let Ω be an open set in $\mathbf{R}^n$, $\varepsilon \in (0,1)$. Notation $K \Subset \Omega$ means that K is compact subset of Ω and $\omega \Subset \Omega$ means that ω is open and $\bar{\omega}$ is compact in Ω. Then, $\mathcal{E}(\Omega)$ is the space of families $f_\varepsilon(\cdot)$ of C^∞ functions on Ω; $\mathcal{E}_M(\Omega)$ is the space of families $G_\varepsilon \in \mathcal{E}(\Omega)$ with the property that for every $K \Subset \Omega$, and $\alpha \in \mathbf{N}_0^n$ there exists $r \in \mathbf{R}$ such that $\sup_{x\in K} |G_\varepsilon^{(\alpha)}(x)| = \mathcal{O}(\varepsilon^r)$; $\mathcal{N}(\Omega)$ is the space of families $G_\varepsilon(x) \in \mathcal{E}_M(\Omega)$ with the property that for every $K \Subset \Omega$, for all $\alpha \in \mathbf{N}_0^n$ and for every $q \in \mathbf{R}$, $\sup_{x\in K} |G_\varepsilon^{(\alpha)}(x)| = \mathcal{O}(\varepsilon^q)$ ($\mathcal{O}$ is Landau symbol). If G_ε does not depend on x we obtain a family of generalized numbers in $\mathbf{C}_M$, and $\mathcal{N}_0$ (which correspond to $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$). Then, $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$, and $\bar{\mathbf{C}} = \mathbf{C}_M/\mathcal{N}_0$. In the case of real numbers we have $\bar{\mathbf{R}} = \mathbf{R}_M/\mathcal{N}_{\mathbf{R}^0}$ with appropriate definitions of $\mathbf{R}_M$ and $\mathcal{N}_{\mathbf{R}^0}$.

For families of smooth functions in $\mathcal{E}_M(\mathbf{R}^n)$, various types of "equalities" are defined.

If $F, G \in \mathcal{G}(\mathbf{R}^n)$, F_ε and G_ε being representatives of F and G, respectively, then $F = [F_\varepsilon] = [G_\varepsilon] = G$ if $F_\varepsilon - G_\varepsilon \in \mathcal{N}(\mathbf{R}^n)$. (Both families define the same generalized function.) If for every $\psi \in C_0^\infty(\mathbf{R}^n)$, $\int_{\mathbf{R}^n} F_\varepsilon(x)\psi(x)\,dx$ and $\int_{\mathbf{R}^n} G_\varepsilon(x)\psi(x)\,dx$ represent the same complex number in $\bar{\mathbf{C}}$, then they are equal in Gd sense ($[F_\varepsilon] = [G_\varepsilon]$ in Gd sense). If for every $\psi \in C_0^\infty(\mathbf{R}^n)$ $\lim_{\varepsilon\to 0}[\int_{\mathbf{R}^n}(F_\varepsilon(x) - G_\varepsilon(x))\psi(x)\,dx] = 0$, then, they are associated in $\mathcal{D}'$ sense, i.e., $[F_\varepsilon] \sim [G_\varepsilon]$.

Generalized constant $B \in \bar{\mathbf{C}}$ is associated to a constant $b \in \mathbf{C}$, $B \approx b$ if $\lim_{\varepsilon\to 0} B_\varepsilon = b$ for its representative B_ε. Let $G, H \in \mathcal{G}(\Omega)$. Elements $G, H \in \mathcal{G}(\Omega)$, are L^∞-associated on Ω if for every $\omega \Subset \Omega$, $\lim_{\varepsilon\to 0} \|G_\varepsilon - H_\varepsilon\|_{L^\infty(\omega)} = 0$, where G_ε and H_ε are representatives of G and H, respectively. If $g \in L^\infty(\Omega)$, then $G \in \mathcal{G}(\Omega)$ is associated to g if $\|g - G_\varepsilon\|_{L^\infty(\omega)} \to 0$, as $\varepsilon \to 0$ for every $\omega \Subset \Omega$ and every representative G_ε of G.

$\mathcal{O}_M(\mathbf{C}^p)$ denotes the space of functions f of $C^\infty(\mathbf{C}^p)$ for which $f^{(\alpha)}$ is slowly increasing at infinity (bounded by a polynomial) for every $\alpha \in \mathbf{N}_0^{2p}$, where $f(z_1,\dots,z_p) = f(x_1,y_1,\dots,x_p,y_p)$ and

$$f^{(\alpha)} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\partial y_1^{\alpha_2}\dots\partial x_p^{\alpha_{2p-1}}\partial y_p^{\alpha_{2p}}}.$$

The embedding of $\mathcal{E}'(\Omega)$ into $\mathcal{G}(\Omega)$ given by $i_{0,\Omega}: w \mapsto (w * \phi_{\varepsilon|\Omega})_{\varepsilon>0}$, where

$$\phi \in \mathcal{S}(\mathbf{R}^n), \quad \int \phi(x)\,dx = 1, \quad \int x^\alpha \phi(x)\,dx = 0, \quad \forall \alpha \in \mathbf{N}_0^n,\ |\alpha| \geqslant 1 \tag{1}$$

(ϕ is called the vision function). Here $*$ is a symbol for the convolution ($f * g = \int f(\cdot - t)g(t)\,dt$). The mapping $i_{0,\Omega}$ can be extended to an embedding of $\mathcal{D}'(\Omega)$ employing the sheaf property of $\mathcal{G}(\Omega)$ along the line of [14, Chapter III, Appendix to Section 9]. Recall, $\mathcal{G}(\Omega)$ is a fine sheaf of differential algebras over open sets of $\mathbf{R}^n$. There is a unique sheaf morphism of complex vector space $i : \mathcal{D}' \to \mathcal{G}$ which extends the canonical embedding $i_{0,\Omega} : \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega)$. Note, i commutes with the operation of differentiation and its restriction to $C^\infty(\Omega)$ constitutes a sheaf morphism of algebras. We will describe this mapping.

If $A \subset \mathbf{R}^n$, then by $(A)_r$ is denoted the set $(A)_r = \{x \in A;\ d(x, CA) > r\}$.

Denote by $\kappa_{\phi_\varepsilon}$ a function in $C^\infty(\Omega)$ which is equal 1 on $(\bar{\Omega})_{2\varepsilon}$ and $\operatorname{supp} \kappa_{\phi_\varepsilon} \subset (\bar{\Omega})_\varepsilon$. If $\bar{\Omega}_{2\varepsilon} = \emptyset$, then $\kappa_{\phi_\varepsilon} \equiv 0$. If $f \in C^0(\Omega)$, then the mapping

$$i : (\phi_\varepsilon, x) \mapsto (f\kappa_{\phi_\varepsilon} * \check{\phi}_\varepsilon)(x) = \int_{\mathbf{R}^n} (\kappa_{\phi_\varepsilon} f)(t + x)\phi_\varepsilon(t)\,dt, \quad x \in \Omega,$$

is an element in $\mathcal{E}_M(\Omega)$. This mapping defines $i : C^0(\Omega) \to \mathcal{G}(\Omega)$. If $f \in \mathcal{D}'(\Omega)$, then the corresponding element in $\mathcal{G}(\Omega)$, $i(f)$, is defined by $[f\kappa_{\phi_\varepsilon} * \check{\phi}_{\varepsilon|\Omega}]$. Following our previous notation, we put $\mathrm{Cd}\, f = i(f)$ (Cd means Colombeau distribution). If an element $F \in \mathcal{G}$ is not of the form $\mathrm{Cd}\, f$ then we write $F \in \mathcal{G} \setminus \mathcal{D}'$.

In order to embed $\mathcal{E}'(\Omega)$ into $\mathcal{G}(\Omega)$ for the sake of $\mathcal{G}$-q.a. behavior at zero we use another embedding: Instead of mollifier ϕ_ε, we use ϕ_{ε^2}, $\varepsilon \in (0, 1)$ $\big(\phi_{\varepsilon^2} = \frac{1}{\varepsilon^{2n}}\phi\big(\frac{\cdot}{\varepsilon}\big)\big)$. If $T \in \mathcal{E}'(\Omega)$, then $I_\phi(T) = [T * \phi_{\varepsilon^2}]$. One embeds $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$, $I_\phi : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega)$, by using the sheaf property of both spaces.

The use of embedding I_ϕ implies that distributions having the q.a. behavior at zero in $\mathcal{D}'(\Omega)$ have the $\mathcal{G}$-q.a. behavior as embedded elements of $\mathcal{G}$. For the sake of $\mathcal{G}$-q.a. at infinity we use usual mollifiers $\phi_\varepsilon = \frac{1}{\varepsilon}\phi\big(\frac{\cdot}{\varepsilon}\big)$.

Recall [10] the construction of the space of Colombeau algebra of tempered generalized functions $\mathcal{G}_t$. Let $I = (0, 1]$. Then,

$$\mathcal{E}_t(\mathbf{R}^n) = \Big\{(u_\varepsilon)_\varepsilon \in \big(C^\infty(\mathbf{R}^n)\big)^I \mid \forall \alpha \in \mathbf{N}_0^n\ \exists N \in \mathbf{N}: \sup_{x \in \mathbf{R}^n} \big(1 + |x|\big)^{-N} \big|\partial^\alpha u_\varepsilon(x)\big| = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\Big\},$$

$$\mathcal{N}_t(\mathbf{R}^n) = \Big\{(u_\varepsilon)_\varepsilon \in \big(C^\infty(\mathbf{R}^n)\big)^I \mid \forall \alpha \in \mathbf{N}_0^n\ \exists p \in \mathbf{N}\ \forall m \in \mathbf{N}: \sup_{x \in \mathbf{R}^n} \big(1 + |x|\big)^{-p} \big|\partial^\alpha u_\varepsilon(x)\big| = \mathcal{O}(\varepsilon^{m}) \text{ as } \varepsilon \to 0\Big\}.$$

Elements of $\mathcal{E}_t(\mathbf{R}^n)$ and $\mathcal{N}_t(\mathbf{R}^n)$ are called moderate, respectively negligible. Algebra of tempered generalized function on $\mathbf{R}^n$ is defined as $\mathcal{G}_t(\mathbf{R}^n) = \mathcal{E}_t(\mathbf{R}^n)/\mathcal{N}_t(\mathbf{R}^n)$. Note $\mathcal{N}_t(\mathbf{R}^n) \cap \mathcal{E}_M(\mathbf{R}^n) \neq \mathcal{N}(\mathbf{R}^n)$.

Let $\theta \in C^\infty(\mathbf{R})$ such that $\theta(x) = 1$ for $|x| > 2$ and $\theta(x) = 0$ for $|x| < 1$. Then, $\theta_\varepsilon(x) = \theta(\varepsilon x) \in \mathcal{N}(\mathbf{R}^n) \cap \mathcal{E}_M(\mathbf{R}^n)$ but $\theta_\varepsilon \notin \mathcal{N}_t(\mathbf{R}^n)$.

Elements of $\mathcal{G}_t(\mathbf{R}^n)$ determine elements of $\mathcal{G}(\mathbf{R}^n)$ in an obvious way.

If $f \in \mathcal{S}'(\mathbf{R}^n)$, then the embedding of f into $\mathcal{G}_t(\mathbf{R}^n)$ is denoted by $f \mapsto \mathrm{Ctd}\, f$. More precisely, as in the case of embedding i we put $\mathrm{Ctd}\, f = [f * \phi_\varepsilon]$, where $\phi \in \mathcal{S}(\mathbf{R}^n)$ satisfies (1). Clearly, for every $\psi \in C_c^\infty(\mathbf{R}^n) = \mathcal{D}(\mathbf{R}^n)$, $N_\varepsilon = (\psi * \phi_\varepsilon - \psi) \in \mathcal{N}(\mathbf{R}^n)$.

For the time being, C will denote a generic constant which is different in different appearances.

3. $\mathcal{G}$-q.a. at zero of a solution to semilinear hyperbolic system

We denote by $\mathcal{K}$ a set of positive measurable functions defined on $(0,1)$ with the property $A^{-1}\varepsilon^{p} \leqslant c(\varepsilon) \leqslant A\varepsilon^{-p}$, $\varepsilon \in (0,1)$, for some $A > 0$ and $p > 0$.

Let Ω denote an open set in $\mathbf{R}^n$ which contains 0.

Definition 1. Let $F \in \mathcal{G}(\Omega)$. It is said that an $F \in \mathcal{G}(\Omega)$ has the $\mathcal{G}$-q.a. behavior at zero (w.r.) to $c(\varepsilon) \in \mathcal{K}$ if there is F_ε, a representative of F, such that for every $\psi \in \mathcal{D}(\Omega)$ there is $C_\psi \in \mathbf{C}$ such that

$$\lim_{\varepsilon\to 0}\left\langle \frac{F_\varepsilon(\varepsilon x)}{c(\varepsilon)}, \psi(x)\right\rangle = C_\psi$$

and $C_\psi \neq 0$ for some ψ.

It is proved in [17] that there exists $g \in \mathcal{D}'(\Omega)$ such that $C_\psi = \langle g, \psi\rangle$, $\psi \in \mathcal{D}(\Omega)$. For the main properties of $\mathcal{G}$-q.a. behavior at zero cf. [17].

If $F_\varepsilon \in \mathcal{N}(\Omega)$, then for every $c \in \mathcal{K}$ and every $\psi \in \mathcal{D}(\Omega)$,

$$\lim_{\varepsilon\to 0^+}\left\langle \frac{F_\varepsilon(\varepsilon x)}{c(\varepsilon)}, \psi(x)\right\rangle = 0.$$

Recall, when we deal with the $\mathcal{G}$-q.a. behavior or expansion (later) at zero we use the embedding $I_\phi : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega)$ (with mollifier $\phi_{\varepsilon^2}(x) = \frac{1}{\varepsilon^{2n}}\phi\big(\frac{x}{\varepsilon^2}\big)$, $x \in \mathbf{R}^n$, $\varepsilon \in (0,1)$).

Let

$$\begin{aligned} &\big(\partial_t + \Lambda(x,t)\partial_x\big)u(x,t) = F\big(x,t,u(x,t)\big), \quad (x,t) \in \mathbf{R}^2,\\ &u(x,0) = \big(u_1(x,0),\dots,u_n(x,0)\big) = \big(a_1(x),\dots,a_n(x)\big) \in \big(\mathcal{G}(\mathbf{R})\big)^n, \end{aligned} \tag{2}$$

be a semilinear strictly hyperbolic system where $\Lambda(x,t)$ is a diagonal matrix with the real distinct smooth functions on the diagonal and $(x,t,u) \mapsto F(x,t,u)$ is a smooth function on $\mathbf{R}^2 \times \mathbf{C}^n$, with the unbounded gradient $\nabla_z F(x,y,z)$. By assumption on matrix ($\|\Lambda\| \leqslant c < \infty$) the characteristic curves globally exist, i.e., for every $(x,t) \in \mathbf{R}^2$ there exists a compact set K such that the characteristic curves which pass through (x,t) start from K. It is proved in [14] that the Cauchy problem (2) is uniquely solvable in the algebra of generalized functions $(\mathcal{G}(K_T))^n$ if: (1) $\mathbf{C}^n \ni u \mapsto F(x,t,u)$ is polynomially bounded together with all derivatives, uniformly for $(x,t) \in K$, for any compact set $K \Subset \mathbf{R}^2$; (2) $\mathbf{C}^n \ni u \mapsto \nabla_u F(x,t,u)$ is globally bounded, uniformly (w.r.) to $(x,t) \in K$, for any compact set $K \Subset \mathbf{R}^2$. Here, K_T is a domain of determinacy bounded by extremal characteristics emanating from the end points of a given compact interval K, and the lines $t = \pm T$, $T > 0$. In the sequel we will assume $\mathring{K}_T$ to be a neighborhood of 0.

We find a regularization for a nonlinear term such that the regularized equation is uniquely solvable in $\mathcal{G}$. Moreover, if a local continuous solution to (2) exists under some mild assumptions, then it is associated to the solution of regularized equation and $\mathcal{G}$-q.a.

behavior at zero of respective solutions are the same. In particular, we consider the wave equation and Euler–Lagrange equation in one space dimension with distributions as initial data.

3.1. *h-regularized system*

Assume that $F(x, y, u, v) \in C^\infty(\mathbf{R}^{2+2n})$ and that $\nabla_{(u,v)}F(x, y, u, v)$ is not necessarily bounded for $(x, y) \in K$ for every $K \Subset \mathbf{R}^2$. Fix a decreasing function $h:(0,1) \to (0,\infty)$ such that

$$h(\varepsilon) = \mathcal{O}\big(|\log\varepsilon|^{1/2}\big), \quad h(\varepsilon) \to \infty, \quad \text{as } \varepsilon \to 0. \tag{3}$$

Let B_r, be the cube $|x| \leqslant r$, $|t| \leqslant r$, $|u| \leqslant r$, and $u = (u_1, v_1, \dots, u_n, v_n)$. Denote by ε_i a decreasing sequence of positive numbers such that $h(\varepsilon_{i+1}) = i$, $i \in \mathbf{N}$. We have $h(\varepsilon) \geqslant i-1$ for $\varepsilon_{i+1} \leqslant \varepsilon < \varepsilon_i$. Set for $i \in \mathbf{N}$,

$$S_i = B_i \cap \big\{(x,t,u,v),\ \big|F(x,t,u,v)\big| \leqslant i-1\big\},$$
$$\bigcap\big\{(x,t,u,v),\ \big|\nabla_{(u,v)}F(x,t,u,v)\big| \leqslant i-1\big\}.$$

Let κ_i be the characteristic function for S_i, $i \in \mathbf{N}$, and $\alpha \in C_0^\infty(\mathbf{R})$, $\int \alpha(t)\,dt = 1$, $\alpha \geqslant 1$, $\alpha = 1$ in a neighborhood of $t = 0$ and $\alpha_\varepsilon(t) = \frac{1}{\varepsilon}\alpha\big(\frac{t}{\varepsilon}\big)$, $t \in \mathbf{R}$. Put $\alpha_\varepsilon(x,t,u,v) = \alpha_\varepsilon(x)\alpha_\varepsilon(t)\alpha_\varepsilon(u_1)\dots\alpha_\varepsilon(u_n)\alpha_\varepsilon(v_1)\dots\alpha_\varepsilon(v_n)$,

$$\kappa_{h(\varepsilon)} = (\kappa_i * \alpha_{h(\varepsilon)^{-1}}), \quad \varepsilon \in [\varepsilon_{i+1}, \varepsilon_i),\ i \in \mathbf{N},$$
$$F^k_{h(\varepsilon)} = F^k\kappa_{h(\varepsilon)}, \quad \varepsilon \in (0, \varepsilon_1),\ k = 1, \dots, n.$$

Then, there exists a constant $C > 0$ such that

$$\|F_{h(\varepsilon)}\|_{L^\infty(\mathbf{R}^{2n+2})} \leqslant Ch(\varepsilon), \qquad \|\nabla_{(u,v)}F_{h(\varepsilon)}\|_{L^\infty(\mathbf{R}^{2n+2})} \leqslant Ch(\varepsilon)^2. \tag{4}$$

The integral curves for (2), which pass through (x_0, t_0) at time $\tau = t_0$, are the solutions to $\frac{\partial}{\partial\tau}\gamma_i(x_0,t_0,\tau) = \lambda_i(\gamma_i(x_0,t_0,\tau),\tau)$, $\gamma_i(x_0,t_0,t_0) = x_0$. In the sequel, we assume that the integral curves $x = \gamma_i(x_0,t_0,\tau)$, $i \in \{1,\dots,n\}$, which are called characteristic curves of the system, exist globally. Using representatives, h-regularized system has the form

$$\big(\partial_t + \Lambda(x,t)\partial_x\big)u_{h(\varepsilon)}(x,t) = F_{h(\varepsilon)}\big(x,t,u_{h(\varepsilon)}(x,t)\big) + d_{1\varepsilon}(x,t), \tag{5}$$

$u_{h(\varepsilon)}(x,0) = A_\varepsilon(x) + d_{2\varepsilon}(x)$, where $A_\varepsilon \in (\mathcal{E}_M(\mathbf{R}))^n$, $d_{1\varepsilon} \in (\mathcal{N}(\mathbf{R}^2))^n$, and $d_{2\varepsilon} \in (\mathcal{N}(\mathbf{R}))^n$.

Proposition 1 [13]. (i) *Assume that every component of the mapping $y \mapsto F(x,t,y)$ belongs to $\mathcal{O}_M(\mathbf{C}^n)$ and has uniform bounds for $(x,t) \in K$ for any $K \Subset \mathbf{R}^2$. Then the regularized system* (5) *has a unique solution in $(\mathcal{G}(\mathbf{R}^2))^n$ whenever the initial data are in $(\mathcal{G}(\mathbf{R}))^n$.*

(ii) *Let the initial data $(a_1,\dots,a_n)$ in* (2) *belong to $(C(\mathbf{R}))^n$ and $K_0 = [-J, J]$, $J > 0$. The solution $u_{h(\varepsilon)}$ to the regularized system* (5) *is L^∞-associated with the continuous local solution u to* (2) *in $\mathring{K}_{T_0}$, for some $T_0 > 0$.*

Proposition 2. (i) *Assume that the conditions of Proposition* 1(i) *are satisfied and that for every characteristic curve* γ *to* (5) (*which pass through* εx *at* $\tau = \varepsilon t$),

$$\lim_{\varepsilon\to 0} \frac{a_{i,\varepsilon}(\gamma_i(\varepsilon x, \varepsilon t, 0))}{c(\varepsilon)} \text{ exists in } \mathcal{D}'(\mathbf{R}^2), \quad i = 1, \dots, n,$$

where $c(\varepsilon) \in \mathcal{K}$, *and*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon|\log\varepsilon|^{1/2}}{c(\varepsilon)} = 0, \tag{6}$$

or

$$\frac{\varepsilon}{c(\varepsilon)}, \quad \frac{\varepsilon|\log\varepsilon|}{c(\varepsilon)} \sup_{(x,t)\in K} \bigl|a_\varepsilon\bigl(\gamma_i(\varepsilon x, \varepsilon t, 0)\bigr)\bigr| \to 0,$$
$$\text{for every } K \Subset \mathbf{R}^2,\ i = 1, \dots, n. \tag{7}$$

Then, the solution $([u_{1,h(\varepsilon)}(x,t)], \dots, [u_{n,h(\varepsilon)}(x,t)])$ *to* (5) *has the* $\mathcal{G}$*-q.a. behavior at zero* (*w.r.*) *to* $c(\varepsilon)$, *i.e.,*

$$\lim_{\varepsilon\to 0}\biggl\langle \frac{u_{i,h(\varepsilon)}(\varepsilon x, \varepsilon t)}{c(\varepsilon)}, \psi(x,t)\biggr\rangle = C_{i,\psi}, \quad C_{i,\psi} \in \mathbf{C},\ \psi \in \mathcal{D}(\mathbf{R}^2),\ i = 1, \dots, n.$$

(ii) *Assume that the conditions of Proposition* 1(ii) *hold as well as* $\frac{\varepsilon}{c(\varepsilon)} \to 0$, $\varepsilon \to 0$. *Assume*

$$\lim_{\varepsilon\to 0^+} \frac{a_i(\gamma_i(\varepsilon x, \varepsilon t, 0))}{c(\varepsilon)} \text{ exists in } \mathcal{D}'\bigl((-J, J) \times (-T_0, T_0)\bigr) \tag{8}$$

for some $T_0 > 0$, *where* $c(\varepsilon) \in \mathcal{K}$. *Then, the local continuous solution to* (2) *has the* $\mathcal{G}$*-q.a. behavior at* $(0,0)$ *which is the same as the* $\mathcal{G}$*-q.a. behavior at zero of the generalized solution to* (5), *where* $A_\varepsilon(x) = [(A_{1\varepsilon}(x), \dots, A_{n\varepsilon}(x))]$, $A_{i\varepsilon}(x) = (a_i\beta_\varepsilon) * \phi_{\varepsilon^2}(x)$, $i = 1, \dots, n$, *and* β_ε *is the characteristic function of* $(-J+\varepsilon, J-\varepsilon)$.

Proof. (i) Let $\psi \in \mathcal{D}(\mathbf{R}^2)$, $\operatorname{supp}\psi \subset K \Subset \mathbf{R}^2$, K_0 and T be such that $K \Subset \mathring{K}_T$. If $i = 1, \dots, n$, $\varepsilon \in (0, \varepsilon_0)$, then there exists the solution to (5) (with $d_{1\varepsilon} = 0$, $d_{2\varepsilon} = 0$) or equivalently to

$$u_{i,h(\varepsilon)}(\varepsilon x, \varepsilon t) = a_{i,\varepsilon}\bigl(\gamma_i(\varepsilon x, \varepsilon t, 0)\bigr) + \int_0^{\varepsilon t} F^i_{h(\varepsilon)}\bigl(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau, u_{i,h(\varepsilon)}\bigl(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau\bigr)\bigr)\, d\tau, \tag{9}$$

$(x,t) \in K_T$, $u_{h(\varepsilon)} = (u_{1,h(\varepsilon)}, \dots, u_{n,h(\varepsilon)}) \in (\mathcal{E}_M(\mathring{K}_T))^n$.

Note that for $0 \leqslant \theta \leqslant 1$, $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned}
&F^i_{h(\varepsilon)}\bigl(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau, u_{i,h(\varepsilon)}\bigl(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau\bigr)\bigr)\\
&\quad = F^i_{h(\varepsilon)}\bigl(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau, 0\bigr)\\
&\qquad + \bigl(u_{1,h(\varepsilon)}\bigl(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau\bigr), \dots, u_{n,h(\varepsilon)}\bigl(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau\bigr)\bigr)\\
&\qquad \times \nabla_u F^i_{h(\varepsilon)}\bigl(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau, \theta u_{i,h(\varepsilon)}\bigl(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau\bigr)\bigr).
\end{aligned}$$

Then (9) becomes

$$
\begin{aligned}
u_{i,h(\varepsilon)}(\varepsilon x,\varepsilon t) = a_{i,\varepsilon}\big(\gamma_i(\varepsilon x,\varepsilon t,0)\big) &+ \int\limits_0^{\varepsilon t} F^i_{h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau,0\big)\,d\tau \\
&+ \int\limits_0^{\varepsilon t} \big(u_{1,h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau\big),\dots,u_{n,h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau\big)\big) \\
&\times \nabla_u F^i_{h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau,\theta u_{i,h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau\big)\big)\,d\tau,
\end{aligned}
$$

where $0 \leqslant \theta \leqslant 1$, $\varepsilon \in (0,\varepsilon_0)$. By Gronwall's inequality, we have for $0 \leqslant \theta \leqslant 1$, $\varepsilon \in (0,\varepsilon_0)$,

$$
\begin{aligned}
&\sup_{(x,t)\in K_T} \big|u_{i,h(\varepsilon)}(\varepsilon x,\varepsilon t)\big| \\
&\quad\leqslant \Big(\sup_{(x,t)\in K_T} \big|a_\varepsilon\big(\gamma_i(\varepsilon x,\varepsilon t,0)\big)\big| + |\varepsilon T| \sup_{(x,t)\in K_T} \big|F^i_{h(\varepsilon)}(x,t,0)\big|\Big) \\
&\qquad\times \exp\Big(\varepsilon n T \sup_{\substack{(x,t)\in K_T\\ u\in \mathbf{C}^n}} \big|\nabla_u F^i_{h(\varepsilon)}(x,t,u)\big|\Big) \\
&\quad\leqslant \Big(\sup_{(x,t)\in K_T} \big|a_\varepsilon\big(\gamma_i(\varepsilon x,\varepsilon t,0)\big)\big| + C\varepsilon\Big) \exp\big(CTn\varepsilon|\ln\varepsilon|\big).
\end{aligned}
$$

The last inequality follows due to the boundedness of $F^i(x,t,0)$ on compact sets and $|\nabla_u F^i_{h(\varepsilon)}| \leqslant C|\log\varepsilon|$ on $\mathbf{R}^{2n+2}$, $i=1,\dots,n$. Thus, we obtain that there exists $C>0$ such that

$$
\sup_{(x,t)\in K_T} \big|u_{i,h(\varepsilon)}(\varepsilon x,\varepsilon t)\big| \leqslant C\Big(\sup_{(x,t)\in K_T} \big|a_\varepsilon\big(\gamma_i(\varepsilon x,\varepsilon t,0)\big)\big| + \varepsilon\Big), \quad \varepsilon\in(0,\varepsilon_0). \tag{10}
$$

We shall estimate the integral part of (9). We have

$$
\begin{aligned}
&\int\limits_0^{\varepsilon t} \big|F^i_{h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau,u_{i,h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau\big)\big)\big|\,d\tau \\
&\quad\leqslant \int\limits_0^{\varepsilon t} \big|F^i_{h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau,0\big)\big|\,d\tau \\
&\qquad+ \int\limits_0^{\varepsilon t} \big|u_{i,h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau\big)\big| \sup_{\substack{(x,t)\in K_T\\ u\in\mathbf{C}}} \big|\nabla_u F^i_{h(\varepsilon)}(x,t,u)\big|\,d\tau.
\end{aligned}
$$

Applying (10) we obtain

$$
\begin{aligned}
&\int\limits_0^{\varepsilon t} \big|F^i_{h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau,u_{i,h(\varepsilon)}\big(\gamma_i(\varepsilon x,\varepsilon t,\tau),\tau\big)\big)\big|\,d\tau \\
&\quad\leqslant C\varepsilon\Big(1+|\log\varepsilon| \sup_{(x,t)\in K_T} \big|a_\varepsilon\big(\gamma_i(\varepsilon x,\varepsilon t,0)\big)\big|\Big).
\end{aligned} \tag{11}
$$

We have

$$
\begin{aligned}
&\left\langle \frac{u_{i,h(\varepsilon)}(\varepsilon x, \varepsilon t)}{c(\varepsilon)}, \psi(x,t) \right\rangle \\
&\quad = \left\langle \frac{a_{i,\varepsilon}(\gamma_i(\varepsilon x, \varepsilon t, 0))}{c(\varepsilon)}, \psi(x,t) \right\rangle \\
&\qquad + \int \int\limits_{(x,t)\in K_T} \int\limits_0^{\varepsilon t} \frac{1}{c(\varepsilon)} F^i_{h(\varepsilon)}\big(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau, u_{i,h(\varepsilon)}\big(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau\big)\big)\, d\tau \\
&\qquad \times \psi(x,t)\, dx\, dt.
\end{aligned}
$$

Now, if (6) holds, by (3) and (4) the assertion follows. Let us consider case (7). Applying (11) we obtain

$$
\begin{aligned}
&\left| \int \int\limits_{(x,t)\in K_T} \int\limits_0^{\varepsilon t} \frac{1}{c(\varepsilon)} F^i_{h(\varepsilon)}\big(\gamma_i(\varepsilon x, \varepsilon t, \tau), \tau, u_{i,h(\varepsilon)}\big(\gamma_i(\varepsilon x, \varepsilon t), \tau\big)\big)\, d\tau\, \psi(x,t)\, dx\, dt \right| \\
&\quad \leqslant \left\langle \frac{C\varepsilon(1 + |\log\varepsilon| \sup_{(x,t)\in K_T} |a_\varepsilon(\gamma_i(\varepsilon x, \varepsilon t, 0))|)}{c(\varepsilon)}, \psi(x,t) \right\rangle \to 0.
\end{aligned}
$$

Thus, the assertion follows by (7).

(ii) We note that assumption (8) implies that

$$
\lim_{\varepsilon\to 0^+} \left\langle \frac{A_{i\varepsilon}(\gamma(\varepsilon x, \varepsilon t, 0))}{c(\varepsilon)}, \psi(x,t) \right\rangle
$$

exists for $\psi \in \mathcal{D}((-a,a)\times(-T_0,T_0))$. This follows by the properties of the convolution in the definition of $A_{i,\varepsilon}$.

Let $K_0 = [-J, J]$, u_i be the exact solution to (2) in K_{T_0} for some $T_0 > 0$ such that u_i is bounded on K_{T_0}, $i = 1, \dots, n$. By the same arguments as in the proof of (i) we have the assertion. Note, $|F^i(x,t,u)|$ and $|F^i(x,t,u_{i,h(\varepsilon)})|$ are bounded on K_{T_0}. $\square$

3.2. Examples

Example 1. Consider the wave equation

$$
u_{tt}(x,t) - \Delta u(x,t) = f\big(x,t,u(x,t)\big), \quad u_{|t=0} = \tilde{u}_0(x), \quad u_{t|t=0} = \tilde{u}_1(x), \tag{12}
$$

$\tilde{u}_0, \tilde{u}_1 \in \mathcal{D}'(\mathbf{R})$, where f is C^∞ function with the unbounded gradient such that $f(x,t,0) = 0$. With the substitution $u_2 = u$, $u_1 = u_t - u_x$, $u_2(x,0) = a_2(x) = \tilde{u}_0(x)$, $u_1(x,0) = a_1(x) = \tilde{u}_1(x) - \tilde{u}_{0x}(x)$, it can be considered as a semilinear strictly hyperbolic system (2) with the diagonal matrix

$$
\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad F = \begin{bmatrix} f(x,t,u_2) \\ u_1 \end{bmatrix}.
$$

The characteristic curves are the straight lines $\gamma_1(s) \equiv x - t + s = 0$, $\gamma_2(s) \equiv x + t - s = 0$. Equivalently to (12) we have, with $a_1(x) = \tilde{u}_1(x) - \tilde{u}_{0x}(x)$, $a_2(x) = \tilde{u}_0(x)$, that

$$u_1(x,t) = a_1(x-t) + \int_0^t f\big(u_2(x-t+s,s)\big)\,ds,$$

$$u_2(x,t) = a_2(x+t) + \int_0^t u_1(x+t-s,s)\,ds.$$

We use the regularization from Section 3.1 and find the generalized solution to

$$(\partial_t + \partial_x)u_{1,\varepsilon} = f_{h(\varepsilon)}(u_{2,\varepsilon}) + d_{1,\varepsilon}(x,t), \qquad (\partial_t - \partial_x)u_{2,\varepsilon} = u_{1,\varepsilon},$$
$$u_{i,\varepsilon}(x,0) = a_{i,\varepsilon}(x) + d_{2i,\varepsilon}, \tag{13}$$

where

$$a_{i\varepsilon}(x) = a_i\beta_\varepsilon * \phi_{\varepsilon^2}(x), \quad i = 1,2, \tag{14}$$

β_ε is the characteristic function of $(-1/\varepsilon, 1/\varepsilon)$ $(d_{1,\varepsilon} \in \mathcal{N}(\mathbf{R}^2)$, $d_{2i,\varepsilon} \in \mathcal{N}(\mathbf{R}))$. We call (13) and the corresponding equation in $(\mathcal{G}(\mathbf{R})^2)^2$ the h-regularized systems related to the h-regularized wave equation. Since (3) holds, the assumption on h implies

$$\sup_{x\in\mathbf{R}}\big|f'_{h(\varepsilon)}(x)\big| \leqslant C|\log\varepsilon| \quad \text{for some } C > 0. \tag{15}$$

By Proposition 1, the h-regularized system is uniquely solvable in $(\mathcal{G}(\mathbf{R}^2))^2$.

Proposition 3. *Let $a_1, a_2 \in \mathcal{D}'(\mathbf{R})$ and $c(\varepsilon) \in \mathcal{K}$. Assume*

$$\lim_{\varepsilon\to 0}\frac{a_2(\varepsilon x + \varepsilon t)}{c(\varepsilon)} \text{ exists in } \mathcal{D}'(\mathbf{R}^2), \tag{16}$$

$$\lim_{\varepsilon\to 0}\frac{\varepsilon|\log\varepsilon|}{c(\varepsilon)}\sup_{(x,t)\in K}\big\{\big|a_{1,\varepsilon}(\varepsilon x - \varepsilon t)\big| + \big|a_{2,\varepsilon}(\varepsilon x + \varepsilon t)\big|\big\} = 0$$
$$\text{for every } K \Subset \mathbf{R}^2 \tag{17}$$

($a_{i\varepsilon}$ is given by (14), $i = 1,2$). Then, the solution $[u_{2,h(\varepsilon)}(x,t)]$ to the h-regularized system (13) has the $\mathcal{G}$-q.a. behavior at zero (w.r.) to $c(\varepsilon)$, i.e.,

$$\lim_{\varepsilon\to 0}\left\langle\frac{u_{2,\varepsilon}(\varepsilon x, \varepsilon t)}{c(\varepsilon)}, \psi(x,t)\right\rangle = C_{2,\psi}, \quad C_{2,\psi} \in \mathbf{C},\ \psi \in \mathcal{D}(\mathbf{R}^2).$$

Remark 1. In fact we shall prove that the generalized solution $u_\varepsilon = u_{2\varepsilon}$ to the h-regularized wave equation has the corresponding $\mathcal{G}$-q.a. behavior.

Proof. We note that (16) holds with $a_{2,\varepsilon}$ instead of a_2. This follows from [26].

Let $\psi \in \mathcal{D}(\mathbf{R}^2)$, $\operatorname{supp}\psi \Subset K_T$ for enough large compact set $K_0 \subset \mathbf{R}$ and $T > 0$. Due to $f(x,t,0) = 0$, we have

$$\left|u_{1,\varepsilon}(\varepsilon x,\varepsilon t)\right| \leqslant \left|a_{1,\varepsilon}(\varepsilon x-\varepsilon t)\right| + \sup_{x\in\mathbf{R}}\left\{\left|f'_{h(\varepsilon)}\right|\right\}\int\limits_0^{\varepsilon t}\left|u_{2,\varepsilon}(\varepsilon x-\varepsilon t+\tau,\tau)\right|d\tau,$$

$$\left|u_{2,\varepsilon}(\varepsilon x,\varepsilon t)\right| \leqslant \left|a_{2,\varepsilon}(\varepsilon x+\varepsilon t)\right| + \int\limits_0^{\varepsilon t}\left|u_{1,\varepsilon}(\varepsilon x+\varepsilon t-\tau,\tau)\right|d\tau,$$

$\varepsilon \in (0,\varepsilon_0)$, for suitable ε_0. Gronwall's inequality and (15) imply

$$\begin{aligned}&\sup_{(x,t)\in K_T}\sqrt{\left|u_{1,\varepsilon}(\varepsilon x,\varepsilon t)\right|^2+\left|u_{2,\varepsilon}(\varepsilon x,\varepsilon t)\right|^2}\\ &\quad\leqslant C\sup_{(x,t)\in K_T}\sqrt{\left|a_{1,\varepsilon}(\varepsilon x-\varepsilon t)\right|^2+\left|a_{2,\varepsilon}(\varepsilon x+\varepsilon t)\right|^2}.\end{aligned}\tag{18}$$

We apply these estimates in the examination of the $\mathcal{G}$-q.a. behavior of $u_{2,\varepsilon}$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\left|\left\langle\frac{u_{2,\varepsilon}(\varepsilon x,\varepsilon t)}{c(\varepsilon)},\psi(x,t)\right\rangle\right| &= \lim_{\varepsilon\to 0}\left\langle\frac{a_2(\varepsilon x+\varepsilon t)}{c(\varepsilon)},\psi(x,t)\right\rangle\\ &\quad+\left\langle\frac{\int_0^{\varepsilon t}u_{1,\varepsilon}(\varepsilon x+\varepsilon t-\tau,\tau)\,d\tau}{c(\varepsilon)},\psi(x,t)\right\rangle.\end{aligned}$$

By assumptions in (16) and (17) and by (18) it follows that

$$\lim_{\varepsilon\to 0}\left\langle\frac{u_{2,\varepsilon}(\varepsilon x,\varepsilon t)}{c(\varepsilon)},\psi(x,t)\right\rangle = C_{2,\psi},\quad C_{2,\psi}\in\mathbf{C},\ \psi\in\mathcal{D}(\mathbf{R}^2).\qquad\square$$

Example 2. As another example of a semilinear hyperbolic system we consider the Euler–Lagrange equation with one space variable. The existence and uniqueness of generalized solutions to this equation are given in [9], especially when $V(x)$ is the delta function. For the Euler–Lagrange equation $\ddot{x}+V'(x)=0$, $x(0)=x_0$, $\dot{x}(0)=\dot{x}_0$, $x_0,\dot{x}_0\in\bar{\mathbf{R}}$, or in the form of representatives $\ddot{x}_\varepsilon+V'_\varepsilon(x_\varepsilon)=0$, $x_\varepsilon(0)=x_{0\varepsilon}$, $\dot{x}_\varepsilon(0)=\dot{x}_{0\varepsilon}$, $x_{0\varepsilon},\dot{x}_{0\varepsilon}\in\mathbf{R}_M$, it is proved in [9] that there exists a unique solution in $\mathcal{G}(\mathbf{R})$, $[V_\varepsilon]$ is a tempered generalized function of L^∞-log type, i.e., $|V''_\varepsilon|\leqslant C|\log\varepsilon|$, $\varepsilon\in(0,1)$.

Let V be a C^∞ function with unbounded second derivative. We find a regularization $V_{h(\varepsilon)}=V\kappa_{h(\varepsilon)}$, such that $|V''_{h(\varepsilon)}(x)|\leqslant C|\log\varepsilon|^{3/2}$, for some $C>0$. Setting $x_\varepsilon=u_{2,\varepsilon}$, $u_{1,\varepsilon}=\dot{u}_{2,\varepsilon}$ we obtain the regularized system

$$\frac{d}{dt}u_1=-\left[V'_{h(\varepsilon)}(u_{2\varepsilon})\right],\quad \frac{d}{dt}u_2=u_1,\quad u_2(0)=[x_{0\varepsilon}],\quad u_1(0)=[\dot{x}_{0\varepsilon}]$$

which corresponds to h-regularized equation

$$\ddot{x}+\left[V'_{h(\varepsilon)}(x)\right]=0,\quad x(0)=[x_{0\varepsilon}],\quad \dot{x}(0)=[\dot{x}_{0\varepsilon}].\tag{19}$$

Then, there is $C>0$ such that $\sup_{t\in[-T,T]}\sqrt{|u_{1,\varepsilon}(\varepsilon t)|^2+|u_{2,\varepsilon}(\varepsilon t)|^2}\leqslant C(x_{0\varepsilon}^2+\dot{x}_{0\varepsilon}^2)^{1/2}$. Using this inequality and $u_{2,\varepsilon}(\varepsilon t)=x_{0\varepsilon}+\int_0^{\varepsilon t}u_{1,\varepsilon}(\varepsilon u)\,du$, we directly obtain

Proposition 4. *Let $V \in C^1(\mathbf{R})$, $V'(0) = 0$, $c(\varepsilon) \in \mathcal{K}$. If the limit $\lim_{\varepsilon\to 0} \frac{x_{0\varepsilon}}{c(\varepsilon)}$ exists, then the solution $[u_{2,\varepsilon}(t)] = [x_\varepsilon(t)]$ to* (19) *has the $\mathcal{G}$-q.a. behavior at zero* (w.r.) *to $c(\varepsilon)$, i.e.,*

$$\lim_{\varepsilon\to 0}\left\langle \frac{x_\varepsilon(\varepsilon t)}{c(\varepsilon)}, \psi(t)\right\rangle = C_\psi, \quad C_\psi \in \mathbf{C},\ \psi \in \mathcal{D}(\mathbf{R}^2).$$

4. $\mathcal{G}$-q.a. expansion at infinity

The $\mathcal{G}$-q.a. behavior at infinity is defined for tempered generalized functions since for $c(1/\varepsilon) \in \mathcal{K}$ and $(\eta)_\varepsilon \in \mathcal{N}_t(\mathbf{R}^n)$ we have

$$\lim_{\varepsilon\to 0^+}\left\langle \frac{\eta_\varepsilon(t)}{c(1/\varepsilon)}, \varphi(t)\right\rangle \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ for every } \varphi \in \mathcal{D}(\mathbf{R}^n).$$

We give the definition of the notion of $\mathcal{G}$-q.a. expansion at infinity. This includes the $\mathcal{G}$-q.a. behavior at infinity.

For the sake of $\mathcal{G}$-q.a. expansion at infinity we use the usual mollifier net $(\phi_\varepsilon)_\varepsilon$.

Recall, if $\theta \in \mathcal{S}(\mathbf{R}^n)$, then $(\theta - \theta * \phi_\varepsilon) \in \mathcal{N}_t(\mathbf{R}^n)$ and if $f \in \mathcal{S}'(\mathbf{R}^n)$, then $\mathrm{Ctd}\, f = [f_\varepsilon]$, where $f_\varepsilon = f * \phi_\varepsilon$, $\varepsilon \in (0,1)$ (ϕ is described in (1)).

Recall, $\mathcal{K}$ is defined in Section 3.

Definition 2. We denote by Λ the set $\mathbf{N}$ or a finite set of the form $\{1, 2, \ldots, N\}$, $N \in \mathbf{N}$. Let $c_k \in \mathcal{K}$, $k \in \Lambda$, such that

$$\lim_{\varepsilon\to 0} \frac{c_{k+1}(1/\varepsilon)}{c_k(1/\varepsilon)} \to 0, \quad k = 1, \ldots, N-1 \text{ (or } k \in \mathbf{N}, \text{ if } \Lambda = \mathbf{N}) \text{ and}$$
$$P_k = [P_{k\varepsilon}] \in \mathcal{G}_t(\mathbf{R}), \quad k \in \Lambda.$$

Then $G = [G_\varepsilon] \in \mathcal{G}_t(\mathbf{R})$ has the $\mathcal{G}$-q.a. expansion at infinity as $\sum_{k\in\Lambda} P_{k\varepsilon}$ (w.r.) to $\{c_k(1/\varepsilon);\ k \in \Lambda\}$ if

$$\frac{(G_\varepsilon - \sum_{k=1}^m P_{k\varepsilon})(x/\varepsilon)}{c_m(1/\varepsilon)} \to 0, \quad \varepsilon \to 0^+,\ \text{in } \mathcal{D}'(\mathbf{R}) \text{ for every } m \in \Lambda.$$

In this case we write

$$G \overset{\text{q.e.c.}}{\sim} \sum_{k\in\Lambda} P_k \quad \text{(w.r.) to } \left\{c_m(1/\varepsilon),\ m \in \Lambda\right\},$$

and say that G has the $\mathcal{G}$-q.a. expansion at infinity in Colombeau sense.

We refer to [22] for corresponding definition at zero.

Proposition 5. *Let $f \in \mathcal{S}'(\mathbf{R})$. Then $f \overset{\text{q.e.}}{\sim} \sum_{k\in\Lambda} A_k P_k$ (w.r.) to $\{c_m(1/\varepsilon),\ m \in \Lambda\}$ if and only if*

$$\mathrm{Ctd}\, f \overset{\text{q.e.c.}}{\sim} \sum_{k\in\Lambda} A_k[P_{k\varepsilon}] \quad \textit{(w.r.) to } \left\{c_m(1/\varepsilon),\ m \in \Lambda\right\}.$$

Proof. Setting up $\alpha \in \mathcal{D}$, $\varepsilon \in (0,1)$, $x \in \mathbf{R}$, we obtain

$$\left\langle \frac{(f * \phi_\varepsilon)(x/\varepsilon) - \sum_{k=1}^m A_k (P_k * \phi_\varepsilon)(x/\varepsilon)}{c_m(1/\varepsilon)}, \alpha(x) \right\rangle$$
$$= \left\langle \frac{\varepsilon(f(x) - \sum_{k=1}^m A_k P_k(x))}{c_m(1/\varepsilon)}, \bigl(\check{\phi}_\varepsilon(t) * \alpha(t\varepsilon)\bigr)(x) \right\rangle$$
$$= \left\langle \frac{f(x/\varepsilon) - \sum_{k=1}^m A_k P_k(x/\varepsilon)}{c_m(1/\varepsilon)}, \psi_\varepsilon(x) \right\rangle,$$

where

$$\psi_\varepsilon(x) = \int_{-\infty}^{\infty} \check{\phi}_\varepsilon(t)\alpha(x - t\varepsilon)\,dt = \int_{-\infty}^{\infty} \check{\phi}(t)\alpha(x - t\varepsilon^2)\,dt \in \mathcal{S}(\mathbf{R}).$$

Due to the convergence $\psi_\varepsilon \xrightarrow{\mathcal{S}} \alpha$ as $\varepsilon \to 0$, this implies

$$\lim_{\varepsilon \to 0} \left\langle \frac{(f * \phi_\varepsilon) - \sum_{k=0}^m A_k (P_k * \phi_\varepsilon)(x/\varepsilon)}{c_m(1/\varepsilon)}, \alpha(x) \right\rangle = \left\langle g(x) - \sum_{k=0}^m A_k P_k(x), \alpha(x) \right\rangle$$

and the assertion follows. □

In the next propositions we give the $\mathcal{G}$-q.a. expansion of an element in $\mathcal{G}_t \setminus \mathcal{S}'$.

Proposition 6. *Let* $\delta^n = \left[\frac{1}{\varepsilon^n}\phi^n\left(\frac{\cdot}{\varepsilon}\right)\right]$, *where* $n \in \mathbf{N}$, $\phi \in C_0^\infty$, $\int \phi = 1$, $\int x^j \phi(x)\,dx = 0$, $j \leqslant N$. *Then*

$$\delta^n(\cdot) = \left[\frac{1}{\varepsilon^n}\phi^n\left(\frac{\cdot}{\varepsilon}\right)\right] \overset{\text{q.e.c.}}{\sim} \sum_{k=0}^{j} (-1)^k \frac{\mu_k}{k!}\left[\frac{1}{\varepsilon^n}\phi^{(k)}\left(\frac{\cdot}{\varepsilon}\right)\right]$$
(w.r.) *to* $\{\varepsilon^{-n+2+2j},\ j = 0, 1, \dots, N\}$.

Proof. Let $\mu_k = \int x^k \phi^n(x)\,dx$, $k \in \mathbf{N}_0$. We have $\varepsilon \to 0$, $j = 0, 1, \dots, N \in \mathbf{N}$,

$$\frac{1}{\varepsilon^{-n+2}} \int \left(\frac{1}{\varepsilon^n}\phi^n\left(\frac{x}{\varepsilon^2}\right) - \frac{\mu_0}{\varepsilon^n}\phi\left(\frac{x}{\varepsilon^2}\right)\right)\psi(x)\,dx \quad \Rightarrow \quad \int \phi^n(x)\,dx - \mu_0 = 0$$
as $\varepsilon \to 0$,
$$\vdots$$
$$\frac{1}{\varepsilon^{-n+2+2j}} \int \left(\frac{1}{\varepsilon^n}\phi^n\left(\frac{x}{\varepsilon^2}\right) - \frac{\mu_0}{\varepsilon^n}\phi\left(\frac{x}{\varepsilon^2}\right) + \cdots + \frac{(-1)^j}{\varepsilon^n (j)!}\mu_j \phi^{(j)}\left(\frac{x}{\varepsilon^2}\right)\right)\psi(x)\,dx \to 0.$$

Thus,

$$\left[\frac{1}{\varepsilon^n}\phi^n\left(\frac{\cdot}{\varepsilon}\right)\right] \overset{\text{q.e.c.}}{\sim} \sum_{k=0}^{j} (-1)^k \frac{\mu_k}{k!}\left[\frac{1}{\varepsilon^n}\phi^{(k)}\left(\frac{\cdot}{\varepsilon}\right)\right]$$
(w.r.) to $\{\varepsilon^{-n+2+2j},\ j = 0, 1, \dots, N \in \mathbf{N}\}$. □

4.1. $\mathcal{G}$-q.a. expansion at infinity to wave equation

As an application, we consider the wave equation

$$\left(\partial_t^2 - \partial_x^2\right)v = 0, \quad v_{|t=0} = a_0(x), \quad v_{t|t=0} = a_1(x), \quad v_{x|x=0} = 0 \tag{20}$$

and give the evaluation of the solution when initial data are $(a_0, a_1) = (\delta^2, \delta^2)$. Note that this Cauchy problem has no sense in Schwartz type spaces. This equation is a singular part of a general wave equation considered in [15]. We find $\mathcal{G}$-q.a. expansion of the solution to (20) along the characteristic curves.

Our intention is to give some information on the behavior of the solution $[v_\varepsilon]$ along the characteristics with argument $(x/\varepsilon, t/\varepsilon)$ which reflect the growth order at infinity, since $\varepsilon \to 0$.

We will apply our theory to the solution of (20). We will use the form of a solution for this equation given in [15]. Note, in [15] it was assumed that the initial values (a_0, a_1) are supported by finitely many points $0 < \xi_1 < \xi_2 < \xi_3 < \cdots < \xi_n$. But, for us it is interesting to consider this equation with $(a_0, a_1) = (\delta^2, \delta^2)$ because with such condition the given equation does not have any sense in the framework of Schwartz distributions. On the other hand we are able to apply the form of the solution of [15] for expressing the solution $[v_\varepsilon]$ which is to be analyzed through generalized asymptotic expansion at infinity. Thus, without a loss of generality we shall give the expansion of $\delta^2(x \pm t)$ for the sake of $\mathcal{G}$-q.a. expansion of the solution $[v_\varepsilon]$ to wave equation since the translation $x - \xi_i \pm t \to x \pm t$ does not influence on $\mathcal{G}$-q.a. expansion. Proposition 6 imply the following useful expansions.

Let $\phi \in C_0^\infty$, $\int \phi = 1$, $\int x^m \phi(x)\,dx = 0$, $m = 0, 1, \ldots, N$, $N \in \mathbf{N}$, and $\delta^2(x \pm t) = \left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{x \pm t}{\varepsilon}\right)\right]$. Then,

$$\delta^2(x \pm t) \overset{\text{q.e.c.}}{\sim} \sum_{k=0}^{j} \frac{(-1)^k}{k!}\mu_k \left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x \pm t}{\varepsilon}\right)\right],$$

where $\mu_k = \int u^k \phi^2(u)\,du$, $k = 1, \ldots, N$, (w.r.) to $\{\varepsilon^{2j},\ j = 0, 1, \ldots, N \in \mathbf{N}\}$. In particular, let $\delta^2(2\cdot) = \left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{2\cdot}{\varepsilon}\right)\right]$. Then,

$$\left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{2\cdot}{\varepsilon}\right)\right] \overset{\text{q.e.c.}}{\sim} \sum_{k=0}^{j} \frac{(-1)^k}{k!}\frac{\mu_k}{2^{k+1}}\left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{\cdot}{\varepsilon}\right)\right] \quad \text{(w.r.) to } \{\varepsilon^{2j},\ j = 0, 1, \ldots, N\}.$$

Let $\delta^2(x + t - 2s) = \left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{x+t-2s}{\varepsilon}\right)\right]$. Then in $\mathbf{R}^3$,

$$\left[\delta^2(x + t - 2s)\right] \overset{\text{q.e.c.}}{\sim} \sum_{k=0}^{j} \frac{(-1)^k}{k!}\mu_k\left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x+t-2s}{\varepsilon}\right)\right] \quad \text{(w.r.) to } \{\varepsilon^{2j},\ j = 0, 1, \ldots, N \in \mathbf{N}\}. \tag{21}$$

In particular, let $\delta^2(2x - 2s) = \left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{2x-2s}{\varepsilon}\right)\right]$. Then in $\mathbf{R}^2$,

$$\left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{2x-2s}{\varepsilon}\right)\right] \overset{\text{q.e.c.}}{\sim} \sum_{k=0}^{j} \frac{(-1)^k}{k!}\frac{\mu_k}{2^{k+1}}\left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x-s}{\varepsilon}\right)\right] \quad \text{(w.r.) to } \{\varepsilon^{2j},\ j = 0, 1, \ldots, N \in \mathbf{N}\}.$$

Consider the singular part v to the wave equation (see [15]), which solves the following linear problem in Colombeau algebra:

$$\begin{cases} (\partial_t^2 - \partial_x^2)v^\varepsilon = 0, \\ v^\varepsilon{}_{|t=0} = a_0^\varepsilon(x), \quad v^\varepsilon_{t|t=0} = a_1^\varepsilon(x), \\ v^\varepsilon_{x|x=0} = 0. \end{cases}$$

Setting $\chi_1^\varepsilon = (\partial_t + \partial_x)v^\varepsilon$, $\chi_2^\varepsilon = (\partial_t - \partial_x)v^\varepsilon$ (cf. [15]), we have the evaluation along the characteristic lines (i) as $x \geqslant t$,

$$v^\varepsilon(t,x) = a_0^\varepsilon(x+t) + \int_0^t \big(a_1^\varepsilon(x+t-2s) - \big(a_0^\varepsilon\big)'(x+t-2s)\big)\,ds;$$

(ii) as $x < t$,

$$v^\varepsilon(t,x) = a_0^\varepsilon(x+t) + \int_0^{(x+t)/2} \big(a_1^\varepsilon(x+t-2s) - \big(a_0^\varepsilon\big)'(x+t-2s)\big)\,ds$$
$$+ \int_{(x+t)/2}^{t} \big(\big(a_0^\varepsilon\big)'(2s-x-t) + a_1^\varepsilon(2s-x-t)\big)\,ds.$$

We shall give the expansion at infinity to the singular part of the wave equation when $x \geqslant t$.

Let $x > t$. Then,

$$v^\varepsilon\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) = \frac{1}{2}\left(a_0^\varepsilon\left(\frac{x+t}{\varepsilon}\right) + a_0^\varepsilon\left(\frac{x-t}{\varepsilon}\right)\right) + \int_0^{t/\varepsilon} a_1^\varepsilon\left(\frac{x+t}{\varepsilon} - 2s\right) ds = A^\varepsilon + B^\varepsilon,$$

where $B^\varepsilon = \frac{1}{\varepsilon}\int_0^t a_1^\varepsilon\big(\frac{x+t-2s}{\varepsilon}\big)\,ds$. Let

$$\left(a_0^\varepsilon\left(\frac{\cdot}{\varepsilon}\right), a_1^\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\right) = \big(\delta_\varepsilon^2, \delta_\varepsilon^2\big), \quad \text{where } \delta^2 = \big[\delta_\varepsilon^2\big] = \left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{\cdot}{\varepsilon}\right)\right].$$

We have

$$[A^\varepsilon] \overset{\text{q.e.c.}}{\sim} \sum_{k=0}^{j} \frac{(-1)^k}{k!}\frac{\mu_k}{2}\left(\left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x+t}{\varepsilon}\right)\right] + \left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x-t}{\varepsilon}\right)\right]\right).$$

Let us see B^ε. Integrand of B^ε has the evaluation (21). Then,

$$[B^\varepsilon] \overset{\text{q.e.c.}}{\sim} \sum_{k=1}^{j} \frac{(-1)^k}{k!}\frac{\mu_k}{2}\left(\left[\frac{1}{\varepsilon^2}\phi^{(k-1)}\left(\frac{x+t}{\varepsilon^2}\right)\right] - \left[\frac{1}{\varepsilon^2}\phi^{(k-1)}\left(\frac{x-t}{\varepsilon^2}\right)\right]\right)$$

(w.r.) to $\{\varepsilon^{2j},\ j=0,1,\dots,N,\ N\in\mathbf{N}\}$. Consequently,

$$\frac{1}{\varepsilon^0}\int\int\left(v^\varepsilon\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)-\frac{\mu_0}{2}\left(\frac{1}{\varepsilon^2}\phi\left(\frac{x+t}{\varepsilon^2}\right)+\frac{1}{\varepsilon^2}\phi\left(\frac{x-t}{\varepsilon^2}\right)\right)\right.$$
$$\left.-\frac{1}{\varepsilon}\int_0^t\frac{\mu_0}{\varepsilon^2}\phi\left(\frac{x+t-2s}{\varepsilon^2}\right)ds\right)\psi(x,t)\,dx\,dt\to 0$$

(w.r.) to $\varepsilon^0=1$, where $\int_0^t\frac{1}{\varepsilon^2}\phi(\frac{x+t-2s}{\varepsilon^2})\,ds\to 0$,

$$\frac{1}{\varepsilon^2}\int\int\left(v^\varepsilon\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)-\frac{\mu_0}{2}\left(\frac{1}{\varepsilon^2}\phi\left(\frac{x+t}{\varepsilon^2}\right)+\frac{1}{\varepsilon^2}\phi\left(\frac{x-t}{\varepsilon^2}\right)\right)\right.$$
$$+\frac{\mu_1}{2}\left(\frac{1}{\varepsilon^2}\phi'\left(\frac{x+t}{\varepsilon^2}\right)+\frac{1}{\varepsilon^2}\phi'\left(\frac{x-t}{\varepsilon^2}\right)\right)-\frac{1}{\varepsilon}\int_0^t\frac{\mu_0}{\varepsilon^2}\phi\left(\frac{x+t-2s}{\varepsilon^2}\right)ds$$
$$\left.+\frac{\mu_1}{2}\left(\frac{1}{\varepsilon}\phi\left(\frac{x-t}{\varepsilon^2}\right)-\frac{1}{\varepsilon}\phi\left(\frac{x+t}{\varepsilon^2}\right)\right)\right)\psi(x,t)\,dx\,dt\to 0$$

(w.r.) to ε^2, where $\int_0^t\frac{1}{\varepsilon^2}\phi(\frac{x+t-2s}{\varepsilon^2})\,ds\to 0$.

Finally, we obtain in $\mathbf{R}^2$,

$$\left[v^\varepsilon(t,x)\right]\overset{\text{q.e.c.}}{\sim}\sum_{k=0}^{j}\frac{(-1)^k}{k!}\frac{\mu_k}{2}\left(\left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x+t}{\varepsilon}\right)\right]+\left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x-t}{\varepsilon}\right)\right]\right)$$
$$+\sum_{k=1}^{j}\frac{(-1)^k}{k!}\frac{\mu_k}{2}\left(\left[\frac{1}{\varepsilon}\phi^{(k-1)}\left(\frac{x+t}{\varepsilon}\right)\right]-\left[\frac{1}{\varepsilon}\phi^{(k-1)}\left(\frac{x-t}{\varepsilon}\right)\right]\right)$$

(w.r.) to $\{\varepsilon^{2j},\ j=0,1,\dots,N\in\mathbf{N}\}$.

Consider the case $x=t$. Then,

$$v^\varepsilon\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}\right)=\frac{1}{2}\left(a_0^\varepsilon\left(\frac{2x}{\varepsilon}\right)+a_0^\varepsilon(0)\right)+\frac{1}{\varepsilon}\int_0^x a_1^\varepsilon\left(\frac{2x-2s}{\varepsilon}\right)ds$$

and

$$v^\varepsilon\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon}\right)=\frac{1}{2}\frac{1}{\varepsilon^2}\phi^2\left(\frac{2x}{\varepsilon^2}\right)+\frac{1}{\varepsilon}\int_0^x\frac{1}{\varepsilon^2}\phi^2\left(\frac{2x-2s}{\varepsilon^2}\right)ds+\frac{1}{2}\frac{1}{\varepsilon^2}\phi^2(0)$$
$$=v_1^\varepsilon\left(\frac{x}{\varepsilon}\right)+\frac{1}{2}\frac{1}{\varepsilon^2}\phi^2(0).$$

We shall evaluate part $v_1^\varepsilon(\frac{x}{\varepsilon})$. We have

$$\frac{1}{\varepsilon^0}\int\left(v_1^\varepsilon\left(\frac{x}{\varepsilon}\right)-\frac{1}{2}\frac{\mu_0}{\varepsilon^2}\phi\left(\frac{x}{\varepsilon^2}\right)-\frac{\mu_0}{2\varepsilon}\int_0^x\frac{1}{\varepsilon^2}\phi\left(\frac{x-s}{\varepsilon^2}\right)ds\right)\psi(x)\,dx\to 0,$$

where $\int_0^x \frac{1}{\varepsilon^2}\phi\left(\frac{x-s}{\varepsilon^2}\right) ds = \frac{1}{2}$ since ϕ is even (see Section 3),

$$\frac{1}{\varepsilon^2}\int \left(v_1^\varepsilon\left(\frac{x}{\varepsilon}\right) - \frac{1}{2}\frac{\mu_0}{\varepsilon^2}\phi\left(\frac{x}{\varepsilon^2}\right) + \frac{1}{4}\frac{\mu_1}{\varepsilon^2}\phi'\left(\frac{x}{\varepsilon^2}\right) - \frac{\mu_0}{2\varepsilon}\int_0^x \frac{1}{\varepsilon^2}\phi\left(\frac{x-s}{\varepsilon^2}\right) ds \right.$$
$$\left. + \frac{1}{\varepsilon}\frac{\mu_1}{4}\left(\phi'\left(\frac{x}{\varepsilon}\right) - \phi'(0)\right)\right)\psi(x)\,dx \to 0,$$

$$\frac{1}{\varepsilon^4}\int \left(v_1^\varepsilon\left(\frac{x}{\varepsilon}\right) - \frac{1}{2}\frac{\mu_0}{\varepsilon^2}\phi\left(\frac{x}{\varepsilon^2}\right) + \frac{1}{4}\frac{\mu_1}{\varepsilon^2}\phi'\left(\frac{x}{\varepsilon^2}\right) - \frac{1}{16}\frac{\mu_2}{\varepsilon^2}\phi''\left(\frac{x}{\varepsilon^2}\right)\right.$$
$$- \frac{\mu_0}{2\varepsilon}\int_0^x \frac{1}{\varepsilon^2}\phi\left(\frac{x-s}{\varepsilon^2}\right) ds + \frac{1}{\varepsilon}\frac{\mu_1}{4}\left(\phi'\left(\frac{x}{\varepsilon}\right) - \phi'(0)\right)$$
$$\left. - \frac{1}{\varepsilon}\frac{\mu_2}{16}\left(\phi''\left(\frac{x}{\varepsilon}\right) - \phi''(0)\right)\right)\psi(x)\,dx \to 0.$$

Thus,

$$v^\varepsilon\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}\right) = v_1^\varepsilon\left(\frac{x}{\varepsilon}\right) + \frac{1}{2\varepsilon^2}\phi^2(0)$$

and

$$[v^\varepsilon] \overset{\text{q.e.c.}}{\sim} \frac{1}{2}\sum_{k=0}^{j}\frac{(-1)^k}{k!}\frac{\mu_k}{2^{k+1}}\left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x}{\varepsilon}\right)\right] + \left[\frac{1}{\varepsilon}\frac{\mu_0}{4}\right]$$
$$- \sum_{k=1}^{j}\frac{(-1)^k}{k!}\frac{\mu_k}{2^{k+1}}\left(\left[\frac{1}{\varepsilon}\phi^{(k)}\left(\frac{x}{\varepsilon}\right) - \frac{1}{\varepsilon}\phi^{(k)}(0)\right]\right) + \left[\frac{1}{2\varepsilon^2}\phi^2(0)\right]$$

(w.r.) to $\{\varepsilon^{2j},\ j = 0, 1, \ldots, N\}$, when $x = t$.

Let $x < t$. Then,

$$v^\varepsilon\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) = a_0^\varepsilon\left(\frac{x+t}{\varepsilon}\right) + \frac{1}{\varepsilon}\int_0^{(x+t)/2}\left(a_1^\varepsilon\left(\frac{x+t-2s}{\varepsilon}\right) - \left(a_0^\varepsilon\right)'\left(\frac{x+t-2s}{\varepsilon}\right)\right) ds$$
$$+ \int_{(x+t)/2}^{t}\left(\left(a_0^\varepsilon\right)'\left(\frac{2s-x-t}{\varepsilon}\right) + a_1^\varepsilon\left(\frac{2s-x-t}{\varepsilon}\right)\right) ds$$

and

$$v^\varepsilon\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) = \frac{1}{2}\left(a_0^\varepsilon\left(\frac{x+t}{\varepsilon}\right) + a_0^\varepsilon\left(\frac{t-x}{\varepsilon}\right)\right)$$
$$+ \frac{1}{\varepsilon}\int_0^{(x+t)/2} a_1^\varepsilon\left(\frac{x+t-2s}{\varepsilon}\right) ds + \frac{1}{\varepsilon}\int_{(x+t)/2}^{t} a_1^\varepsilon\left(\frac{2s-x-t}{\varepsilon}\right) ds.$$

Setting

$$\left(a_1^\varepsilon\left(\frac{\cdot}{\varepsilon}\right), a_0^\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\right) = \left(\delta_\varepsilon^2, \delta_\varepsilon^2\right)$$

we obtain

$$v^\varepsilon\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) = \frac{1}{2}A^\varepsilon + \frac{1}{\varepsilon}B^\varepsilon + \frac{1}{\varepsilon}C^\varepsilon.$$

Similarly as in previous part we obtain

$$[A^\varepsilon] \overset{\text{q.e.c.}}{\sim} \sum_{k=0}^{j} \frac{(-1)^k}{k!}\mu_k\left(\left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x+t}{\varepsilon}\right)\right] + \left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{t-x}{\varepsilon}\right)\right]\right),$$

$$[B^\varepsilon] \overset{\text{q.e.c.}}{\sim} \frac{1}{4}\mu_0 + \sum_{k=1}^{j} \frac{(-1)^k}{k!}\frac{\mu_k}{2}\left(\left[\frac{1}{\varepsilon^2}\phi^{(k-1)}\left(\frac{x+t}{\varepsilon}\right)\right] - \left[\frac{1}{\varepsilon^2}\phi^{(k-1)}(0)\right]\right),$$

$$[C^\varepsilon] \overset{\text{q.e.c.}}{\sim} -\frac{1}{4}\mu_0 - \sum_{k=1}^{j} \frac{(-1)^k}{k!}\frac{\mu_k}{2}\left(\left[\frac{1}{\varepsilon^2}\phi^{(k-1)}\left(\frac{t-x}{\varepsilon}\right)\right] - \left[\frac{1}{\varepsilon^2}\phi^{(k-1)}(0)\right]\right).$$

Finally, for $x < t$ we obtain (in $\mathbf{R}^2$)

$$\begin{aligned}[v^\varepsilon] \overset{\text{q.e.c.}}{\sim}\ & \frac{1}{2}\sum_{k=0}^{j} \frac{(-1)^k}{k!}\mu_k\left(\left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{x+t}{\varepsilon}\right)\right] + \left[\frac{1}{\varepsilon^2}\phi^{(k)}\left(\frac{t-x}{\varepsilon}\right)\right]\right) \\ & + \sum_{k=1}^{j} \frac{(-1)^k}{k!}\frac{\mu_k}{2}\left(\left[\frac{1}{\varepsilon}\phi^{(k-1)}\left(\frac{x+t}{\varepsilon}\right)\right] - \left[\frac{1}{\varepsilon}\phi^{(k-1)}\left(\frac{t-x}{\varepsilon}\right)\right]\right)\end{aligned}$$

(w.r.) to $\{\varepsilon^{2j},\ j = 0, \ldots, N,\ N \in \mathbf{N}\}$.

5. *Sc*-asymptotic, respectively *Sc*-boundedness

5.1. *Sc-asymptotic in Colombeau algebra*

First, we shall recall the definition of regularly varying function (cf. [4]). A function $\rho : (a, \infty) \to \mathbf{R}$, $a \in \mathbf{R}$, is called regularly varying at infinity if it is positive, measurable and there exists a real number α such that for each $x > 0$, $\lim_{k\to\infty} \frac{\rho(kx)}{\rho(k)} = x^\alpha$. If $\alpha = 0$, ρ is called slowly varying at infinity and for such a function the letter L is used [25].

Recall [4] $\rho : (a, \infty) \to \mathbf{R}$ is regularly varying at infinity if and only if it can be written as $\rho(x) = x^\alpha L(x)$, $x > a$, for some real number α and some slowly varying function L at infinity. (L is slowly varying if $\frac{L(kx)}{L(k)} \to 1$ as $k \to \infty$, for every $x > a$.)

The same conclusions hold for regularly varying functions at zero. Then, we just have ρ (and L) defined on $(0, a)$ for some a (cf. [4]).

Given a generalized function $U \in \mathcal{G}(\mathbf{R}^n)$ with a representative $u_\varepsilon \in \mathcal{E}_M(\mathbf{R}^n)$ and $h \in \mathbf{R}^n$ one defines a translation $\tau_h U$, $h \in \mathbf{R}$, by its representative $\tau_h(u_\varepsilon)(x) = U_\varepsilon(x-h)$. Clearly, $\tau_h U \in \mathcal{G}(\mathbf{R}^n)$.

Let $\mathbf{R}^n_+ = \{(\xi_1, \dots, \xi_n);\ \xi_i > 0,\ i = 1, \dots, n\}$. By $\sum(\mathbf{R}^n_+)$ we denote the set of functions $c : \mathbf{R}^n_+ \to \mathbf{R}_+$ which are positive continuous and equal 1 in $\mathbf{R}^n_+ \setminus B(0,r)$, where $B(0,r) = \{x \in \mathbf{R}^n,\ \|x\| < r\}$, $r > 0$.

Proposition 7. (a) *Let $R_\varepsilon \in \mathcal{N}_t(\mathbf{R}^n)$. Then for every $c \in \sum(\mathbf{R}^n_+)$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n_+$, and for every $\psi \in \mathcal{D}(\mathbf{R}^n)$,*

$$\lim_{\varepsilon \to 0} \left\langle \frac{R_\varepsilon(x_1 + \xi_1/\varepsilon, x_2 + \xi_2/\varepsilon, \dots, x_n + \xi_n/\varepsilon)}{c(1/\varepsilon)}, \psi(x_1, \dots, x_n) \right\rangle = 0.$$

(b) *Let $F \in \mathcal{G}(\mathbf{R}^n)$, $c \in \sum(\mathbf{R}^n_+)$ and*

$$\lim_{\varepsilon \to 0} \left\langle \frac{F_\varepsilon(x + \xi/\varepsilon)}{c(1/\varepsilon)}, \psi(x) \right\rangle = C_\psi, \tag{22}$$

$C_\psi \neq 0$ for some $\psi \in \mathcal{D}(\mathbf{R}^n)$, holds. Then, (22) *holds for every representative F_ε of F.* (*The limit is independent of a representative*.)

(c) *Sc-boundedness does not depend on representatives.*

Proof. (a) is obvious since $R_\varepsilon \in \mathcal{N}_t(\mathbf{R}^n)$. (b) Suppose that there exist two representatives $F_{1\varepsilon}, F_{2\varepsilon}$ with the same property. Then, $F_{1\varepsilon} - F_{2\varepsilon} \in \mathcal{N}_t(\mathbf{R}^n)$ and due to (a) it tends to zero. (c) follows from (a). □

Definition 3. Let $F \in \mathcal{G}_t(\mathbf{R}^n)$. Then, F has the *Sc*-asymptotic in $\mathbf{R}^n_+$ related to some $c \in \sum(\mathbf{R}^n_+)$ if there exists a representative F_ε of F, and $a \in \mathbf{R}^n$, such that for every $\psi \in \mathcal{D}(\mathbf{R}^n)$, and every $\xi \in \mathbf{R}^n_+$,

$$\lim_{\varepsilon \to 0} \left\langle \frac{F_\varepsilon(t + \xi/\varepsilon)}{c(1/\varepsilon)}, \psi(t) \right\rangle \tag{23}$$

exists and does not depend on $\xi \in \mathbf{R}^n_+$.

By the Banach–Steinhaus theorem, we know that the existence of limit in (23) implies also that there exists $U \in \mathcal{D}'(\mathbf{R}^n)$ such that

$$\left\langle \frac{F_\varepsilon(t + \xi/\varepsilon)}{c(1/\varepsilon)}, \psi(t) \right\rangle \to \langle U, \psi \rangle, \quad \psi \in \mathcal{D}(\mathbf{R}^n).$$

Then, we write

$$F_\varepsilon(t + \xi/\varepsilon) \overset{\mathrm{Sc}}{\sim} c(1/\varepsilon) \cdot U(t), \quad \xi \in \mathbf{R}^n_+,\ \varepsilon \to 0.$$

We shall give the definition of *Sc*-boundedness.

Definition 4. An $F \in \mathcal{G}_t(\mathbf{R}^n)$ is Sc-bounded in $\mathbf{R}^n_+$ related to some $c \in \sum(\mathbf{R}^n_+)$ if there exists a representative F_ε of F and $a \in \mathbf{R}^n$, such that for every $\psi \in \mathcal{D}(\mathbf{R}^n)$, and every $\xi \in \mathbf{R}^n_+$,

$$\lim_{\varepsilon\to 0} \sup \left| \left\langle \frac{F_\varepsilon(t+\xi/\varepsilon)}{c(1/\varepsilon)}, \psi(t) \right\rangle \right| < \infty.$$

We shall give properties of Sc-asymptotic and Sc-boundedness by starting with classical properties of S-asymptotic which are analogous to the corresponding one for $\mathcal{G}$-q.a. behavior (cf. [21]).

Let $a \in \mathbf{R}^n$. We denote by η_a a function of the form $\eta_a(t) = \eta_{a_1}(t_1)\dots\eta_{a_n}(t_n)$, where $\eta_a(t) \in C^\infty(\mathbf{R}^n)$,

$$\eta_a(t) = \begin{cases} 1, & t > a+\mu, \\ 0, & t < a-\mu, \end{cases}$$

for some $\mu > 0$.

Proposition 8. (a) *Let $f \in \mathcal{D}'(\mathbf{R})$ and $c(h)$, $h > 0$, be a positive and continuous. Assume that for every $\psi \in \mathcal{D}(\mathbf{R})$,*

$$\lim_{h\to\infty} \left\langle \frac{f(x+h)}{c(h)}, \psi(x) \right\rangle \text{ exists} \tag{24}$$

and it is different from zero for some ψ.

(i) *The limit distribution g is of the form $g(x) = C\exp(\alpha x)$ and $c(h) = \exp(\alpha h)L(\exp(h))$, where L is a regularly varying function*;
(ii) *If $f \in \mathcal{S}'(\mathbf{R})$ and the limit in (24) exists with $(\eta_a f)$ instead of f, for every $\psi \in \mathcal{S}(\mathbf{R})$, then $g(x) = C \neq 0$ and $c(h) = L(\exp(h))$, $h > h_0$.*

(b) *Let $f \in \mathcal{S}'(\mathbf{R})$, $\varepsilon > \varepsilon_0$. Then, the following statements are equivalent*:

(i) *$(\eta_a f) \overset{S}{\sim} c(1/\varepsilon)g$ in $\mathcal{S}'(\mathbf{R})$, $\varepsilon \to 0$, in $\mathbf{R}_+$ related to some $c \in \sum(\mathbf{R}_+)$;*
(ii) *$f \overset{S}{\sim} c(1/\varepsilon)g$ in $\mathcal{D}'(\mathbf{R})$, $\varepsilon \to 0$ and $\left\{\frac{(\eta_a f)(\cdot+1/\varepsilon)}{c(1/\varepsilon)},\ \varepsilon > 0\right\}$ is bounded in $\mathcal{S}'(\mathbf{R})$ in $\mathbf{R}_+$ related to some $c \in \sum(\mathbf{R}_+)$;*
(iii)
$$\lim_{\varepsilon\to 0} \frac{((\eta_a f) * \phi_\varepsilon)(\cdot + 1/\varepsilon)}{c(1/\varepsilon)} = g_\varepsilon \quad \text{in } \mathcal{S}'(\mathbf{R}),\ g_\varepsilon \neq 0, \tag{25}$$

and $\frac{(\eta_a f)(\cdot+1/\varepsilon)}{c(1/\varepsilon)}$ is bounded in $\mathcal{S}'(\mathbf{R})$ in $\mathbf{R}_+$ related to some $c \in \sum(\mathbf{R}_+)$.

(c) *Let $f \in \mathcal{S}'(\mathbf{R})$, $a \in \mathbf{R}$, $\varepsilon < \varepsilon_0$. Then, the following statements are equivalent*:

(i) *$(\eta_a f)$ is S-bounded in $\mathcal{S}'(\mathbf{R})$ related to some $c \in \sum(\mathbf{R}_+)$;*
(ii) *$\frac{((\eta_a f)*\phi_\varepsilon)(\cdot+1/\varepsilon)}{c(1/\varepsilon)}$ is bounded in $\mathcal{S}'(\mathbf{R})$ related to some $c \in \sum(\mathbf{R}_+)$.*

Proof. (a) is proved in [23].

(b) (i) $\Rightarrow$ (iii) Clearly, $\frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}$ is S-bounded in $\mathbf{R}_+$ related to some $c \in \sum(\mathbf{R}_+)$ in $\mathcal{S}'(\mathbf{R})$. Let us prove (25). Let $\alpha \in \mathcal{S}(\mathbf{R})$, $\varepsilon > 0$. We have

$$\left\langle \frac{((\eta_a f) * \phi_\varepsilon)(x+1/\varepsilon)}{c(1/\varepsilon)}, \alpha(x) \right\rangle = \left\langle \frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}, \big(\check{\phi}_\varepsilon * \alpha(x)\big) \right\rangle$$

$$= \left\langle \frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}, \int_{-\infty}^{\infty} \phi_\varepsilon(x-t)\alpha(x)\,dx \right\rangle$$

$$= \left\langle \frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}, \int_{-\infty}^{\infty} \frac{1}{\varepsilon}\phi_\varepsilon\left(\frac{x-t}{\varepsilon}\right)\alpha(x)\,dx \right\rangle$$

$$= \left\langle \frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}, \int_{-\infty}^{\infty} \phi(-u)\alpha(t-\varepsilon u)\,du \right\rangle,$$

where we used substitution $(x-t)/\varepsilon = -u$. Thus,

$$\left\langle \frac{((\eta_a f) * \phi_\varepsilon)(x+1/\varepsilon)}{c(1/\varepsilon)}, \alpha(x) \right\rangle = \left\langle \frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}, \psi_\varepsilon(x) \right\rangle, \tag{26}$$

where

$$\psi_\varepsilon = \int_{-\infty}^{\infty} \phi(-u)\alpha(\cdot - \varepsilon u)\,du. \tag{27}$$

Note ψ_ε is a net in $\mathcal{S}(\mathbf{R})$ which converges to $\alpha \in \mathcal{S}(\mathbf{R})$ in $\mathcal{S}$. Thus, (26) and (i) imply

$$\lim_{\varepsilon\to 0}\left\langle \frac{(f * \phi_\varepsilon)(x+1/\varepsilon)}{c(1/\varepsilon)}, \alpha(x) \right\rangle = \langle g, \alpha\rangle, \quad \alpha \in \mathcal{S}(\mathbf{R}).$$

(iii) $\Rightarrow$ (i) Let $\alpha \in \mathcal{S}(\mathbf{R})$ and ψ_ε be defined by (27). We have

$$\left\langle \frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}, \alpha(x) \right\rangle$$

$$= \left\langle \frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}, \psi_\varepsilon(x) \right\rangle + \left\langle \frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}, (\alpha - \psi_\varepsilon)(x) \right\rangle$$

$$= \left\langle \frac{((\eta_a f) * \phi_\varepsilon)(x+1/\varepsilon)}{c(1/\varepsilon)}, \alpha(x) \right\rangle + \left\langle \frac{(\eta_a f)(x+1/\varepsilon)}{c(1/\varepsilon)}, (\alpha - \psi_\varepsilon)(x) \right\rangle.$$

Since the set

$$\left\{\psi_\varepsilon;\ \psi_\varepsilon = \int \alpha(\cdot - \varepsilon u)\check{\phi}(u)\,du,\ \alpha \in \mathcal{S}(\mathbf{R}),\ \varepsilon \in (0,1)\right\}$$

is dense in $\mathcal{S}(\mathbf{R})$, the boundedness of $\frac{(\eta_a f)(\cdot+1/\varepsilon)}{c(1/\varepsilon)}$ in $\mathcal{S}'(\mathbf{R}_+)$, implies the assertion.

(c) We shall prove (i) $\Rightarrow$ (ii). Let $\psi \in \mathcal{S}(\mathbf{R})$. We have

$$\left\langle \frac{((\eta_a f) * \phi_\varepsilon)(x + 1/\varepsilon)}{c(1/\varepsilon)}, \psi(x) \right\rangle = \left\langle \frac{(\eta_a f)(x + 1/\varepsilon)}{c(1/\varepsilon)}, (\check{\phi}_\varepsilon * \psi)(x) \right\rangle < \infty$$

since $(\check{\phi}_\varepsilon * \psi)(\cdot) = \int_{-\infty}^{\infty} \phi(u)\psi(\cdot - \varepsilon u)\, du$, $\varepsilon \in (0, 1)$, is bounded in $\mathcal{S}(\mathbf{R})$. $\square$

Proposition 9. (a) *Let $f \in \mathcal{S}'(\mathbf{R})$ and $c(t) = L(\exp(t))$, $t \geqslant t_0$. Then, the following conditions are equivalent*:

(1) $$\lim_{\varepsilon \to 0} \frac{f(\cdot + 1/\varepsilon)}{c(1/\varepsilon)} = M \quad \text{in } \mathcal{D}'(\mathbf{R}),\ M \neq 0;$$

(2) *Generalized function $F = \mathrm{Ctd}\, f = [f_\varepsilon] \in \mathcal{G}_t(\mathbf{R})$ (with representative $f_\varepsilon = f * \phi_\varepsilon$) has Sc-asymptotic in $\mathbf{R}_+$ (w.r.) to $c(1/\varepsilon)$ related to some $c \in \sum(\mathbf{R}_+)$ with the limit M in $\mathcal{D}'(\mathbf{R})$, for every $n \in \mathbf{N}$.*

(b) *If $f \in \mathcal{S}'(\mathbf{R})$ is S-bounded in $\mathbf{R}_+$ related to some $c \in \sum(\mathbf{R}_+)$ than $F = \mathrm{Ctd}\, f = [f_\varepsilon]$ is Sc-bounded in $\mathbf{R}_+$ related to some $c \in \sum(\mathbf{R}_+)$.*

Proof. Let $\psi \in \mathcal{D}(\mathbf{R})$. Then, by the equivalence of strong and weak convergence,

$$\lim_{\varepsilon \to 0} \left\langle \frac{(f * \phi_\varepsilon)(x + 1/\varepsilon)}{c(1/\varepsilon)}, \psi(x) \right\rangle = \lim_{\varepsilon \to 0} \left\langle \frac{f(x + 1/\varepsilon)}{c(1/\varepsilon)}, (\check{\phi}_\varepsilon * \psi)(x) \right\rangle = \langle M, \psi \rangle.$$

This implies (1) $\longleftrightarrow$ (2).

(b) Follows from Proposition 8(c). $\square$

The previous proposition can be formulated in the multidimensional case with some cone.

In the next proposition we shall prove that multiplication of distributions in $\mathcal{G}$ preserves the *Sc*-asymptotic.

Proposition 10. (a) *Let $g, T \in \mathcal{S}'(\mathbf{R}^n)$ and $g_\varepsilon, T_\varepsilon \in \mathcal{E}_M(\mathbf{R}^n)$ be their representatives (as above,* $\mathrm{Ctd}\, g = [g_\varepsilon]$, $\mathrm{Ctd}\, T = [T_\varepsilon]$). *Let $c(1/\varepsilon), c_1(1/\varepsilon) \in \sum(\mathbf{R}_+)$. Assume that for every $\xi \in \mathbf{R}^n_+$ and $\alpha \in \mathbf{N}^n_0$,*

$$\frac{1}{c(1/\varepsilon)} g_\varepsilon^{(\alpha)}(\cdot + \xi/\varepsilon) \to V^{(\alpha)}(\cdot) \tag{28}$$

as $\varepsilon \to 0$, uniformly on compact sets. Moreover, assume that

$$T_\varepsilon(t + \xi/\varepsilon) \overset{\mathrm{Sc}}{\sim} c(1/\varepsilon) U, \quad \xi \in \mathbf{R}^n_+.$$

Then,

$$g_\varepsilon(t + \xi/\varepsilon) T_\varepsilon(t + \xi/\varepsilon) \overset{\mathrm{Sc}}{\sim} c(1/\varepsilon) c_1(1/\varepsilon) U V, \quad \xi \in \mathbf{R}^n_+.$$

(b) *Let* g, T *be as in part* (a) *and let* $[T_\varepsilon]$ *be Sc-bounded in* $\mathbf{R}^n_+$ *related to* $c(1/\varepsilon)$. *Assume instead of* (28) *a weaker condition*: *For every compact set* $K \subset \mathbf{R}^n$, *every* $\alpha \in \mathbf{N}^n_0$ *and every* $\xi \in \mathbf{R}^n_+$,

$$\sup_{\substack{t\in K\\ \varepsilon\in(0,1)}} \frac{1}{c_1(1/\varepsilon)}\left|g_\varepsilon^{(\alpha)}(\cdot+\xi/\varepsilon)\right| < \infty. \tag{29}$$

Then, $[g_\varepsilon T_\varepsilon]$ *is Sc-bounded in* $\mathbf{R}^n_+$ *related to* $c(1/\varepsilon)c_1(1/\varepsilon)$.

Proof. We will prove only assertion (a). By assumption (28), for every $\phi \in \mathcal{D}(\mathbf{R}^n)$,

$$\frac{1}{c_1(1/\varepsilon)} g_\varepsilon(\cdot+\xi/\varepsilon)\phi \to V\phi \in \mathcal{D}(\mathbf{R}^n)$$

as $\varepsilon \to 0$, in the sense of convergence in $\mathcal{D}(\mathbf{R}^n)$ $(\xi \in \mathbf{R}^n_+)$. We have

$$\begin{aligned}
&\frac{1}{c_1(1/\varepsilon)c(1/\varepsilon)}\big\langle g_\varepsilon(t+\xi/\varepsilon)T_\varepsilon(t+\xi/\varepsilon), \phi(t)\big\rangle \\
&\quad= \left\langle \frac{1}{c(1/\varepsilon)}T_\varepsilon(t+\xi/\varepsilon), \frac{1}{c_1(1/\varepsilon)}g_\varepsilon(t+\xi/\varepsilon)\phi(t)\right\rangle \\
&\quad= \left\langle \frac{1}{c(1/\varepsilon)}T_\varepsilon(t+\xi/\varepsilon) - U(t), \frac{1}{c_1(1/\varepsilon)}g_\varepsilon(t+\xi/\varepsilon)\phi(t)\right\rangle \\
&\qquad+ \left\langle U(t), \frac{1}{c_1(1/\varepsilon)}g_\varepsilon(t+\xi/\varepsilon)\phi(t)\right\rangle.
\end{aligned}$$

Since the set $\{1/c_1(1/\varepsilon)g_\varepsilon(\cdot+\xi/\varepsilon)\phi;\ \varepsilon \in (0,1)\}$ is bounded in $\mathcal{D}(\mathbf{R}^n)$, the first summand converges to zero. Clearly, the second one converges to $\langle U, V\phi\rangle$ as $\varepsilon \to 0$. This proves the assertion. □

Proposition 11. (i) *Let* $T_1, T_2 \in \mathcal{S}'(\mathbf{R}^n)$ *such that for some* $a \in \mathbf{R}^n_+$, $(\eta_a T_1) = (\eta_a T_2)$. *Let* $T_{1\varepsilon} = T_1 * \delta_\varepsilon$, $T_{2\varepsilon} = T_2 * \delta_\varepsilon$, $\delta_\varepsilon = \frac{1}{\varepsilon^n}\phi\big(\frac{\cdot}{\varepsilon}\big)$, $\phi \in \mathcal{D}(\mathbf{R}^n)$, $\int \phi(x)\,dx = 1$. *If* $T_{1\varepsilon}(t+\xi/\varepsilon) \overset{\mathrm{Sc}}{\sim} c(1/\varepsilon)U(t)$, $\xi \in \mathbf{R}^n_+$, *then* $T_{2\varepsilon}(x+\xi/\varepsilon) \overset{\mathrm{Sc}}{\sim} c(1/\varepsilon)U(t)$, $\xi \in \mathbf{R}^n_+$.

(ii) *Let* $T_1, T_2 \in \mathcal{G}_t(\mathbf{R}^n_+)$ *and let* $T_1 = T_2$, *in* Gtd *sense. If* $T_1(x+\xi/\varepsilon) \overset{\mathrm{Sc}}{\sim} c(1/\varepsilon)U(t)$, *then* $T_2(x+\xi/\varepsilon) \overset{\mathrm{Sc}}{\sim} c(1/\varepsilon)U(t)$.

(iii) *If* $T \in \mathcal{G}$ *and* $T(t+\xi/\varepsilon) \overset{\mathrm{Sc}}{\sim} c(1/\varepsilon)U(t)$, $\varepsilon \to 0$, *then for every* $k \in \mathbf{N}^n_0$, $T^{(k)}(t+\xi/\varepsilon) \overset{\mathrm{Sc}}{\sim} c(1/\varepsilon)U^{(k)}(t)$, $\varepsilon \to 0$.

Proof. We will prove only (ii) and (iii).

(ii) Let $T_{1\varepsilon}, T_{2\varepsilon}$ are the corresponding representatives. Then, since $T_{1\varepsilon} - T_{2\varepsilon} \in \mathcal{N}_t(\mathbf{R}^n)$ then for $\xi \in \mathbf{R}^n_+$,

$$\lim_{\varepsilon\to 0}\left\langle \frac{T_{1\varepsilon}(t+\xi/\varepsilon) - T_{2\varepsilon}(t+\xi/\varepsilon)}{c(1/\varepsilon)}, \phi(t)\right\rangle = \lim_{\varepsilon\to 0}\left\langle \frac{d_\varepsilon(t+\xi/\varepsilon)}{c(1/\varepsilon)}, \phi(t)\right\rangle = 0,$$

since $d_\varepsilon(t+\xi/\varepsilon) \in \mathcal{N}_t(\mathbf{R}^n)$ (cf. Proposition 7).

(iii) With $\xi \in \mathbf{R}_+^n$ and $\phi \in \mathcal{D}(\mathbf{R}^n)$, we have

$$\lim_{\varepsilon\to 0}\left\langle \frac{T_\varepsilon^{(k)}(t+\xi/\varepsilon)}{c(1/\varepsilon)}, \phi(t)\right\rangle = (-1)^k \lim_{\varepsilon\to 0}\left\langle \frac{(T_\varepsilon)(t+\xi/\varepsilon)}{c(1/\varepsilon)}, \phi^{(k)}(t)\right\rangle$$
$$= (-1)^k\langle U(t), \phi^{(k)}(t)\rangle = \langle U^{(k)}(t), \phi(t)\rangle. \qquad \square$$

Similarly, we have

Proposition 12. (1) *Let $T_1, T_2 \in \mathcal{S}'(\mathbf{R}^n)$, such that for some $a \in \mathbf{R}_+^n$, $(\eta_a T_1) = (\eta_a T_2)$. Let $T_{1\varepsilon} = T_1 * \phi_\varepsilon$ and $T_{2\varepsilon} = T_2 * \phi_\varepsilon$, where $\phi \in \mathcal{D}(\mathbf{R}^n)$, $\int \phi(x)\,dx = 1$. If $[T_{1\varepsilon}]$ is Sc-bounded in $\mathbf{R}_+^n$ related to $c(1/\varepsilon)$, then $[T_{2\varepsilon}]$ is Sc-bounded in $\mathbf{R}_+^n$ related to $c(1/\varepsilon)$. The converse also holds.*

(2) *Let $T_1, T_2 \in \mathcal{G}_t(\mathbf{R}^n)$ such that T_1 and T_2 are equal in* Gtd *sense, then the Sc-boundedness of T_1 implies the Sc-boundedness of T_2.*

(3) *If $T \in \mathcal{G}_t(\mathbf{R}^n)$ is Sc-bounded, then $T^{(k)}$, $k \in \mathbf{N}_0^n$, is also Sc-bounded.*

Proof. (1)–(3) follows from the corresponding definitions and Proposition 7(a). $\square$

5.2. *Application to nonlinear problems*

5.2.1. *Semilinear parabolic equation*

Consider the Cauchy problem for the semilinear parabolic equation (cf. [3])

$$\partial_t u - \Delta u + g(u) = 0, \quad t > 0,\ x \in \mathbf{R}^n, \tag{30}$$

where $g(u)$ is locally Lipschitz real valued function and the initial data are

$$u(0,\cdot) = \mu \in \mathcal{M}^k(\mathbf{R}^n) \subset \mathcal{D}'(\mathbf{R}^n), \quad k \in \mathbf{N}_0,$$

where $\mathcal{M}^k(\mathbf{R}^n)$ is the strong dual of the Banach space $C_b^k(\mathbf{R}^n)$ of all $C^k(\mathbf{R}^n)$ functions with bounded derivatives till the order k. ($\mathcal{M}^1(\mathbf{R}^n) = \mathcal{M}(\mathbf{R}^n)$ is the space of Radon measures.) As examples, we can take

$$\mu = \sum_{j=1}^{\infty}\sum_{|\alpha|\leqslant k} b_{j\alpha}\partial_x^\alpha \delta(\cdot - \xi^j), \quad k \in \mathbf{N}_0,\ b_{j\alpha} \in \mathbf{R},\ \xi^j \in \mathbf{R}^n,\ j \geqslant 1,$$
$$\sum_{j=1}^{\infty} |b_{j\alpha}| < \infty,\ |\alpha| \leqslant k > 0, \tag{31}$$

or

$$\mu = D^k\psi, \quad \psi \in L^p(\mathbf{R}^n),\ \text{for some } k \geqslant 0,\ 1 \leqslant p \leqslant \infty. \tag{32}$$

We will use $\mu_\varepsilon(\cdot) = \mu * \frac{1}{\varepsilon^n}\phi\left(\frac{\cdot}{\varepsilon}\right)$, where $\phi \in \mathcal{D}(\mathbf{R}^n)$, $\phi(x) \geqslant 0$, $\int_{\mathbf{R}^n}\phi(x)\,dx = 1$. Linear Cauchy problem in $\mathcal{G}(\mathbf{R}_+ \times \mathbf{R}^n)$, $\partial_t u - \Delta u = 0$, $(t,x) \in \mathbf{R}_+ \times \mathbf{R}^n$, $[u_\varepsilon(0,\cdot)] = [\mu_\varepsilon(\cdot)]$, considered in the framework of representatives has the unique solution $U^0_{\mu_\varepsilon}(t,x) = e^{t\Delta}\mu_\varepsilon$ (heat semigroup), where

$$E_n(t,x) = \begin{cases} \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4t}\right), & t > 0,\ x \in \mathbf{R}^n, \\ 0, & t < 0, \end{cases}$$

and $U^0_{\mu_\varepsilon}(t,\cdot) = (E_n(t,\cdot) * \mu_\varepsilon)(\cdot) = \langle E_n(t,\cdot - y), \mu_\varepsilon(y)\rangle$, is the fundamental solution of the heat operator (cf. [3]). Note, for every $t > 0$, $E_n(t,\cdot) \in \mathcal{S}(\mathbf{R}^n)$. So we will consider S-asymptotic and Sc-asymptotic at infinity in $\mathcal{S}'(\mathbf{R}^n)$ and $\mathcal{G}_t(\mathbf{R}^n)$, respectively.

Cauchy problem (30) can be rewritten in integral form as $u_\varepsilon(t,\cdot) = K(u_\varepsilon)(t,\cdot) = e^{t\Delta}\mu_\varepsilon + K_0(u_\varepsilon)(t,\cdot)$ where $K_0(u_\varepsilon)(t,\cdot) = \int_0^t e^{(t-\tau)\Delta} g(u_\varepsilon(\tau,\cdot))\,d\tau$. Thus,

$$u_\varepsilon(t,\cdot) = \int_{\mathbf{R}^n} E_n(t,x-y)\mu_\varepsilon(y)\,dy + \int_0^t E_n(t-\tau,\cdot) * g\big(u_\varepsilon(\tau,\cdot)\big)\,d\tau. \tag{33}$$

Proposition 13. *Assume that $[\mu_\varepsilon]$ has Sc-asymptotic at infinity in $\mathcal{G}_t(\mathbf{R}^n)$ (w.r.) to $c(1/\varepsilon)$ in $\mathbf{R}^n_+$. The solution $[u_\varepsilon(t,x)]$ to the problem* (30) *has the same Sc-asymptotic as μ_ε if*

$$\frac{K_0(u_\varepsilon)(t,\cdot+\xi/\varepsilon)}{c(1/\varepsilon)} \to 0 \quad \text{as } \xi \in \mathbf{R}^n_+,\ \varepsilon \to 0. \tag{34}$$

In particular, this is true if g is bounded and $c(1/\varepsilon) \to \infty$.

Proof. Let $\xi \in \mathbf{R}^n_+$. By (33) we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{u_\varepsilon(t,x+\xi/\varepsilon)}{c(1/\varepsilon)} &= \lim_{\varepsilon\to 0} \frac{1}{c(1/\varepsilon)} \big\langle E_n(t,x+\xi/\varepsilon - y), \mu_\varepsilon(y)\big\rangle \\
&\quad + \lim_{\varepsilon\to 0} \frac{K_0(u_\varepsilon)(t,x+\xi/\varepsilon)}{c(1/\varepsilon)} \\
&= \lim_{\varepsilon\to 0} \frac{1}{c(1/\varepsilon)} \big\langle E_n(t,x-y), \mu_\varepsilon(y+\xi/\varepsilon)\big\rangle
\end{aligned}$$

due to (34). $\square$

We will examine the *Sc*-asymptotic of generalized functions which serve as initial data.

Delta waves (31) have the *Sc*-asymptotic with the limit zero with respect to any $c(1/\varepsilon) \in \mathcal{K}$. Clearly, since $\delta_\varepsilon = \big[\frac{1}{\varepsilon^n}\phi\big(\frac{\cdot}{\varepsilon}\big)\big]$ we have, for $\xi \in \mathbf{R}^n_+$, $\psi \in \mathcal{S}(\mathbf{R}^n)$,

$$\begin{aligned}
\left\langle \frac{\delta_\varepsilon(x+\xi/\varepsilon)}{c(1/\varepsilon)}, \psi(x) \right\rangle &= \left\langle \frac{[1/\varepsilon^n \phi_\varepsilon((x+\xi/\varepsilon)/\varepsilon)]}{c(1/\varepsilon)}, \psi(x) \right\rangle \\
&= \left\langle \phi_\varepsilon(u), \frac{\psi(u-\xi/\varepsilon)}{c(1/\varepsilon)} \right\rangle \to 0
\end{aligned}$$

(we used substitution $(x+\xi/\varepsilon)/\varepsilon = u$). The same holds for derivatives of delta distribution.

In the case (32) in $\mathbf{R}^n_+$, we have

$$\begin{aligned}
\left\langle \frac{\mu(x+\xi/\varepsilon)}{c(1/\varepsilon)}, \psi \right\rangle &= \left\langle \frac{D^k\psi(x+\xi/\varepsilon)}{c(1/\varepsilon)}, \psi(x) \right\rangle = (-1)^k \left\langle \frac{\psi(x+\xi/\varepsilon)}{c(1/\varepsilon)}, D^k\phi(x) \right\rangle \\
&= \big\langle g^{(k)}(x), \phi(x) \big\rangle,
\end{aligned}$$

where we suppose that

$$\lim_{\varepsilon\to 0}\frac{\psi(\cdot+\xi/\varepsilon)}{c(1/\varepsilon)}=g(\cdot),\quad \xi\in\mathbf{R}^n_+$$

(i.e., $\psi(\cdot+\xi/\varepsilon)\overset{\mathrm{Sc}}{\sim}c(1/\varepsilon)g(\cdot)$).

Now we shall discuss *Sc*-boundedness of the solution to (33) if we have *Sc*-boundedness of the initial data. Suppose that $g(u)$ has

(1) sublinear growth, $|g(u)|\leqslant C|u|^s$, $s<1$;
(2) superlinear growth, $|g(u)|\leqslant C|u|^s$, $s>1$,

and (a) $c(1/\varepsilon)\to\infty$; (b) $c(1/\varepsilon)\to 0$.

(1) *Sublinear growth, in case* (a) *for* $c(1/\varepsilon)$. From

$$\left|\frac{K_0(u_\varepsilon)(t,x+\xi/\varepsilon)}{c(1/\varepsilon)}\right|\leqslant\int_0^t\int_{-\infty}^{\infty}E_n(t-\tau,x-m)\left|\frac{g(u_\varepsilon)(\tau,\xi/\varepsilon+m)}{c(1/\varepsilon)}\right|d\tau\,dm$$

we have

$$\left|\frac{K_0(u_\varepsilon)(t,x+\xi/\varepsilon)}{c(1/\varepsilon)}\right|\leqslant C\int_0^t\int_{-\infty}^{\infty}E_n(t-\tau,x-m)\left|\frac{u_\varepsilon^s(\tau,\xi/\varepsilon+m)}{c(1/\varepsilon)}\right|d\tau\,dm$$

and by (33),

$$\left|\frac{u_\varepsilon(t,x+\xi/\varepsilon)}{c(1/\varepsilon)}\right|\leqslant\left|\frac{U_{\mu\varepsilon}(t,x+\xi/\varepsilon)}{c(1/\varepsilon)}\right| +\left(\frac{1}{c(1/\varepsilon)}\right)^{1-s}\int_0^t\int_{-\infty}^{\infty}\left|\frac{u_\varepsilon(\tau,m+\xi/\varepsilon)}{c(1/\varepsilon)}\right|^s d\tau\,dm,$$

since by [12], $|E_n(t,\cdot)|\leqslant 1$.

It means that it is impossible that $\frac{u_\varepsilon(t,\cdot+\xi/\varepsilon)}{c(1/\varepsilon)}$ "oscillates in infinity" if μ_ε is *Sc*-bounded. Thus, $\frac{u_\varepsilon(t,\cdot+\xi/\varepsilon)}{c(1/\varepsilon)}$ is *Sc*-bounded if μ is *Sc*-bounded. The *Sc*-boundedness of the solution is the consequence of the *Sc*-boundedness of the initial data.

(2) *Superlinear growth in case* (b) *for* $c(1/\varepsilon)$. Since $|g(u)|\leqslant C|u|^s$, $s>1$, we have

$$\left|\frac{K_0(u)(t,x+1/\varepsilon)}{c(1/\varepsilon)}\right|\leqslant c^{s-1}(1/\varepsilon)\int_0^t\int_{-\infty}^{\infty}E_n(t-\tau,x-m)\left|\frac{u(\tau,\xi/\varepsilon+m)}{c(1/\varepsilon)}\right|^s d\tau\,dm.$$

Then, we have the same conclusion as in previous case. The analysis of quoted cases is our next goal.

5.2.2. *Semilinear parabolic equation with nonlinear conservative term*

We transfer the properties of *Sc*-asymptotic, respectively *Sc*-boundedness to the solution of semilinear parabolic equation with nonlinear conservative term and initial data from

$L^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$. Consider Cauchy problem for the semilinear parabolic equation with nonlinear conservative term (cf. [2])

$$\partial_t u - \Delta u + \partial_x \vec{g}(u) = 0, \quad t > 0,\ x \in \mathbf{R}^n,\ \mu = D^k \psi, \tag{35}$$

where $\vec{g}(u) = (g_1(u), \ldots, g_n(u)) \in C^1(\mathbf{R} : \mathbf{R}^n)$, $\partial_x \cdot \vec{g}(u) = \vec{g}\,'(u) \cdot \nabla u = \sum_{j=1}^n g_j'(u)\partial_{x_j} u$, $k \geqslant 0$, $D = (-\Delta)^{1/2}$, $\psi \in L^p(\mathbf{R}^n)$ for some $1 \leqslant p \leqslant \infty$.

Let $E_n(t,x)$, $t > 0$, $x \in \mathbf{R}^n$, be the fundamental solution of the heat operator as in previous example. Then,

$$U_\mu^0(t,x) = \big(E_n(t,\cdot) * \mu\big)(x) \in C\big([0,\infty) : \mathcal{S}'(\mathbf{R}^n)\big)$$

is the solution to the linear Cauchy problem (cf. [2])

$$\partial_t u - \Delta u = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^n, \quad u(0,\cdot) = \mu \in \mathcal{S}'(\mathbf{R}^n).$$

Cauchy problem (35) can be rewritten as an integral equation

$$\begin{aligned} u_\varepsilon(t,\cdot) = K(u_\varepsilon)(t,\cdot) &= U_{\mu_\varepsilon}^0(t,\cdot) + K_0(u_\varepsilon)(t,\cdot) \\ &= E_n(t,\cdot) * D^k\psi + K_0(u_\varepsilon)(t,\cdot), \end{aligned} \tag{36}$$

where

$$K_0(u_\varepsilon)(t,\cdot) = \int_0^t \nabla E_n(t-\tau,\cdot) * \vec{g}\big(u_\varepsilon(\tau,\cdot)\big)\,d\tau. \tag{37}$$

We shall prove that *Sc*-asymptotic of the solution when *Sc*-asymptotic of the part (37) (w.r.) to $c(1/\varepsilon)$ tends to zero.

If

$$\frac{K_0(u_\varepsilon)(t,\cdot+\xi/\varepsilon)}{c(1/\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0,\ \xi \in \mathbf{R}_+^n,$$

we obtain

$$\begin{aligned} \frac{u_\varepsilon(t,\cdot+\xi/\varepsilon)}{c(1/\varepsilon)} &= \frac{(E_n(t,\cdot+1/\varepsilon) * D^k\psi)(x)}{c(1/\varepsilon)} = \left\langle \frac{E_n(t,\cdot+\xi/\varepsilon-y)}{c(1/\varepsilon)}, D^k\psi(y) \right\rangle \\ &= \left\langle E_n(t,\cdot-m)\frac{D^k\psi(m+\xi/\varepsilon)}{c(1/\varepsilon)} \right\rangle. \end{aligned}$$

If the initial data has *Sc*-asymptotic, i.e., if for $\xi \in \mathbf{R}_+^n$,

$$\lim_{\varepsilon \to 0} \left\langle \frac{D^k\psi(m+\xi/\varepsilon)}{c(1/\varepsilon)}, \phi(m) \right\rangle = \big\langle D^k\psi_0(m), \phi(m) \big\rangle,$$

then

$$\lim_{\varepsilon \to 0} \frac{u(t,\cdot+\xi/\varepsilon)}{c(1/\varepsilon)} = \big(E_n(t,\cdot) * D^k\psi_0\big)(\cdot), \quad \xi \in \mathbf{R}_+^n.$$

We have just proved the following proposition.

Proposition 14. *Sc-asymptotic at infinity of the solution $u_\varepsilon(t,x)$ to the problem* (36) *is the consequence of the Sc-asymptotic of the initial data, if it exists, in a case when*

$$\frac{K_0(u_\varepsilon)(t,\cdot+\xi/\varepsilon)}{c(1/\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0,\ \xi \in \mathbf{R}^n_+.$$

In particular, it holds if g is bounded and $c(1/\varepsilon) \to \infty$.

Remark 2. Similar discussion for *Sc*-boundedness as after Proposition 13 holds.

6. *Sc*-asymptotic expansion at infinity

Finally, as an example we give the *Sc*-asymptotic expansion of δ^2 at infinity. The similar characterization and application as in a case of $\mathcal{G}$-q.a. expansion hold and they are omitted. The first we give the definition of *Sc*-asymptotic expansion at infinity.

Definition 5. We denote by Λ the set $\mathbf{N}$ or a finite set of the form $\{1,2,\dots,N\}$, $N \in \mathbf{N}$.

Let $c_k \in \mathcal{K}$, $k \in \Lambda$, such that

$$\lim_{\varepsilon\to 0} \frac{c_{k+1}(1/\varepsilon)}{c_k(1/\varepsilon)} \to 0, \quad k = 1,\dots,N-1 \text{ (or } k \in \mathbf{N},\ \text{if } \Lambda = \mathbf{N}),$$

and $P_k = [P_{k\varepsilon}] \in \mathcal{G}_t(\mathbf{R})$, $k \in \Lambda$. Then $G = [G_\varepsilon] \in \mathcal{G}_t(\mathbf{R})$ has the *Sc*-asymptotic expansion (the strong *Sc*-asymptotic expansion) at infinity as $\sum_{k\in\Lambda} P_{k\varepsilon}$ (w.r.) to $\{c_k(1/\varepsilon);\ k \in \Lambda\}$ if

$$\frac{(G_\varepsilon - \sum_{k=1}^m P_{k\varepsilon})(x+1/\varepsilon)}{c_m(1/\varepsilon)} \to 0, \quad \varepsilon \to 0^+,\ \text{in } \mathcal{D}'(\mathbf{R}) \text{ for every } m \in \Lambda$$

$$\left(\frac{(G_\varepsilon - \sum_{k=1}^m P_{k\varepsilon})(x+1/\varepsilon)}{c_m(1/\varepsilon)} \to 0, \quad \varepsilon \to 0^+,\ \text{for every } x \in \mathbf{R},\ x \neq 0 \text{ and every } m \in \Lambda\right).$$

In this case we write

$$G \overset{\text{Sc.e.}}{\sim} \sum_{k\in\Lambda} P_k \quad \text{(w.r.) to } \{c_m(1/\varepsilon),\ m \in \Lambda\}$$

$$\left(G \overset{\text{sSc.e.}}{\sim} \sum_{k\in\Lambda} P_k \quad \text{(w.r.) to } \{c_m(1/\varepsilon),\ m \in \Lambda\}\right)$$

and say that G has the *Sc*-asymptotic expansion at infinity in Colombeau sense.

One can simply prove that this definition does not depend on representatives. We have the following proposition.

Proposition 15. *Let $f \in \mathcal{S}'(\mathbf{R})$. Then $f \overset{\text{S.e.}}{\sim} \sum_{k\in\Lambda} A_k P_k$ (w.r.) to $\{c_m(1/\varepsilon),\ m \in \Lambda\}$ if and only if*

$$\operatorname{Ctd} f \overset{\text{Sc.e.}}{\sim} \sum_{k\in\Lambda} A_k[P_{k\varepsilon}] \quad \text{(w.r.) to } \{c_m(1/\varepsilon),\ m \in \Lambda\}.$$

Proof. Setting up $\alpha \in \mathcal{D}(\mathbf{R})$, $\varepsilon > 0$, we obtain

$$\begin{aligned}
&\left\langle \frac{(f * \phi_\varepsilon)(x + 1/\varepsilon) - \sum_{k=1}^m A_k (P_k * \phi_\varepsilon)(x + 1/\varepsilon)}{c_m(1/\varepsilon)}, \alpha(x) \right\rangle \\
&\quad = \left\langle \frac{(f(t) - \sum_{k=1}^m A_k P_k(t))}{c_m(1/\varepsilon)}, (\check{\phi}_\varepsilon * \alpha)(t - 1/\varepsilon) \right\rangle \\
&\quad = \left\langle \frac{f(x + 1/\varepsilon) - \sum_{k=1}^m A_k P_k(x + 1/\varepsilon)}{c_m(1/\varepsilon)}, \psi_\varepsilon(x) \right\rangle,
\end{aligned}$$

where

$$\psi_\varepsilon(x) = \int_{-\infty}^{\infty} \check{\phi}_\varepsilon(t)\alpha(x - t)\,dt = \int_{-\infty}^{\infty} \check{\phi}(t)\alpha(x - t\varepsilon)\,dt, \quad x \in \mathbf{R},\ \varepsilon \in (0, 1).$$

This implies, due to the convergence of $\psi_\varepsilon \xrightarrow{\mathcal{S}} \alpha$, $\varepsilon \to 0$,

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left\langle \frac{((f * \phi_\varepsilon) - \sum_{k=0}^m A_k P_k * \phi_\varepsilon)(x + \xi/\varepsilon)}{c_m(1/\varepsilon)}, \alpha(x) \right\rangle \\
&\quad = \left\langle g(x) - \sum_{k=0}^m A_k P_k(x), \alpha(x) \right\rangle
\end{aligned}$$

and the assertion follows. $\square$

In the next propositions we give the *Sc*-asymptotic expansion of an element in $\mathcal{G}_t(\mathbf{R}) \setminus \mathcal{S}'(\mathbf{R})$.

Proposition 16. *Let* $\delta^2 = \left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{\cdot}{\varepsilon}\right)\right]$, *where* $\phi \in C_0^\infty$, $\int \phi = 1$, $\int x^n \phi(x)\,dx = 0$, $n \leqslant N$, $N \in \mathbf{N}$. *Then*

$$\delta^2(\cdot + h) = \left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{\cdot + h}{\varepsilon}\right)\right] \overset{\text{Sc.e.}}{\sim} \sum_{j=0}^{n} \varepsilon^j \frac{\mu_j}{j!} \sum_{i=0}^{n-j} (-1)^i \frac{h^i}{i!} \left[\frac{1}{\varepsilon}\phi\left(\frac{\cdot}{\varepsilon}\right)\right]^{(i+j)} \tag{38}$$

(*w.r.*) *to the scale* $\{\varepsilon^{-n},\ n = 0, 1, \ldots, N \in \mathbf{N}\}$, *where*

$$\mu_j = \int u^j \phi^2(u)\,du, \quad j = 1, \ldots, N.$$

Setting $h = 1/\varepsilon$ *we obtain*

$$\delta_\varepsilon^2(\cdot + \xi/\varepsilon) \overset{\text{S.e.}}{\sim} \sum_{j=0}^{n} \frac{\mu_j}{j!} \sum_{i=0}^{n-j} (-1)^i \varepsilon^{j-i} \frac{1}{i!} \left[\frac{1}{\varepsilon}\phi\left(\frac{\cdot}{\varepsilon}\right)\right]^{(i+j)}$$

instead of (38).

Proof. Let

$$I = \frac{1}{\varepsilon^p} \int \frac{1}{\varepsilon^2}\phi^2\left(\frac{x + h}{\varepsilon}\right)\psi(x)\,dx.$$

After substitution $(x+h)/\varepsilon = u$ we obtain $I = \frac{1}{\varepsilon^{p+1}} \int \phi^2(u)\psi(\varepsilon u - h)\,du$. Then, we expand ψ at the point εu and then at zero. We have

$$I = \frac{1}{\varepsilon^{p+1}} \int \phi^2(u)\big(\psi(\varepsilon u) - h\psi'(\varepsilon u) + h^2/2!\psi''(\varepsilon u) - h^3/3!\psi'''(\varepsilon u) + h^4/4!\psi^{(4)}(\varepsilon u) + \cdots\big)\,du,$$

and

$$\begin{aligned}
I = \frac{1}{\varepsilon^{p+1}} \int \phi^2(u)\big\{&\psi(0) + \varepsilon u\psi'(0) + (\varepsilon u)^2/2!\psi''(0) + (\varepsilon u)^3/3!\psi'''(0) + \cdots \\
&- h\big(\psi'(0) + \varepsilon u\psi''(0) + (\varepsilon u)^2/2!\psi'''(0) + (\varepsilon u)^3/3!\psi^{(4)}(0) + \cdots\big) \\
&+ h^2/2!\big(\psi''(0) + \varepsilon u\psi'''(0) + (\varepsilon u)^2/2!\psi^{(4)}(0) + \cdots\big) \\
&- h^3/3!\big(\psi'''(0) + \varepsilon u\psi^{(4)}(0) + \cdots\big) \\
&+ h^4/4!\big(\psi^{(4)}(0) + \varepsilon u\psi^{(5)}(0) + \cdots\big) - \cdots\big\}, \\
I = \varepsilon^{-p-1}\big\{&\mu_0\big(\psi(0) - h\psi'(0) + h^2/2!\psi''(0) - h^3/3!\psi'''(0) + \cdots\big) \\
&+ \varepsilon\mu_1\big(\psi'(0) - h\psi''(0) + h^2/2!\psi'''(0) + \cdots\big) \\
&+ \varepsilon^2/2!\mu_2\big(\psi''(0) - h\psi'''(0) + h^2/2!\psi^{(4)}(0) + \cdots\big) \\
&+ \varepsilon^3/3!\mu_3\big(\psi'''(0) - h\psi^{(4)}(0) + h^2/2!\psi^{(5)}(0) - h^3/3!\psi^{(6)}(0) + \cdots\big) + \cdots\big\}.
\end{aligned}$$

By ordering the terms we obtain

$$\begin{aligned}
I &= \varepsilon^{-p-1}\Bigg(\mu_0 \sum_{i=0}^{n}(-1)^i \frac{h^i}{i!}\psi^{(i)}(0) + \varepsilon/1!\mu_1 \sum_{i=0}^{n-1}(-1)^i \frac{h^i}{i!}\psi^{(i+1)}(0) \\
&\quad + \varepsilon^2/2!\mu_2 \sum_{i=0}^{n-2}(-1)^i \frac{h^i}{i!}\psi^{(i+2)}(0) + \varepsilon^3/3!\mu_3 \sum_{i=0}^{n-3}(-1)^i \frac{h^i}{i!}\psi^{(i+3)}(0) + \cdots \\
&\quad + \varepsilon^n/n!\mu_n \sum_{i=0}^{n-n}(-1)^i \frac{h^i}{i!}\psi^{(i+n)}(0)\Bigg) \\
&= \varepsilon^{-p-1} \sum_{j=0}^{n} \varepsilon^j \frac{\mu_j}{j!} \sum_{i=0}^{n-j}(-1)^i \frac{h^i}{i!}\psi^{(i+j)}(0).
\end{aligned}$$

Thus, we obtain (38). Let $h = 1/\varepsilon$. Then

$$\delta_\varepsilon^2(x \cdot + 1/\varepsilon) \overset{\text{Sc.e.}}{\sim} \sum_{j=0}^{n} \frac{\mu_j}{j!} \sum_{i=0}^{n-j}(-1)^i \varepsilon^{j-i} \frac{1}{i!}\left[\frac{1}{\varepsilon}\phi\left(\frac{\cdot}{\varepsilon}\right)\right]^{(i+j)}$$

(w.r.) to the scale $\varepsilon^0, \varepsilon^{-1}, \ldots, \varepsilon^{-n}$, $n \leqslant N \in \mathbf{N}$. □

This expansion also has applications in expansion of the solution to some types of PDEs. Techniques are the similar as the techniques described in previous sections and will be omitted.

Remark 3. Note that similar asymptotic expansion established in Proposition 16, is also given in [8, Theorem 22, p. 96 and formula (3.3.13), p. 97]. Recall them.

Let $f \in \mathcal{E}'(\mathbf{R})$ and let $\{\mu_n\}$ be its moment space. Then,

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n!\lambda^{n+1}} \quad \text{as } \lambda \to \infty,$$

in the sense that for any $\phi \in \mathcal{E}(\mathbf{R})$ we have

$$\langle f(\lambda x), \phi(x) \rangle = \sum_{n=0}^{N} \frac{\mu_n \phi^{(n)}(0)}{n!\lambda^{n+1}} + \mathcal{O}\left(\frac{1}{\lambda^{N+2}}\right) \quad \text{as } \lambda \to \infty.$$

As an example of the moment asymptotic expansion in the space $\mathcal{E}'(\mathbf{R})$ is given

$$\delta^{(k)}(\lambda x - x_0) \sim \sum_{n=0}^{\infty} \frac{(-1)^n x_0^n \delta^{(n+k)}(x)}{n!\lambda^{n+k+1}} \quad \text{as } \lambda \to \infty.$$

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