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Incompressible surfaces in handlebodies☆

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Abstract

Let F be a compact surface and let I be the unit interval. This paper gives a standard form for all 2-sided incompressible surfaces in the 3-manifold $F \times I$. It also supplies a simple sufficient condition for when 2-sided surfaces in this form are incompressible. Since $F \times I$ is a handlebody when F has boundary, the paper applies to incompressible surfaces in handlebodies. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction and notation

Let M be a 3-dimensional manifold and let $X \subset M$ be a properly embedded surface. A *compression disk* for $X \subset M$ is an embedded disk $D \subset M$ such that $\partial D \subset X$, $\mathrm{int}(D) \subset (M \backslash X)$, and ∂D is an essential loop in X. The surface $X \subset M$ is *incompressible* if there are no compression disks for $X \subset M$ and no component of X is a sphere that bounds a ball. If $X \subset M$ is connected and 2-sided, then X is incompressible if and only if the induced map $\pi_1(X) \to \pi_1(M)$ is injective and X is not a sphere that bounds a ball. See, for example, [2, Chapter 6].

Let F be a compact surface and let I be the unit interval [0, 1]. The manifold $F \times I$ is foliated by copies of I, which can be thought of as vertical flow lines.

This paper shows that every properly embedded, 2-sided, incompressible surface in $F \times I$ can be isotoped to a standard form, called "near-horizontal position". A surface in near-horizontal position is transverse to the flow on $F \times I$ in the I direction, except at isolated intervals, where it coincides with flow lines. Near each of these intervals, the projection of the surface to F looks like a bow-tie, where the center point of the bow-tie is the projection of the interval. A surface in

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near-horizontal position can be described combinatorially by listing its boundary curves and the number of times it crosses each line in a certain finite collection of flow lines.

Not every 2-sided surface in near-horizontal position is incompressible. This paper gives a simple sufficient condition that insures that a 2-sided surface in near-horizontal position is incompressible. To deal with surfaces that do not meet this condition, the paper outlines an easy algorithm for deciding when a 2-sided surface in near-horizontal position is incompressible. The sufficient condition and the algorithm are based on an analysis of graphs immersed in surfaces.

When F is a compact surface with boundary, then $F \times I$ is a handlebody. So the paper applies, in particular, to incompressible surfaces in handlebodies.

Throughout this paper, all maps are continuous unless otherwise stated. If E is a topological space, then $|E|$ denotes the number of components of E. The symbol I denotes the unit interval $[0, 1]$. If M is a manifold, then ∂M refers to the boundary of M and $\mathrm{int}(M)$ refers to its interior. The symbol F refers to a compact surface, and p denotes the projection map $F \times I \to F$.

The surface $X \subset M$ is a *proper embedding* if $\mathrm{int}(X) \subset \mathrm{int}(M)$, $\partial X \subset \partial M$, and the intersection of X with a compact subset of M is a compact subset of X. The map $G: X \times I \to M$ is a *proper isotopy* between $G|_{X \times 0}$ and $G|_{X \times 1}$ if for all $t \in I$, $G|_{X \times t}$ is a proper embedding. In this paper, all surfaces in 3-manifolds are intended to be properly embedded and all isotopies are intended to be proper isotopies.

If $i: S^1 \to F$ is a map of a loop into a surface, the image $i(S^1)$ is *essential* in F if the induced map $i_*: \pi_1(S^1) \to \pi_1(F)$ is injective.

A connected surface $X \subset M$ is *boundary parallel* if X separates M and there is a component K of $M \backslash X$ such that $(\mathrm{closure}(K), X)$ is homeomorphic to $(X \times I, X \times 0)$.

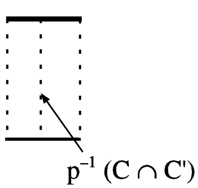
Call a map $g: M_1 \to M_2$ between topological spaces an *immersion* if g is locally injective; that is, for each point $x \in M_1$, there is a neighborhood U of x such that g takes distinct points of U to distinct points of M_2.

2. Definition of near-horizontal position

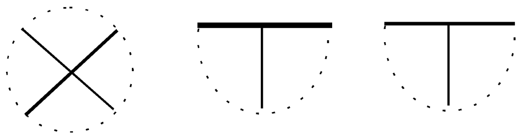
Let X be a surface in $F \times I$. Let $C \subset F$ be the union of loops and arcs $p(X \cap (F \times 1))$, let $C' = p(X \cap (F \times 0))$, and let $B = X \cap (\partial F \times I)$.

X is in *near-horizontal position* if

1. C and C' intersect transversely,
2. each component of B is an arc with one endpoint on $\partial F \times 1$ and one endpoint on $\partial F \times 0$,
3. $p|_B: B \to \partial F$ is injective,
4. $p|_{X \backslash p^{-1}(C \cap C')}$ is a local homeomorphism, and
5. for each point $z \in C \cap C'$, there is a neighborhood $U \subset F$ of z such that $p^{-1}(U) \cap X$ either looks (in X) like the region



or else looks (in X) like a union of regions of the form



Here, thick lines are used to draw $p^{-1}(C)\cap X$ and thin lines are used to draw $p^{-1}(C')\cap X$. Dotted lines are used for boundary arcs of $p^{-1}(U)\cap X$ that are not part of $p^{-1}(C)\cap X$ or $p^{-1}(C')\cap X$. Call the dashed vertical line of $p^{-1}(C\cap C')$ in the first picture a *vertical twist line*.

3. Examples of surfaces in near-horizontal position

Suppose that X is a surface in near-horizontal position and let C, C', and $p(B)$ be the unions of loops and arcs of F described above. By definition of near-horizontal position, C, C', and $p(B)$ have the following properties:

1. C and C' intersect transversely.
2. Each arc of $p(B)$ has one endpoint in ∂C and one endpoint in $\partial C'$.
3. $\partial(p(B)) = \partial C \cup \partial C'$.

Furthermore, a surface X in near-horizontal position determines a function

$$N: \text{components of } F\backslash(C\cup C') \rightarrow \text{non-negative integers}$$

defined as follows. For each component r of $F\backslash(C\cup C')$ and each point $y\in \mathrm{int}(r)$, $N(r)$ counts the number of times X intersects $y\times I$. It is easy to check that N has the following properties:

4. If r is a component of $F\backslash(C\cup C')$ with an edge in $p(B)$, then $N(r)=1$.
5. If r is a component of $F\backslash(C\cup C')$ with an edge in $\partial F\backslash p(B)$, then $N(r)=0$.
6. If r_1 and r_2 are two components which meet along an arc of C or an arc of C', then $|N(r_1)-N(r_2)|=1$.
7. If r_1, r_2, r_3, and r_4 meet at a common vertex, then either

   $$\{N(r_1), N(r_2), N(r_3), N(r_4)\} = \{0, 1\}$$

   or the set $\{N(r_1), N(r_2), N(r_3), N(r_4)\}$ contains 3 distinct numbers.

The first possibility occurs if and only if the common vertex is the projection of a vertical twist line.

Conversely, suppose that C and C' are two unions of disjoint arcs and loops in F, that $p(B)$ is a union of disjoint arcs of ∂F, and that

$$N: \text{components of } F\backslash(C\cup C') \rightarrow \text{non-negative integers}$$

is a numbering scheme satisfying conditions (1)–(7) above. Then the information $(C, C', p(B), N)$ determines a unique surface in near-horizontal position.

Fig. 1 depicts a genus 1 surface with 4 boundary loops, in near-horizontal position in $\mathbb{R}^2 \times I$. The projection of the surface to $\mathbb{R}^2$ is drawn at left. Vertical twist lines, which project to points, are marked with dots. Loops of C are drawn with thick lines, and loops of C' are drawn with thin lines. The shading reflects the fact that the surface is 2-sided, with stripes on one side and solid color on the other. The corresponding combinatorial description is given at right.

4. Putting incompressible surfaces in near-horizontal position

The first major result of this paper is the following theorem.

Theorem 4.1. *Let F be a compact surface. Suppose that $X \subset F \times I$ is a properly embedded*, 2-*sided*, *incompressible surface. Then X is isotopic to a surface in near-horizontal position.*

Remark. In general, it is not possible to isotope a 2-sided incompressible surface $X \subset F \times I$ to avoid all vertical tangencies. For example, it is not hard to see that if $X \subset F \times I$ is a connected, separating surface that is not homeomorphic to a subsurface of F, then X must have vertical tangencies.

Since the proof of Theorem 4.1 is long and detailed, it is postponed until the end. See Section 11.

5. A sufficient condition

It is not true that every surface in near-horizontal position is incompressible. For example, the surface in Fig. 1 compresses in $\mathbb{R}^2 \times I$. The rest of this paper addresses the problem of when a 2-sided surface in near-horizontal position is incompressible. The analysis begins with the notion of cross-over curves.

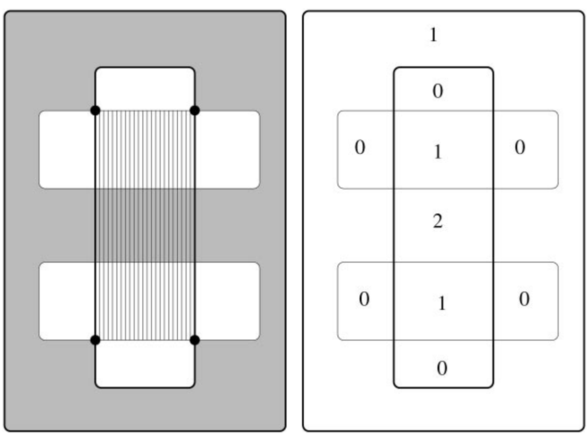


Fig. 1. An example of a surface in near-horizontal position.

Suppose that $X \subset F \times I$ is a surface in near-horizontal position. Let $C = p(\partial X \cap (F \times 1))$ and $C' = p(\partial X \cap (F \times 0))$.

Let Z be the quotient of X obtained by collapsing each vertical twist line to a point. The quotient Z is a surface with a set of singular points $p(V)$, where V is the union of twist lines of X. The projection $p: X \to F$ induces a natural immersion of Z in F.

Let $\mathscr{Y}$ be the abstract union of the closures of components of $C \backslash p(V)$, and let $\mathscr{Y}'$ be the abstract union of the closures of components of $C' \backslash p(V)$. For each curve $\gamma \subset \mathscr{Y} \cup \mathscr{Y}'$, let $i(\gamma)$ be the image of γ in F.

Form a quotient $\mathscr{Q}$ of $\mathscr{Y} \cup \mathscr{Y}'$ by identifying the endpoint $e \in \partial\alpha$ and $f \in \partial\beta$, for arcs $\alpha \subset \mathscr{Y}$ and $\beta \subset \mathscr{Y}'$, if $i(e) = i(f)$ and Z lies entirely to one side of $i(\alpha) \cup i(\beta)$ near $i(e)$. See Fig. 2.

The map $i: \mathscr{Y} \cup \mathscr{Y}' \to F$ induces a map $\mathscr{Q} \to F$ which will also be denoted by i. For each component $\gamma \subset \mathscr{Q}$, the arc or loop $i(\gamma)$ is a *cross-over curve* of X. See Fig. 3.

Each cross-over curve of X is a 1-manifold which is immersed in F away from points of $p(V)$. Each cross-over curve has no triple points and has double points contained in the set $C \cap C'$. Each arc of $C \backslash p(V)$ and $C' \backslash p(V)$ is traversed exactly once by a cross-over curve.

The following technical lemma about cross-over curves will be used in Section 7.

Lemma 5.1. *Let $X \subset F \times I$ be a 2-sided surface in near-horizontal position. Let V be the union of the vertical twist lines and let $i(\gamma)$ be a cross-over curve. Then $i(\gamma)$ passes through each point of $p(V)$ at most once. Furthermore, if $i(\gamma)$ is an arc, then the endpoints of $i(\gamma)$ lie in distinct components of $p(\partial X \cap (\partial F \times I))$.*

Proof. Suppose that X is colored red on one side and blue on the other. Call a component A of $X \backslash p^{-1}(C \cup C')$ *red-side-up* if the flow lines from $F \times 0$ to $F \times 1$ that pass through A cross from the blue side of A to the red side. Call the component *blue-side-up* if the flow lines cross from the red side to the blue side, instead. Since A is transverse to the flow lines by condition 4 in the definition of near-horizontal position, the notion of red-side-up and blue-side-up is well-defined.

The curve γ can be written as the union of arcs

$$\alpha_1 \cup \beta_1 \cup \alpha_2 \cup \cdots \cup \alpha_n \cup \beta_n$$

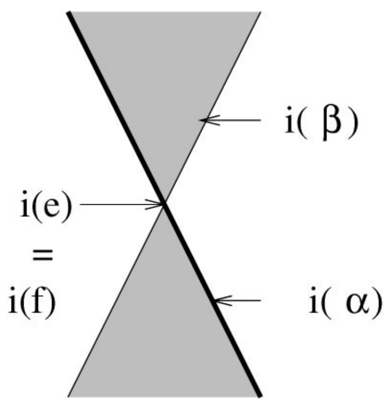


Fig. 2. Z lies entirely to one side of $i(\alpha) \cup i(\beta)$ near $i(e)$.

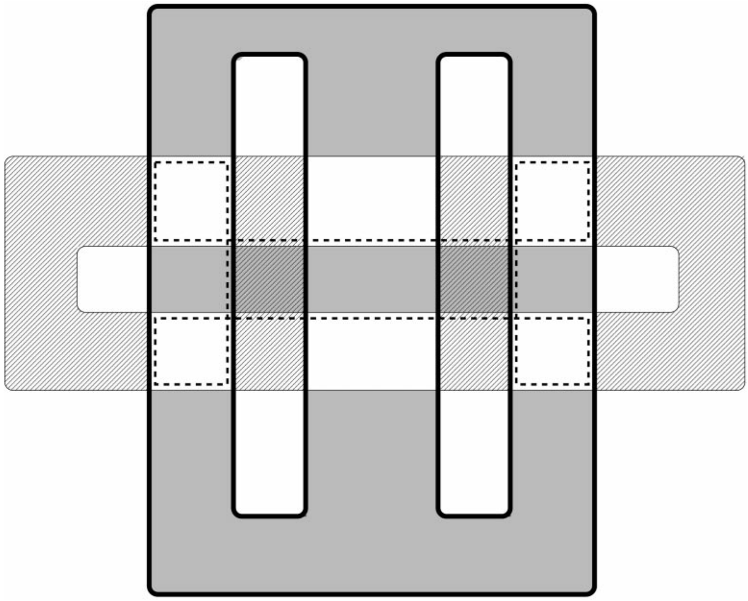


Fig. 3. A cross-over loop.

or as

$$\alpha_1 \cup \beta_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$$

or as

$$\beta_1 \cup \alpha_2 \cup \cdots \cup \alpha_n \cup \beta_n$$

where each α_i is an arc of $C \backslash p(V)$ and each β_i is an arc of $C' \backslash p(V)$. Notice that at each vertical twist line, one piece of $X \backslash p^{-1}(C \cup C')$ that is red-side-up meets another piece of $X \backslash p^{-1}(C \cup C')$ that is blue-side-up. By definition of cross-over curves, each α_i meets β_i at a point of $p(V)$ and each β_i meets α_{i+1} at a point of $p(V)$. It follows that for any i, if $\alpha_i \times 1$ is part of the boundary of a blue-side-up component of $X \backslash p^{-1}(C \cup C')$, then $\beta_i \times 0$ is part of the boundary of a red-side-up component of $X \backslash p^{-1}(C \cup C')$, and $\alpha_{i+1} \times 1$ is part of the boundary of a blue-side-up component, and so on. Thus, either every $\alpha_i \times 1$ is part of the boundary of a blue-side-up component of $X \backslash p^{-1}(C \cup C')$ and every $\beta_i \times 0$ is part of the boundary of a red-side-up component, or vice versa.

From this analysis, it is possible to conclude that $i(\gamma)$ passes through each point of $p(V)$ at most once. Otherwise γ would contain two arcs α_i and α_j on opposite sides of a twist line such that $\alpha_i \times 1$ is part of the boundary of a red-side-up component of $X \backslash p^{-1}(C \cup C')$ and $\alpha_j \times 1$ is part of the boundary of a blue-side-up component.

If $i(\gamma)$ is an arc, it is possible to conclude that the endpoints of $i(\gamma)$ lie in distinct components of $p(\partial X \cap (\partial F \times I))$. Suppose, instead, that the two endpoints bound the same arc $p(q)$, where q is a component of $\partial X \cap (\partial F \times I)$. Assume without loss of generality that q is part of the boundary of a piece of $X \backslash p^{-1}(C \cup C')$ that is red-side-up. By condition 2 in the definition of near-horizontal position, one endpoint of q lies on $\partial F \times 1$ and the other lies on $\partial F \times 0$. Therefore, γ must have the form

$$\alpha_1 \cup \beta_1 \cup \alpha_2 \cup \cdots \cup \alpha_n \cup \beta_n.$$

That is, its first arc lies in C and its last arc lies in C'. But since q bounds a piece of $X \backslash p^{-1}(C \cup C')$ that is red-side-up, so do both $\alpha_1 \times 1$ and $\beta_n \times 0$, contradicting the discussion above. □

The following theorem gives a sufficient condition for a 2-sided surface in near-horizontal position to be incompressible.

Theorem 5.2. *Suppose that $X \subset F \times I$ is a 2-sided surface in near-horizontal position. Suppose that for each cross-over curve $i(\gamma)$ of X,*

1. *$i_*: \pi_1(\gamma) \to \pi_1(F)$ is injective, and*
2. *$i(\gamma)$ lifts to an embedded curve in the covering space of F corresponding to $i_*\pi_1(\gamma)$.*

Then X is incompressible in $F \times I$.

In particular, if $X \subset F \times I$ is a 2-sided surface in near-horizontal position with no twist lines, and if all loops in C and C' are essential in F, then X is incompressible.

Remark. If X has no disk components, then condition (1) is a necessary condition. To see this, notice that it is possible to embed each cross-over loop γ in X so that $p(\gamma)$ is isotopic to $i(\gamma)$ by a small isotopy. In this embedding, $\gamma \subset X$ is either parallel to a loop of ∂X or else passes through a vertical twist line in exactly one point. In the first case, $\gamma \subset X$ is essential in X unless X has disk components. In the second case, $\gamma \subset X$ is essential in X. Therefore, if $X \subset F \times I$ is incompressible and has no disk components, γ must be essential in $F \times I$, and therefore, $p(\gamma) \simeq i(\gamma)$ must be essential in F.

6. Immersions of surfaces

The proof of Theorem 5.2 uses facts about immersions of surfaces in surfaces, which are interesting in their own right.

Suppose that $\theta: b \to F$ is an immersion of a loop b into a surface F. F has a covering space $\mathscr{F} \xrightarrow{q} F$ such that $q_*\pi_1(\mathscr{F}) = \theta_*\pi_1(b)$. If $\theta(b)$ is essential in F, then $\mathscr{F}$ is a cylinder; otherwise $\mathscr{F}$ is a plane. The map $\theta: b \to F$ lifts to a map $\tilde{\theta}: b \to \mathscr{F}$. The loop $\tilde{\theta}(b)$ may or may not be embedded in $\mathscr{F}$.

Lemma 6.1. *Suppose that Σ is a compact surface with boundary, that F is a compact surface, and that $\theta: \Sigma \to F$ is an immersion. Then the following two statements are equivalent:*

1. *For each boundary loop $b \subset \partial\Sigma$, $\theta(b)$ lifts to an embedded loop in the covering space of F corresponding to $\theta_*\pi_1(b)$.*
2. *There is a covering space $\hat{F}$ of F and a lift $\tilde{\theta}: \Sigma \to \hat{F}$ such that $\tilde{\theta}(\Sigma)$ is embedded in $\hat{F}$.*

Corollary 6.2. *Suppose that* Σ *is a compact surface with boundary which is not a disk, that* F *is a surface, and that* $\theta : \Sigma \to F$ *is an immersion. Suppose that for each boundary loop* $b \subset \partial\Sigma$, $\theta(b)$ *lifts to an embedded loop in the covering space corresponding to* $\theta_* \pi_1(b)$. *Then the induced map* $\theta_* : \pi_1(\Sigma) \to \pi_1(F)$ *is injective if and only if for each boundary loop* b *of* Σ, $\theta(b)$ *is essential in* F.

The assumption that Σ is not a disk is unimportant, since if Σ is a disk, then $\theta_* : \pi_1(\Sigma) \to \pi_1(F)$ is automatically injective.

The corollary does not hold without the assumption that the boundary loops lift to embedded loops. Fig. 4 shows a genus 1 surface with 2 boundary loops that is immersed in a pair of pants. The immersed surface has essential boundary curves but does not induce an injective map on fundamental groups.

Proof of Lemma 6.1. It is easy to see that statement (2) implies statement (1). Suppose that statement (1) holds. For each loop $b \subset \partial\Sigma$, let $\mathscr{F}_b$ be the covering space of F corresponding to $\theta_* \pi_1(b)$. Let $q_b : \mathscr{F}_b \to F$ be the covering map, and let $\tilde{\theta}_b(b)$ be the lift of b to $\mathscr{F}_b$. By assumption, $\tilde{\theta}_b(b)$ is embedded.

If $\theta(b)$ is not essential in F, then $\mathscr{F}_b$ is a plane, and $\mathscr{F}_b \backslash \tilde{\theta}_b(b)$ consists of a disk and a cylinder. If $\theta(b)$ is essential in F, then $\mathscr{F}_b$ is a cylinder and $\mathscr{F}_b \backslash \tilde{\theta}_b(b)$ consists of two cylinders.

Let $N(b)$ be a collar neighborhood of b in Σ. Since θ is an immersion and q_b is a covering map, there is a collar neighborhood $N(\tilde{\theta}_b(b))$ of $\tilde{\theta}_b(b)$ in $\mathscr{F}_b$ and a component K of $N(\tilde{\theta}_b(b)) \backslash \tilde{\theta}_b(b)$ such that $q_b(K) = \theta(N(b) \backslash b)$. Let A_b be the component of $\mathscr{F}_b \backslash \tilde{\theta}_b(b)$ that does not contain K. Thus, q_b sends points of A_b near $\tilde{\theta}_b(b)$ to points of F on the opposite side of $\theta(b)$ from $\theta(\Sigma)$.

Construct the covering space $\hat{F}$ of F by gluing each component A_b onto Σ along the boundary curve b. That is, let

$$\hat{F} = \left(\Sigma \cup \bigcup_{b \in \partial\Sigma} A_b \right) \Big/ \{ x \sim \tilde{\theta}_b(x) | \forall x \in b, \forall b \subset \partial\Sigma \}.$$

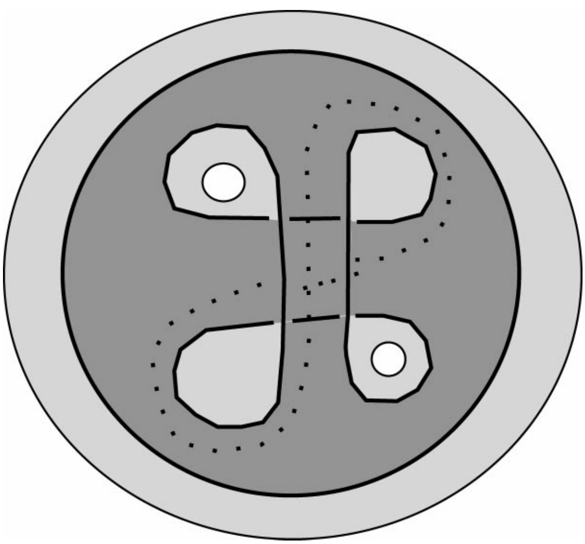


Fig. 4. This immersed surface is not π_1-injective. The dotted line shows a curve that is essential in the genus 1 surface but inessential in the pair of pants.

Define $\hat{q}:\hat{F}\to F$ by

$$\hat{q}=\begin{cases}\theta(x) & \text{if } x\in\Sigma,\\ q_b(x) & \text{if } x\in A_b.\end{cases}$$

Since θ is an immersion of a compact surface and each q_b is a covering map, the map $\hat{q}$ is a covering map. The obvious embedding of Σ in $\hat{F}$ is the desired lift $\hat{\theta}$. □

Proof of Corollary 6.2. Suppose that there is a boundary loop b of Σ such that $\theta(b)$ is not essential in F. Since Σ is not a disk, b is essential in Σ. Therefore, b represents an element of $\pi_1(\Sigma)$ which lies in the kernel of the map $\theta_*:\pi_1(\Sigma)\to\pi_1(F)$.

Conversely, suppose that for each boundary loop b of Σ, $\theta(b)$ is essential in F. By Lemma 6.1, there is a covering space $\hat{F}$ of F and a lift $\hat{\theta}:\Sigma\to\hat{F}$ such that $\hat{\theta}(\Sigma)$ is embedded in $\hat{F}$. For each loop $b\in\partial\Sigma$, $\theta(b)$ is essential in F, and therefore $\hat{\theta}(b)$ is essential in $\hat{F}$. So $\hat{\theta}(\Sigma)\subset\hat{F}$ is an embedded subsurface with essential boundary. Consequently, $\hat{\theta}_*:\pi_1(\Sigma)\to\pi_1(\hat{F})$ is injective. So $\theta_*:\pi_1(\Sigma)\to\pi_1(F)$ is injective as well. □

Suppose that $\theta:b\to F$ is an immersion of a circle into F. A *singular monogon* for $\theta(b)$ is a subarc $\alpha\subset b$ with distinct endpoints, such that $i(\alpha)$ is a closed loop in F which is not essential. The loop $i(\alpha)$ may or may not be embedded.

An essential immersed loop with no singular monogons always lifts to an embedded loop in an appropriate covering space, as the following lemma shows.

Lemma 6.3. *Suppose that b is a circle and F is a surface. Suppose that $\theta:b\to F$ is an immersion such that $\theta(b)$ is essential. Then $\theta(b)$ lifts to an embedded loop in the covering space corresponding to $\theta_*\pi_1(b)$ if and only if $\theta(b)$ has no singular monogons.*

Proof. Let $\mathscr{F}_b$ be the covering space of F corresponding to $\theta_*\pi_1(b)$ and let $\hat{\theta}:b\to\mathscr{F}_b$ be the lift of b to $\mathscr{F}_b$. Consider the universal cover $\mathbb{R}$ of b and the universal cover $\mathbb{R}^2$ of F. Since $\theta(b)$ is essential, θ lifts to a map $\tilde{\theta}:\mathbb{R}\to\mathbb{R}^2$. Notice that $\theta(b)$ has singular monogons if and only if $\tilde{\theta}$ has singular monogons, and $\tilde{\theta}$ has singular monogons if and only if $\tilde{\theta}$ has double points. But $\tilde{\theta}$ has double points if and only if $\hat{\theta}(b)$ has double points; that is, if and only if $\hat{\theta}(b)$ is not embedded. □

Lemma 6.3, together with the proof of Corollary 6.2, imply the following restatement of Corollary 6.2.

Corollary 6.4. *Suppose that Σ is a compact surface with boundary which is not a disk, that F is a surface, and that $\theta:\Sigma\to F$ is an immersion. Suppose that for each boundary loop $b\subset\partial\Sigma$, $\theta(b)$ has no singular monogons. Then the induced map $\theta_*:\pi_1(\Sigma)\to\pi_1(F)$ is injective if and only if for each boundary loop $b\subset\partial\Sigma$, $\theta(b)$ is essential in F.*

7. Proof of the sufficient condition

Proof of Theorem 5.2. Let $X\subset F\times I$ be a surface that satisfies the hypotheses of the theorem. The following argument proves that the induced map $\pi_1(X)\to\pi_1(F\times I)$ is injective, which implies that X is incompressible in $F\times I$.

As in Section 5, let Z be the quotient of X obtained by collapsing each twist line in X to a point. Notice that each cross-over curve of X embeds in Z. Let Σ_X be the surface obtained by gluing a collar $S^1 \times [0, \varepsilon]$ or $[0, 1] \times [0, \varepsilon]$ to each cross-over loop or arc, respectively, embedded in Z. See Fig. 5. The projection map $p: X \to F$ extends to an immersion $j: \Sigma_X \to F$.

Since the quotient map $q: X \to Z$ and the inclusion $Z \hookrightarrow \Sigma_X$ are homotopy equivalences, $q_*: \pi_1(X) \to \pi_1(\Sigma_X)$ is an isomorphism. The projection map $p: F \times I \to F$ is also a homotopy equivalence, so $p_*: \pi_1(F \times I) \to \pi_1(F)$ is also an isomorphism. Therefore, $\pi_1(X) \to \pi_1(F \times I)$ is injective if and only if $(p|_X)_*: \pi_1(X) \to \pi_1(F)$ is injective. Similarly, $j_*: \pi_1(\Sigma_X) \to \pi_1(F)$ is injective if and only if $j_* \circ q_*: \pi_1(X) \to \pi_1(F \times I)$ is injective. Since $p|_X = j \circ q$, it follows that $\pi_1(X) \to \pi_1(F \times I)$ is injective if and only if $j_*: \pi_1(\Sigma_X) \to \pi_1(F)$ is injective.

$$\begin{array}{ccc} X & \longrightarrow & F \times I \\ q \big\downarrow \simeq & & \simeq \big\downarrow p \\ \Sigma_X & \underset{j}{\longrightarrow} & F \end{array}$$

The case when $\partial C = \partial C' = \emptyset$ is most straightforward. In this case, for each boundary loop $b \in \partial \Sigma_X$, $j(b)$ is parallel to a cross-over loop $i(\gamma) \subset F$. By assumption, all cross-over loops are essential in F and lift to embedded loops in appropriate covering spaces of F. Therefore by Corollary 6.2. $j_*: \pi_1(\Sigma_X) \to \pi_1(F)$ is an injection. Thus, $\pi_1(X) \to \pi_1(F \times I)$ is an injection, as wanted.

The proof of the general case follows. Let the triplet $(X', F', F' \times I)$ be an isomorphic copy of $(X, F, F \times I)$, which can be thought of as its mirror image. Let $B = X \cap (\partial F \times I)$, and let $B' = X' \cap (\partial F' \times I)$. Let $(\hat{X}, \hat{F}, \hat{F} \times I)$ be the triplet constructed from $(X \cup X', F \cup F', (F \times I) \cup (F' \times I))$ by identifying X and X' along corresponding points of B and B', identifying F and F' along corresponding points of $p(B)$ and $p(B')$, and identifying $F \times I$ and $F' \times I$ along corresponding points of $p(B) \times I$ and $p(B') \times I$.

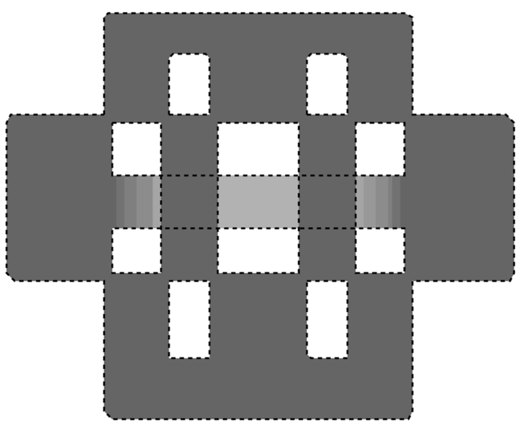


Fig. 5. An example of Σ_X, where X corresponds to the surface in Fig. 3.

The surface $\hat{X} \subset \hat{F} \times I$ is in near-horizontal position. Furthermore, since $X \subset F \times I$ is 2-sided by assumption, $\hat{X} \subset \hat{F} \times I$ is also 2-sided.

Notice that the inclusion $X \subset \hat{X}$ induces an injective map $\pi_1(X) \to \pi_1(\hat{X})$. Therefore, to prove that $\pi_1(X) \to \pi_1(F \times I)$ is injective, it is enough to show that $\pi_1(\hat{X}) \to \pi_1(\hat{F} \times I)$ is injective. Since $\hat{X} \cap \partial \hat{F} = \emptyset$, it will suffice to show that $\hat{X} \subset \hat{F} \times I$ satisfies the hypothesis of the theorem and apply the previous case.

Observe that $\hat{X} \subset \hat{F}$ has the same twist lines as $X \subset F$. Furthermore, each cross-over loop of $\hat{X}$ is either a cross-over loop of $X \subset F$, a cross-over loop of $X' \subset F'$, or the union of a cross-over arc of X with the corresponding cross-over arc of X'.

Let $i(\gamma)$ be a cross-over loop of $\hat{X}$. Suppose first that $i(\gamma)$ is a cross-over loop of $X \subset F$. Then $i(\gamma)$ is essential in F by hypothesis. Since $\pi_1(F) \to \pi_1(\hat{F})$ is injective, $i(\gamma)$ is essential in $\hat{F}$ as well. Let $\mathscr{F}$ and $\hat{\mathscr{F}}$ be covering spaces of F and $\hat{F}$, respectively, with fundamental group $i_*(\pi_1(\gamma))$. Then $\mathscr{F}$ appears as a subsurface of $\hat{\mathscr{F}}$. Since $i(\gamma)$ lifts to an embedding in $\mathscr{F}$, $i(\gamma)$ lifts to an embedding in $\hat{\mathscr{F}}$ as well. The same arguments apply when $i(\gamma)$ is a cross-over loop of X'.

Suppose, instead, that $i(\gamma)$ is the union of a cross-over arc $i(\delta)$ of X with the corresponding cross-over arc $i(\delta')$ of X'. Let b be a component of B such that $p(b)$ shares an endpoint with $i(\gamma)$. Since $\hat{X}$ is 2-sided, $i(\delta)$ has endpoints in distinct arcs of B, by Lemma 5.1. Therefore, $p(b)$ intersects $i(\gamma)$ exactly once. Since $p(b)$ is an arc of $\hat{F}$ with endpoints in $\partial \hat{F}$, it follows that $i(\gamma)$ is an essential loop of $\hat{F}$. Let $\hat{\mathscr{F}}$ be the covering space of $\hat{F}$ with fundamental group $i_*(\pi_1(\gamma))$. Let $\mathscr{F}$ be the universal covering space of F and let $\mathscr{F}'$ be the universal cover of F'. The surface $\hat{\mathscr{F}}$ can be formed by gluing together $\mathscr{F}$ and $\mathscr{F}'$ along pieces of their boundary in such a way that the lift of $i(\delta)$ to $\mathscr{F}$ and the lift of $i(\delta')$ to $\mathscr{F}'$ glue up to form a lift of $i(\gamma)$ to $\hat{\mathscr{F}}$. Since the lifts of $i(\delta)$ and $i(\delta')$ are embedded, so is the lift of $i(\gamma)$. $\square$

8. Immersed graphs and switch moves

As observed in the proof of Theorem 5.2, $\pi_1(X) \to \pi_1(F \times I)$ is injective if and only if $j_*: \pi_1(\Sigma_X) \to \pi_1(F)$ is injective, where $j: \Sigma_X \to F$ is an immersion of surfaces. Therefore, to decide when a 2-sided surface in near-horizontal position is incompressible, it is enough to solve the following problem. (For brevity, call a continuous map $f: X \to Y$ *loop-injective* if the induced homomorphism $\pi_1(X) \to \pi_1(Y)$ has trivial kernel.)

Problem 1. Find an algorithm to determine if an immersion of a compact surface in a surface is loop-injective.

In fact, if Σ is a compact surface without boundary, then any immersion $i: \Sigma \to F$ is a covering space. Therefore, by covering space theory [4], it is automatically loop injective.

Suppose that Σ is a compact surface with boundary. Then an immersion $i: \Sigma \to F$ can be deformed to an immersion of a compact graph in F. One way to deform $i(\Sigma)$ is to triangulate Σ and repeatedly collapse triangles with edges in $\partial\Sigma$ until a one-dimensional "spine" is reached.

Therefore, Problem 1 can be solved by figuring out Problem 2.

Problem 2. Find an algorithm to determine if an immersion of a compact graph in a surface is loop-injective.

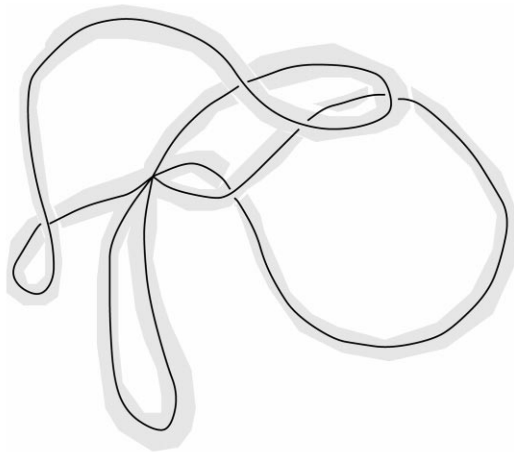


Fig. 6. An immersed graph and its regular neighborhood.

For simplicity, only "general position" immersed graphs are considered in the following discussion. Suppose that $i: G \to F$ is a immersion of a graph G into a surface F. Say that $x \in F$ is a *double point* if $i^{-1}(x)$ contains exactly two points. The immersed graph $i(G)$ is in *general position* if

1. for each $x \in F$, $i^{-1}(x)$ contains at most two points, and
2. at each double point $x \in F$, two edges of G intersect transversely.

Suppose $i: G \to F$ is a general position immersed graph. Let $i: \Sigma \to F$ be the immersed surface obtained as a regular neighborhood of $i(G)$ (Fig. 6). Corollary 6.4 shows that Problem 2 is easy to solve if $i(\Sigma)$ has no singular monogons in its immersed boundary loops. To solve the problem in general, it will help to define a move to get rid of monogons.

Suppose that there is a boundary loop $b \subset \partial\Sigma$ that contains a singular monogon. Let x be the double point of $i: G \to F$ corresponding to the vertex of the monogon. Let e and f be the edges of G that contain preimages of x. Orient the loop b, and orient e and f to agree with the orientation of b near x. (This is possible even when $e = f$.) A *switch move* on $i(G)$ is performed by

1. cutting e and f along $i^{-1}(x)$,
2. regluing in the way that disrepects orientation, and
3. isotoping the resulting immersed graph to lie in general position.

See Fig. 7.

A switch move preserves loop-injectivity:

Lemma 8.1. *Suppose that the immersed graph $i'(G')$ is obtained from the immersed graph $i(G)$ by a monogon switch move. Then $i'_*: \pi_1(G') \to \pi_1(F)$ is injective if and only if $i_*: \pi_1(G) \to \pi_1(F)$ is injective.*

Proof. Glue a disk D to G by identifying ∂D to the arc of the monogon in G. By definition, the map i sends $\partial D \subset G$ to a non-essential loop of F. Therefore, the map i can be extended across D to a map $j: G \cup D \to F$. It is easy to see that $i: G \to F$ and $i': G' \to F$ are both deformation retracts of the map $j: G \cup D \to F$. See Fig. 8. □

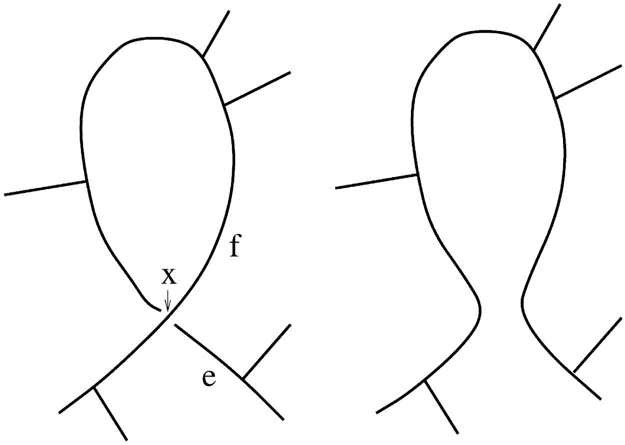


Fig. 7. A switch move.

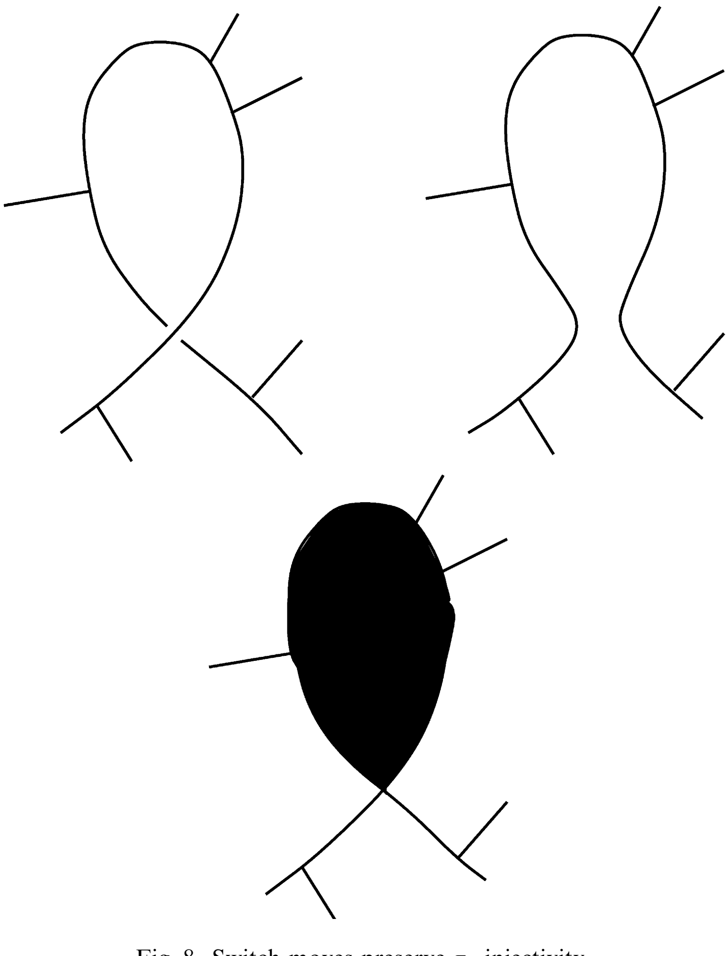


Fig. 8. Switch moves preserve π_1-injectivity.

9. An algorithm

When G is a circle, work of Peter Scott and Joel Hass [1] yields a well-known algorithm for solving Problem 2, which is included here for completeness.

Theorem 2.7 of [1] states:

Theorem. *Let $f: S^1 \to F$ be a general position loop on an orientable surface F which is homotopic to an embedding but is not an embedding. Then there is an embedded monogon or digon for f.*

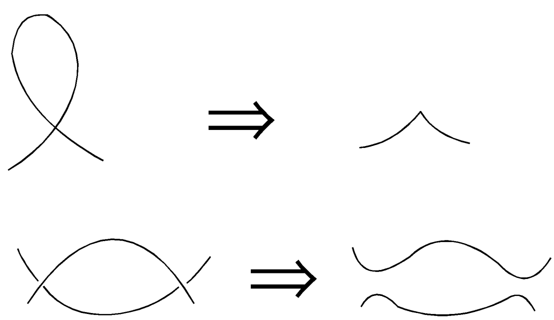
An embedded monogon for $f: S^1 \to F$ is a subarc $\alpha \subset S^1$ such that $f(\alpha)$ is a closed loop, $f|_{\mathrm{int}(\alpha)}$ is injective, and $f(\alpha)$ bounds a disk of F. An embedded digon is a pair of disjoint subarcs α and β of S^1 such that $f|_\alpha$ and $f|_\beta$ are embeddings, and $f(\alpha \cup \beta)$ bounds a disk D of F with

$$f(\alpha) \cup f(\beta) = \partial D \quad \text{and} \quad f(\alpha) \cap f(\beta) = f(\partial\alpha) = f(\partial\beta)$$

By this theorem, if F is orientable and $f: S^1 \to F$ has no embedded monogons or digons, then $i(S^1)$ must either be essential or bound an embedded disk in F. This makes the following algorithm work.

Algorithm when $G = S^1$ and F is orientable

1. Decide if $i(S^1)$ has any embedded monogons or digons.
   - If so, remove one by cutting and pasting, and repeat Step 1.



   - If not, continue.
2. Decide if $i(S^1)$ bounds an embedded disk in F
   - If so, i IS NOT loop-injective.
   - Otherwise, i IS loop-injective.

This algorithm terminates in finitely many steps, since each application of Step 1 eliminates at least one double point.

If F is non-orientable, then Theorem 3.5 of [1] still applies.

Theorem. *Let $f: S^1 \to F$ be a general position loop on a surface F. If f has excess self-intersection, then f has a singular monogon or a weak digon.*

A singular monogon is defined in Section 6. A weak digon is a pair of subarcs α and β such that f identifies the endpoints of α with those of β and the loop formed by $f|_\alpha$ and $f|_\beta$ is null-homotopic in F.

The algorithm for when F is orientable can be modified as follows:

Algorithm when $G=S^1$ and F is non-orientable

1. Decide if $i(S^1)$ has any singular monogons or weak digons.
   Singular monogons can be located as follows:
   (a) orient S^1,
   (b) choose a double point,
   (c) form two loops by cutting and pasting $i(S^1)$ at this double point, in a way that respects the orientation of S^1, and
   (d) apply this algorithm recursively on these two loops to decide if either is non-essential in F (and therefore a singular monogon).
   Weak digons can be located, similarly, starting with two double points.
   - If $i(G)$ has any singular monogons or weak digons, then remove one by cutting and pasting, and repeat Step 1.
   - If not, continue.
2. Decide if $i(G)$ bounds an embedded disk in F.
   - If so, i IS NOT loop-injective.
   - Otherwise, i IS loop-injective.

The algorithm when G is any compact graph is also straightforward.

Algorithm in general

1. Decide whether G is a tree.
   - If it is, then i: $G \to F$ IS loop-injective.
   - Otherwise, continue.
2. Draw Σ, the immersed surface that is a regular neighborhood of $i(G)$, and look for singular monogons in the immersed boundary loops of Σ. Singular monogons can be located using one of the previous two algorithms.
   - If there is a singular monogon, perform a switch move and repeat Step 2 with the new general position immersed graph.
   - Otherwise, continue.
3. Decide whether all immersed boundary loops of Σ are essential, using one of the previous two algorithms.
   - If so, then $i: G \to F$ IS loop-injective.
   - Otherwise, $i: G \to F$ IS NOT loop-injective.

The conclusions in Step 3 follow from Corollary 6.4.

Notice that the general algorithm terminates after finitely many steps, since each switch move decreases the number of double points of the immersed graph.

See Figs. 9 and 10 for some examples of this algorithm.

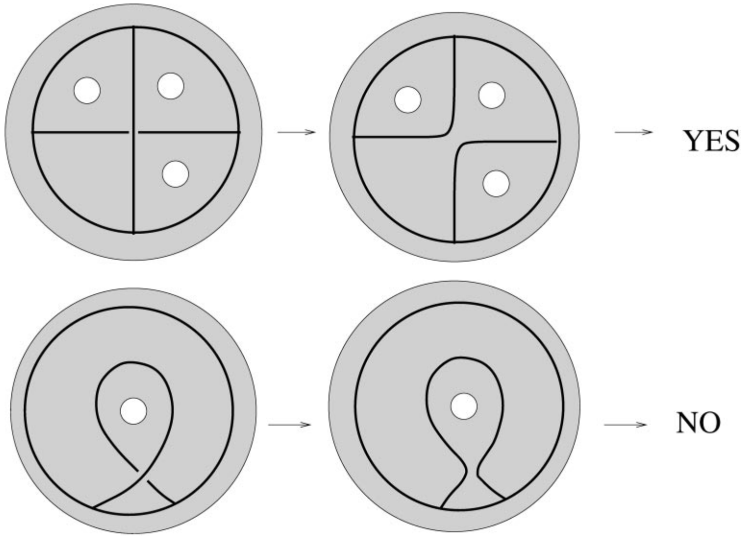


Fig. 9. Two examples of the algorithm. F is a thrice-punctured disk in the first example, and a once-punctured disk in the second.

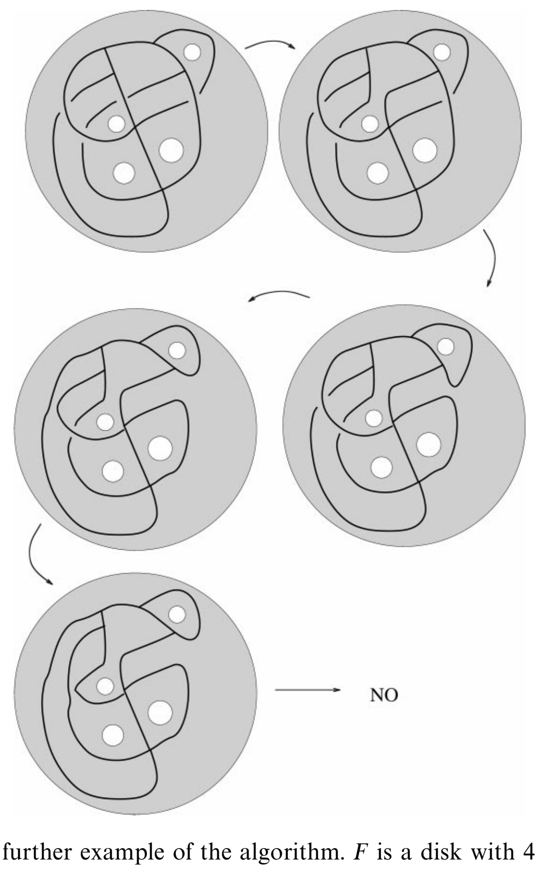


Fig. 10. A further example of the algorithm. F is a disk with 4 punctures.

10. Classification

The results of this paper suggest an alternative method to normal surface theory [3] for finding the incompressible surfaces in a handlebody (up to isotopy). Here is a sketch of the procedure.

1. Write the handlebody as $F \times I$, where F is a punctured disk.
2. List all possible collections of curves C, C', and $p(B)$ that satisfy conditions (1)–(3) in the definition of near-horizontal position, given in Section 2. Collections of boundary loops that are isotopic need only be listed once.
3. Assign numbers to the regions of $F \backslash (C \cup C')$ according to the rules set out in Section 3. The numbers determine surfaces in near-horizontal position.
4. Determine which of these surfaces are incompressible, using the algorithm of Section 9.

One way to accomplish Step 2 above is outlined as follows. Fix a system of arcs $\{\alpha_i\} \subset F$ such that $\{\alpha_i \times I\}$ divides $F \times I$ into solid pairs of pants. See Fig. 11. Notice that each component of $F \backslash \{\alpha_i\}$ is a hexagon with sides that alternate between arcs of $\partial F \backslash \{\partial \alpha_i\}$ and arcs of $\{\alpha_i\}$. Create a collection of curves C, C', and $p(B)$ using the following steps.

- Assign a non-negative integer to each arc of $\partial F \backslash \{\partial \alpha_i\}$ and an ordered pair of non-negative integers to each arc of $\{\alpha_i\}$. Assign numbers in such a way that for each hexagon H of $F \backslash \{\alpha_i\}$, the numbers along the three arcs of $\partial F \backslash \{\partial \alpha_i\}$ that are sides of H, together with the first coordinates in the ordered pairs along the three arcs of $\{\alpha_i\}$ that are sides of H, all add up to an even number. The sum should also be even if the second coordinates in the ordered pairs are used instead of the first coordinates.
  The number assigned to an arc of $\partial F \backslash \{\partial \alpha_i\}$ will specify how many segments of $p(B)$ are contained in that arc. The first coordinate and second coordinate in the ordered pair assigned to an arc of $\{\alpha_i\}$ will specify how many times the curves of C and C', respectively, intersect that arc.
- For each arc α_i, draw m_i black points and n_i white points on α_i, where (m_i, n_i) is the ordered pair assigned to α_i. For each arc β_j of $\partial F \backslash \{\partial \alpha_i\}$, draw k_j gray points on the arc, where k_j is the number assigned to β_j.
- For each hexagon of $F \backslash \{\alpha_i\}$, draw non-intersecting black arcs in the hexagon such that the endpoints of the black arcs are exactly the union of the black points and the gray points. This can be done in finitely many ways, up to isotopy. Similarly, draw non-intersecting white arcs in the hexagon such that the endpoints of the white arcs are exactly the union of the white points and

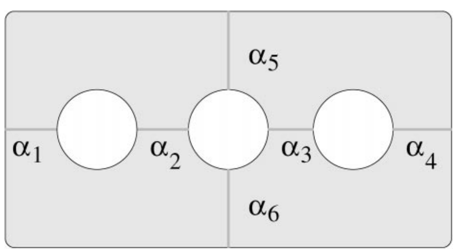


Fig. 11. A system of arcs $\{\alpha_i\}$.

the gray points. The black arcs and the white arcs may intersect, but should be chosen to intersect transversely. See Fig. 12.

- Black arcs from neighboring hexagons fit together and white arcs from neighboring hexagons fit together to create a set of curves on F. Call the black curves C, the white curves C', and stretch the gray points into small intervals to get a set of arcs $p(B)$. The curves C, C', and $p(B)$ satisfy properties (1), (2), and (3) in the definition of near-horizontal position.

11. Proof of near-horizontal position

This final section contains the proof of Theorem 4.1.

Proof of Theorem 4.1. Notice that it is possible to isotope X (leaving it fixed outside a neighborhood of ∂X), so that ∂X satisfies requirements 1–3 in the definition of near-horizontal position. Therefore, assume that C intersects C' transversely, that each component of B is an arc with one endpoint in $\partial F \times 1$ and one endpoint in $\partial F \times 0$, and that $p|_B$ is an embedding, where C, C', and B are defined as above.

Suppose that X contains components $X_1, X_2, \dots X_n$ that are boundary parallel disks. Each X_i can be isotoped so that $p|_{X_i}$ is an embedding and so that $p(\partial X_i)$ is disjoint from $p(\partial X \backslash \partial X_i)$. Therefore, if $X \backslash (X_1 \cup X_2 \cup \cdots \cup X_n)$ can be isotoped to near-horizontal position, so can X. So without loss of generality, assume that X has no components that are boundary parallel disks. Since X is incompressible, this assumption insures that all loops in C and C' are essential in F.

All isotopies in the rest of the proof will leave ∂X fixed.

Let A be the union of vertical strips and annuli $C \times [0, 1]$ and let $A' = C' \times [0, 1]$. The plan of the proof is as follows. First, a notion of "pseudo-transverse" is defined and a measure of complexity is given for surfaces that are pseudo-transverse to $A \cup A'$. Then three moves are described which decrease this complexity. In Step 1, X is isotoped so that it is pseudo-transverse to A and A'. In Step 2, X is isotoped using the three moves as many times as possible. Next, six claims are verified

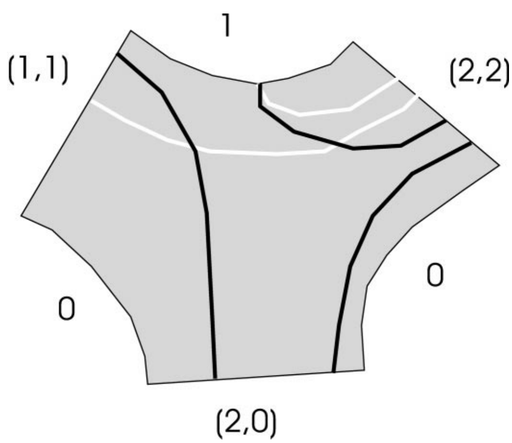
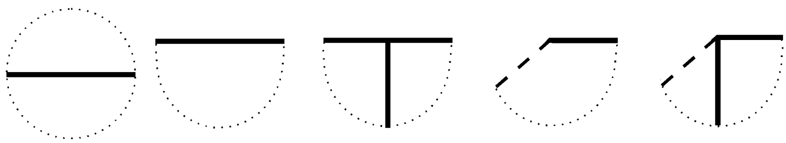


Fig. 12. Black and white arcs in a hexagon.

about the position of X. In Step 3, X is isotoped so that the projection map p is injective when restricted to any arc or loop of $X \cap (A \backslash A')$ and $X \cap (A' \backslash A)$. In Step 4, X is isotoped so that p is locally injective everywhere except at twist lines. This completes the proof of the theorem.

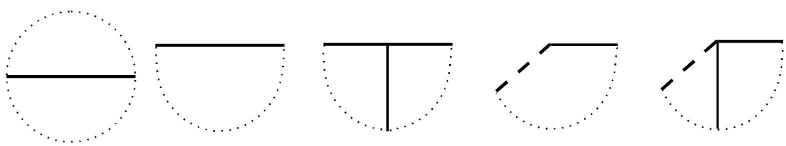
Say that $X \subset F \times I$ is *pseudo-transverse* to $A \cup A'$ if the following three conditions hold:

1. For any point $z \in (C - C')$ there is a neighborhood $U \subset F$ of z such that $p^{-1}(U) \cap X$ is a disjoint union of regions of the form



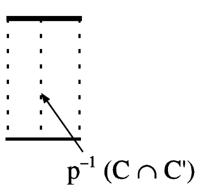
Here, thick lines represent arcs of $p^{-1}(C) \cap X$ and dashed lines represent arcs of B. Dotted lines represent arcs in the boundary of $p^{-1}(U) \cap X$ that are not part of $p^{-1}(C) \cap X$ or B. One of the last two pictures occurs if and only if $z \in \partial C$. Note that the third and fifth pictures do not occur when the surfaces $p^{-1}(C)$ and X are actually transverse along their common boundary. These two pictures reflect the possibility that the surfaces could be tangent at $z \in \partial C$, where their intersection forms a half-saddle critical point.

2. For any point $z \in (C' - C)$, there is a neighborhood U of z such that $p^{-1}(U) \cap X$ is a disjoint union of regions of the form

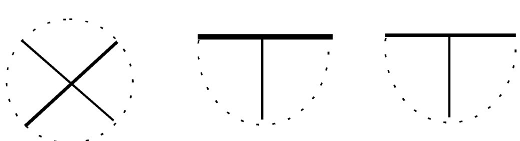


Here, thin lines represent arcs of $p^{-1}(C') \cap X$ and dashed and dotted lines are used as above. One of the last two pictures occurs if and only if $z \in \partial C'$.

3. For any point $z \in (C \cap C')$, there is a neighborhood U of z such that $p^{-1}(U) \cap X$ is either a single region of the form



or else a disjoint union of regions of the form



Suppose X is pseudo-transverse to $A \cup A'$. Define the complexity of X by

$$\zeta(X) = (|X \cap (A \cap A')|, \operatorname{rank} H_1(X \cap A) + \operatorname{rank} H_1(X \cap A'))$$

ordered lexicographically. Consider the following three moves.

Move 1. Suppose D is a disk of A such that $D \cap X = \partial D$ and $\partial D \cap A' = \emptyset$. Then D can be used to isotope X relative to ∂X and decrease $\zeta(X)$. The isotopy leaves X in the class of pseudo-transverse embeddings. If the roles of A and A' are interchanged, an analogous move is possible.

Explanation of Move 1. Since X is incompressible in $F \times I$, ∂D bounds a disk D' in X (see Fig. 13). The set $D \cup D'$ forms a sphere in $F \times I$, which must be embedded since $\operatorname{int}(D) \cap X = \emptyset$. Since $F \times I$ is irreducible, the sphere bounds a ball, which can be used to isotope X relative to ∂X. If $\partial D \cap \partial X = \emptyset$, then D' can be pushed entirely off of ∂D, and one component of $X \cap A$ is eliminated. Components of $X \cap (A \cap A')$, components of $X \cap A'$, and additional components of $X \cap A$ may also be removed if $\operatorname{int}(D') \cap (A \cup A') \neq \emptyset$, but no new components of any kind are added. Therefore, $\zeta(X)$ goes down. If, instead, $\partial D \cap \partial X \neq \emptyset$, then the isotopy of X relative to ∂X must leave $\partial D \cap \partial X$ fixed. But this isotopy still decreases the rank of $H_1(X \cap A)$ without increasing the rank of $H_1(X \cap A')$ or the number of components of $X \cap (A \cap A')$.

The explanation is analogous if the roles of A and A' are interchanged.

Move 2. Suppose E is a disk in $F \times I$ whose boundary consists of two arcs α and σ. Suppose that $E \cap X = \sigma$ and that $E \cap A = E \cap A' = \alpha$. Then E can be used to isotope X relative to ∂X and decrease $\zeta(X)$. The isotopy leaves X in the class of pseudo-transverse embeddings.

Explanation of Move 2. Consider three cases depending on how many points of $\partial\sigma$ lie in ∂X. See Figs. 14 and 15.

Case 1. Both endpoints of σ lie in $\operatorname{int}(X)$. Then X can be isotoped in a neighborhood of σ so that it moves through E and slips entirely off of α. This isotopy decreases by two the number of components of $X \cap (A \cap A')$.

Case 2. One endpoint of σ lies in $\operatorname{int}(X)$ and one endpoint lies in ∂X. Now X cannot be isotoped relative to ∂X entirely off α since the endpoint of $\partial\sigma$ in ∂X must remain fixed. But X can still be pushed off $\operatorname{int}(\alpha)$ and off the free endpoint, lowering the number of components of $X \cap (A \cap A')$ by one.

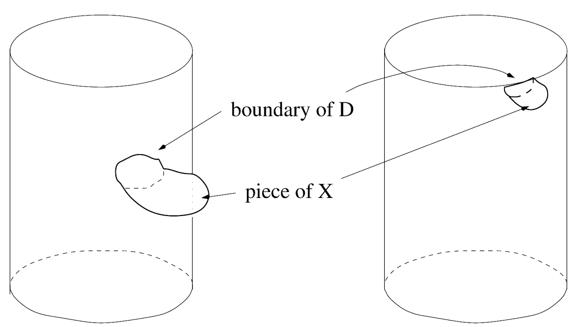


Fig. 13. Move 1 disks.

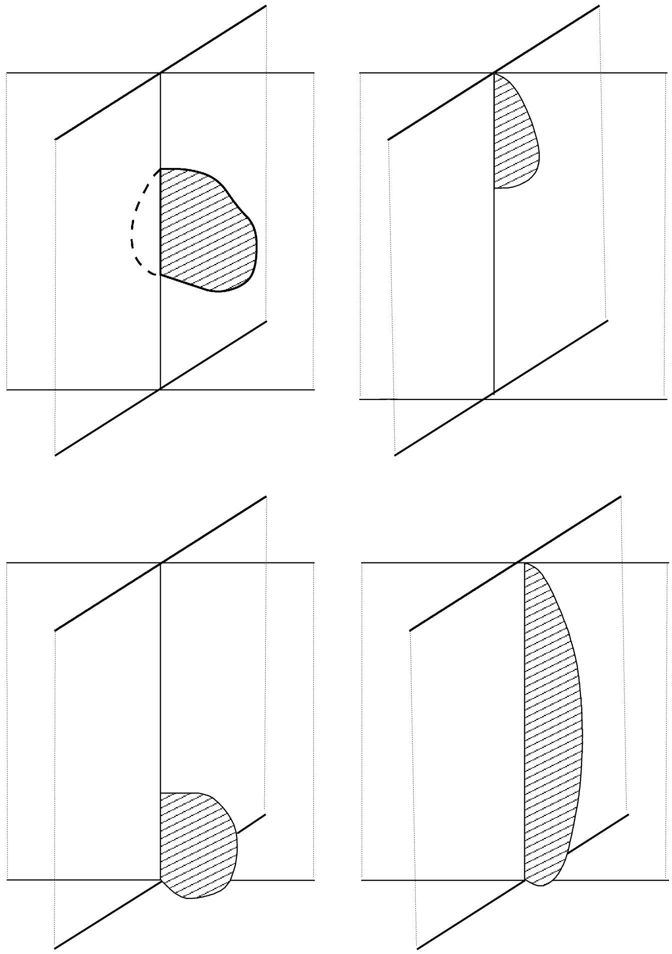


Fig. 14. Move 2 disks.

Case 3. Both endpoints of σ lie in ∂X. Notice that one endpoint must lie in $X \cap (F \times 1)$ and one must lie in $X \cap (F \times 0)$, since α is a vertical line connecting them. In this case X can be isotoped relative to ∂X in a neighborhood of σ to move σ directly onto the vertical line α and produce a twist around this vertical line resembling the first picture in condition (3) of the definition of pseudo–transverse embeddings. This isotopy decreases the number of components of $X \cap (A \cap A')$ by one, since it transforms the two endpoints of $\partial\sigma$ into a single vertical line.

Move 3. Suppose that W is an annulus contained in A such that $W \cap A' = \emptyset$, that ∂W is the boundary of an annulus of $X \backslash A$, and that the loops of ∂W do not bound disks in A. Then W can be used to isotope X and decrease $\zeta(X)$. The isotopy leaves X in the class of pseudo-transverse embeddings. A similar move is possible if the roles of A and A' are interchanged (Fig. 16).

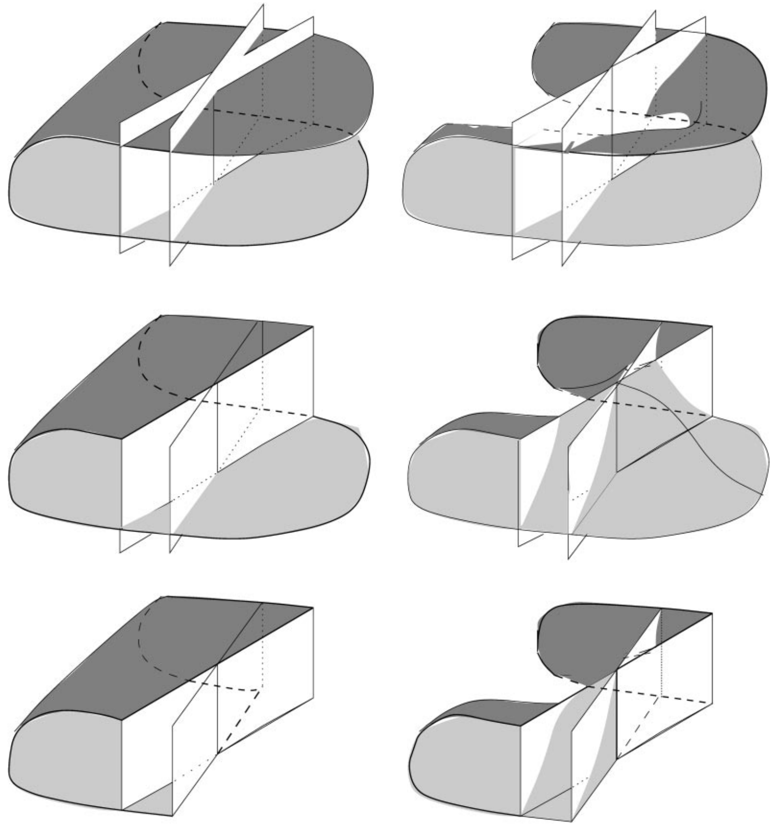


Fig. 15. Using move 2 disks to isotope X.

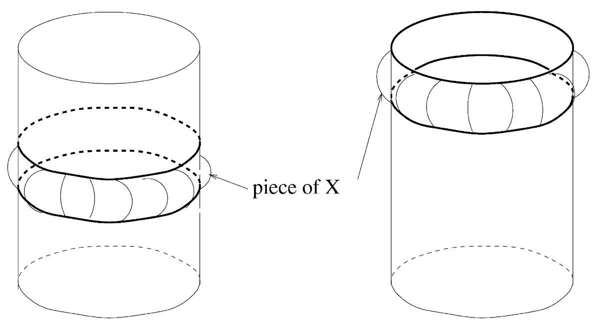


Fig. 16. Move 3 annuli.

Explanation of Move 3. Let G be the annulus of $X\setminus A$ such that $\partial G = \partial W$, and let $F_0 \times I$ be the closure of the component of $F \times I \setminus A$ that contains G. Let A_0 be the component of A that contains W.

The main claim behind Move 3 is that $W \cup G$ bounds a solid torus in $F_0 \times I$ and that the loops of ∂W are longitudes for this solid torus. This claim is easiest to prove when W is embedded in A_0 in such a way that for each point $x \in p(A_0)$, $(x \times I) \cap \partial W$ contains exactly two points, one for each loop of ∂W. However, even if W is embedded in A_0 in a more complicated way, $W \cup G$ can be isotoped within $F_0 \times I$ until ∂W has this property, since the loops of ∂W have degree 1 in A_0 by assumption. If the isotoped version of $W \cup G$ bounds a solid torus and the loops of ∂W are longitudes, then the same is true for the original version. Therefore, it will suffice to prove the claim under the assumption that for each point $x \in p(A_0)$, $(x \times I) \cap \partial W$ contains two points.

By assumption, $p(A_0)$ is essential in F. So F_0 is not a disk. Therefore, it is possible to find a properly embedded arc σ in F_0, with at least one endpoint on $p(A_0)$, which is not homotopic relative boundary to an arc in $p(A_0)$. Isotope G relative to ∂G so that it is transverse to $\sigma \times I$. The intersection of $\sigma \times I$ with G is a union of arcs and loops.

It is possible to isotope G relative to ∂G to remove any loops by the following argument. Pick a loop λ of $(\sigma \times I) \cap G$ that is innermost in $\sigma \times I$. Let D be the disk it bounds in $\sigma \times I$. The loop λ cannot be homotopic in G to ∂G, since $p(\partial G)$ is essential in F. So λ must bound a disk D' in G. The set $D \cup D'$ forms a sphere in $F \times I$, which must be embedded since λ is innermost in $\sigma \times I$. Since $F \times I$ is irreducible, the sphere bounds a ball which can be used to isotope G relative to ∂G to remove the loop λ. All loops of $(\sigma \times I) \cap G$ can be removed similarly.

For each endpoint $x \in \partial\sigma \cap p(A_0)$, $(x \times I) \cap \partial W$ consists of two points by assumption: one for each loop of ∂W. Therefore, $(\sigma \times I) \cap G$ consists of one or two arcs. Let β be an arc of $(\sigma \times I) \cap G$. Suppose that β stretches between distinct components of $\partial\sigma \times I$. Then β will necessarily have both endpoints in the same loop of ∂W, and so β will be homotopic in G relative boundary into ∂W. Therefore $p(\beta)$ will be homotopic in F_0 relative boundary into $p(A_0)$. Since β stretches across $\sigma \times I$, σ will be homotopic in F_0 relative boundary into $p(A_0)$ as well. But this violates the choice of σ.

So β does not stretch between distinct components of $\partial\sigma \times I$. Instead, it has both endpoints on the same vertical interval of $\partial\sigma \times I$. Thus β, together with a subinterval of $\partial\sigma \times I$, bounds a subdisk E of $\sigma \times I$.

Since ∂E intersects each loop of ∂W in one point, E does not bound a disk in $W \cup G$. Instead, E is a compression disk for the torus $W \cup G \subset F_0$.

Since $F_0 \times I$ is irreducible and ∂W is not contained in a ball, $W \cup G$ must bound a solid torus. Furthermore, since ∂E intersects each loop of ∂W in one point, the loops of ∂W are longitudes of the solid torus, as claimed. It follows that the solid torus can be used to isotope X relative to ∂X by pushing G through W.

Notice that ∂W and ∂X share at most one component. If ∂W and ∂X are disjoint, then the isotopy removes at least two components of $X \cap A$. If ∂W and ∂X share a component, the isotopy removes at least one component of $X \cap A$. In either case, the isotopy decreases the rank of $H_1(X \cap A)$ without increasing the rank of $H_1(X \cap A')$ or the number of components of $X \cap (A \cap A')$.

The explanation is analogous if the roles of A and A' are interchanged.

Call the type of disk used in Move 1 a move 1 disk, the type of disk used in Move 2 a move 2 disk, and the type of annulus used in Move 3 a move 3 annulus.

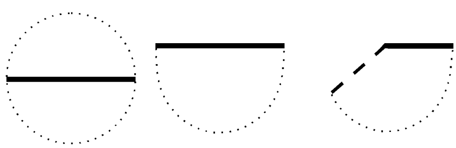
Step 1: Isotope X relative to ∂X so that it is pseudo-transverse to $A \cup A'$. This can be accomplished, for example, by making X honestly transverse to $A \cup A'$. Then the only pictures that can occur are the first, second and fourth pictures in condition (1), the first, second and fourth pictures in condition (2) and the second, third and fourth pictures in condition (3) of the definition of pseudo-transverse.

Step 2: Suppose $F \times I$ contains a move 1 disk, a move 2 disk, or a move 3 annulus. Use it to isotope X. Repeat this step as often as necessary, until there are no more such disks or annuli. The process must terminate after finitely many moves, since each move decreases $\zeta(X)$.

At this stage, X already has a neat posture with respect to $A \cup A'$. In particular, the following claims hold, where K is any component of $X \backslash (A \cup A')$ and $L \times I$ is the component of $(F \times I) \backslash (A \cup A')$ that contains K.

Claim 1. *Suppose that μ is an arc contained in $X \cap (A \backslash A')$ with both endpoints in $A \cap A'$. Then either the endpoints of μ go to distinct vertical lines of $(A \cap A')$ or else μ wraps all the way around an annulus of A. Likewise for arcs of $X \cap (A' \backslash A)$. See Fig.* 17.

Claim 2. *For any point $z \in C \backslash C'$, there is a neighborhood U of z such that $p^{-1}(U) \cap X$ is a disjoint union of neighborhoods of the form*:



Likewise for points of $(C' \backslash C)$. In other words, the third and fifth pictures in condition (1) *and the third and fifth pictures in condition* (2) *in the definition of pseudo-transverse do not occur.*

Claim 3. *Every circle of ∂K has non-zero degree in the cylinder of $\partial L \times I$ that contains it.*

Claim 4. *$\pi_1(K) \to \pi_1(L \times I)$ is injective.*

Claim 5. *Either $\pi_1(K) \to \pi_1(L \times I)$ is surjective, or else K is an annulus that is parallel to $\partial L \times I$.*

Claim 6. *If K is an annulus, then the two loops of ∂K go to two distinct cylinders of $\partial L \times I$.*

Proof of Claim 1. Let μ be an arc contained in $X \cap (A \backslash A')$ and suppose both endpoints of μ lie in one vertical line of $A \cap A'$. Suppose that μ does not wrap all the way around an annulus of A, and let α be the segment of $A \cap A'$ that connects the endpoints of μ. Then $\alpha \cup \mu$ bounds a half-disk E in A.

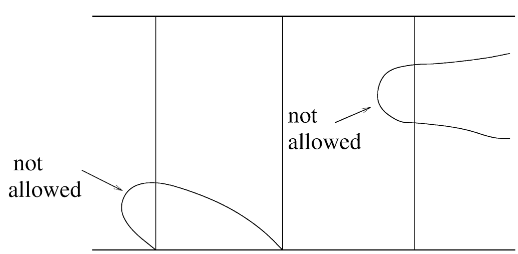


Fig. 17. Endpoints of μ go to distinct vertical lines of $A \cap A'$.

Notice that $E \cap A' = \alpha$. The set $E \cap X$ cannot contain any closed loops of $X \cap A$, since any innermost such loop would bound a move 1 subdisk of E, which should have been removed in Step 2. But $E \cap X$ may contain other arcs besides μ with endpoints on α. (See Fig. 18.) By replacing μ and E with an arc and subdisk closer to α if necessary, assume that $E \cap X = \mu$. Nudge E relative α off of A to get a new disk E' bounded by the arcs α and μ', where $\mu' \subset X$ and $\mathrm{int}(\mu') \subset \mathrm{int}(X)$. Since $E \cap X = \mu$ and $E \cap A' = \alpha$, this can be done so that $E' \cap X = \mu'$ and $E' \cap A' = \alpha$. Also, $E' \cap A = \alpha$. So E' is a move 2 disk, in violation of Step 2. Thus, the endpoints of μ must lie in distinct components of $A \cap A'$ after all.

The same argument applies to arcs contained in $X \cap (A' \backslash A)$.

Proof of Claim 2. Supose the third or the fifth picture of condition (1) does occur. (The argument is similar if the third or the fifth picture of condition (2) occurs.) Let α be the segment of $p^{-1}(U) \cap X \cap A$ drawn vertically in these pictures, extended in A until it first hits ∂X or $A \cap A'$. See Fig. 19. Label the first endpoint of α as $\partial_1 \alpha$ and the second endpoint as $\partial_2 \alpha$. So $\partial_1 \alpha$ lies on $C \times 1$.

Suppose first that $\partial_2 \alpha$ lies on ∂X but not on $A \cap A'$. Recall that $\partial X = (C \times 1) \cup (C' \times 0) \cup B$; therefore $(\partial X \cap A) \backslash (A \cap A') \subset ((C \times 1) \cap A) \cup (B \cap A)$. Since $p|_B$ is an embedding, it follows that

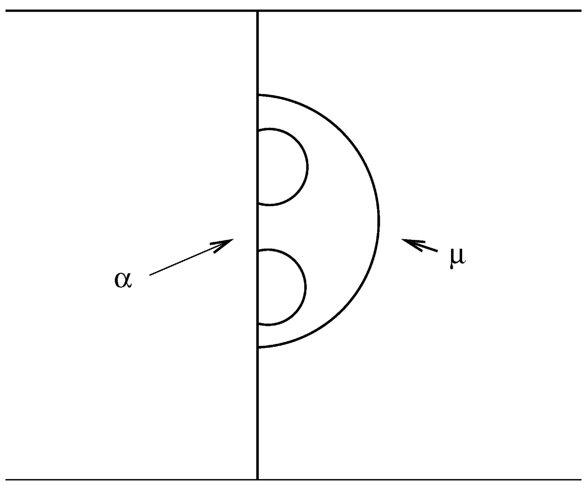


Fig. 18. $E \cap X$ may contain additional arcs.

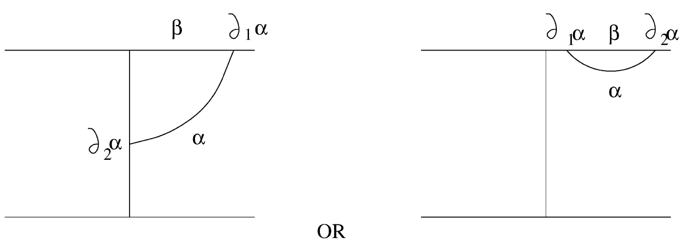


Fig. 19. The arc α in the proof of Claim 2.

$B \cap A \subset C \times 1$. So $\partial_2 \alpha$ lies in $\partial X \cap (C \times 1)$. Thus, α cuts off an arc β of $C \times 1$ such that $\alpha \cup \beta$ bounds a move 1 disk. This disk should already have been removed in Step 2.

Next, suppose that $\partial_2 \alpha$ lies on $A \cap A'$, and let β be the arc of ∂X such that $\partial_1 \beta = \partial_1 \alpha$ and $\partial_2 \beta$ lies on the same vertical line of $A \cap A'$ as $\partial_2 \alpha$, and so that $\alpha \cup \beta$ does not wrap all the way around an annulus of A. Then $\alpha \cup \beta$ is an arc contained in $X \cap A$ with both endpoints in the same vertical line which should not exist by Claim 1.

Proof of Claim 3. Let K be a component of $X \backslash (A \cup A')$ and let $L \times I$ be the component of $(F \times I) \backslash (A \cup A')$ that contains it. Suppose that a circle γ of ∂K gets sent to a cylinder G of $\partial L \times I$ by degree 0.

Suppose first that γ is contained in a single component of $A \backslash A'$ or $A' \backslash A$. (This happens, in particular, if G is an annulus of A or A'.) Since γ has degree 0 in G, it must bound a disk in G, which can be assumed to have interior disjoint from X by replacing γ with an innermost loop if necessary. But this disk is a move 1 disk, so it should have been removed already in Step 2. See Fig. 20.

Now suppose that γ is not contained in a single component of $A \backslash A'$ or $A' \backslash A$. Since $p|_B$ is an embedding, γ must be disjoint from $\partial F \times I$. Again, γ bounds a disk in G. The vertical lines of $A \cap A'$ cut the disk into subdisks. Consider an outermost subdisk and the arc $\sigma \subset \gamma$ that forms half its boundary. This arc σ is contained in $X \cap (A \backslash A')$ (or in $X \cap (A' \backslash A)$) and both its endpoints lie in the same vertical line. Furthermore, σ cannot wrap all the way around an annulus of A (or A'). Claim 1 says that such arcs do not exist.

Proof of Claim 4. The following diagram commutes, and the map $\pi_1(X) \to \pi_1(F \times I)$ is injective. So it will suffice to show that $\pi_1(K) \to \pi_1(X)$ is injective.

$$\begin{array}{ccccc} \pi_1(K) & \longrightarrow & \pi_1(L \times I) & \longrightarrow & \pi_1(F \times I) \\ & \searrow & & \nearrow & \\ & & \pi_1(X) & & \end{array}$$

If $\pi_1(K) \to \pi_1(X)$ does not inject, then some loop in ∂K must bound a disk D in $X \backslash K$. The following argument shows that D contains at least one loop ω of $X \cap A$ or $X \cap A'$.

Since ∂D bounds D on one side and K on the other, ∂D must be disjoint from ∂X. So ∂D must intersect A or A'. Assume without loss of generality that X intersects A. Possibly ∂D itself is contained in $X \cap A$. In this case set $\omega = \partial D$. Otherwise, take an arc of $\partial D \cap (X \cap A)$ and let ω be the

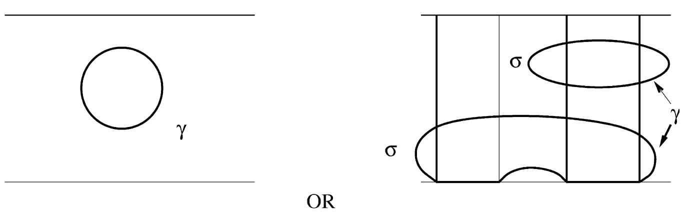


Fig. 20. Suppose a circle γ of ∂K gets sent to a cylinder of $\partial L \times I$ with degree 0.

component of $X \cap A$ that contains it. Notice that ω must lie in D, since it cannot intersect $\mathrm{int}(K)$. Since $D \cap \partial X = \emptyset$, ω is a closed loop rather than an arc.

Since $\omega \subset D$, ω is null-homotopic in $F \times I$. Since each annulus in A π_1-injects into $F \times I$, ω must bound a disk E in A. If $E \cap A' = \emptyset$, then E is a move 1 disk, which should already have been removed. If $E \cap A' \neq \emptyset$, then $\partial E \cap (A \backslash A')$ will contain an arc whose endpoints lie in the same vertical line of $A \cap A'$, but does not wrap all the way around A. Claim 1 says that such arcs do not exist.

Proof of Claim 5. From Claim 4, the map $\pi_1(K) \to \pi_1(L \times I)$ is injective. Therefore, it is possible to write $\pi_1(L \times I)$ either as an amalgamated product or as an HNN extension over $\pi_1(K)$, depending on whether or not K separates L. In fact, K separates $L \times I$, because the inclusion of $L \times 1$ into $L \times I$ induces an isomorphism of fundamental groups, which could not happen if $\pi_1(L \times I)$ were an HNN extension. Let M_1 be the component of $(L \times I) \backslash K$ that contains $L \times 1$, and let M_2 be the other component. Then the composition of maps

$$\pi_1(L \times 1) \to \pi_1(M_1) \to \pi_1(L \times I)$$

is an isomorphism. So $\pi_1(M_1) \to \pi_1(L \times I)$ is surjective. But

$$\pi_1(L \times I) = \pi_1(M_1) *_{\pi_1(K)} \pi_1(M_2)$$

so the injection $\pi_1(K) \to \pi_1(M_2)$ must be an isomorphism. By the h-cobordism theorem [2, Theorem 10.2], $(M_2, K) \cong (K \times I, K \times 1)$.

Now there are two possibilities: either $L \times 0 \subset M_1$ or $L \times 0 \subset M_2$. If $L \times 0 \subset M_1$, then $\partial M_2 \backslash K$ is a subset of $\partial L \times I$. So K is parallel to a subsurface of $\partial L \times I$, and therefore must be a disk or an annulus. By Claim 3, K must be a boundary parallel annulus.

If $L \times 0 \subset M_2$, then it follows as above that the composition

$$\pi_1(L \times 0) \to \pi_1(M_2) \to \pi_1(L \times I)$$

is an isomorphism. Therefore, $\pi_1(M_2) \to \pi_1(L \times I)$ is surjective. Also, $\pi_1(M_2) \to \pi_1(L \times I)$ is injective since

$$\pi_1(L \times I) = \pi_1(M_1) *_{\pi_1(K)} \pi_1(M_2)$$

So $\pi_1(M_2) \to \pi_1(L \times I)$ is an isomorphism. In addition, $\pi_1(K) \to \pi_1(M_2)$ is an isomorphism, from above. Therefore, $\pi_1(K) \to \pi_1(L \times I)$ is an isomorphism, and the claim is proved.

Proof of Claim 6. Let K be any annulus component of $X \backslash (A \cup A')$, let $L \times I$ be the corresponding component of $(F \times I) \backslash (A \cup A')$, and suppose that both circles of ∂K go to the same cylinder G of $\partial L \times I$. Since both circles have degree ± 1 in $\partial L \times I$, they bound an annulus W in $\partial L \times I$. Notice that W is disjoint from $\partial F \times I$, since by assumption, $p|_B : B \to \partial F$ is an embedding. If W is entirely contained in A or A', then W is a move 3 annulus, which should have been removed in Step 2. So W must consist of alternating rectangles of A and A'.

The union $W \cup K$ forms a torus, which is embedded in $L \times I$ since $\mathrm{int}(K) \cap (\partial L \times I) = \emptyset$. Since K is an annulus and $\pi_1(K) \to \pi_1(L \times I)$ injects by Claim 5, the argument in the explanation of Move 3 can be used to show that $W \cup K$ bounds a solid torus T in $L \times I$ and that the loops of ∂W are longitudes of this torus.

Construct a move 2 disk as follows. Start with a vertical arc α of $W \cap (A \cap A')$. Connect its endpoints with an embedded arc σ of K so that $\alpha \cup \sigma$ is null homotopic in T. This can be done because each component of ∂K generates $\pi_1(T)$, so it is possible to replace a poor choice of σ by one that wraps around ∂K an additional number of times and get a good choice of σ. Now $\alpha \cup \sigma$ bounds an embedded disk E in T, which can be assumed to have interior disjoint from X by replacing it with a subdisk if necessary. In addition, $E \cap A = E \cap A' = \alpha$, so E is a move 2 disk. But Step 2 already eliminated all disks of this form.

Step 3: Recall that $p: F \times I \to F$ is the projection map. In this step, X is isotoped relative to ∂X so that for any component γ of $X \cap (A \backslash A')$ or of $X \cap (A' \backslash A)$, $p|_\gamma$ is a local homeomorphism onto its image. The following discussion considers arc components first and loop components next.

Take any arc μ of $X \cap (A \backslash A')$. If $\text{int}(\mu) \cap \partial X \neq \emptyset$, then $\mu \subset \partial X$ by Claim 2. The map p is already injective on arcs of $\partial X \cap A$ and $\partial X \cap A'$, so μ can be left alone. If $\text{int}(\mu) \cap \partial X = \emptyset$, then isotope μ relative to $\partial \mu$ to an embedded arc μ' in $A \backslash A'$ such that $p|_{\mu'}$ is a homeomorphism onto its image. This is possible because by Claim 1, either μ wraps all the way around an annulus of A or else the endpoints of μ lie in distinct vertical lines of $A \cap A'$. The isotopy can be done in such a way that μ' does not intersect any other arcs of $X \cap (A \backslash A')$ that may lie in the same vertical rectangle. Perturb X in a neighborhood of μ to extend the isotopy on μ.

Pick another arc of $X \cap (A \backslash A')$ and repeat the procedure. When all the arcs of $X \cap (A \backslash A')$ have been pulled taut, continue with arcs of $X \cap (A' \backslash A)$.

Next, consider any loop λ of $X \cap A$ that does not intersect A'. By Claim 3, the loop λ has degree 1 in A, so it can be isotoped to a loop λ' such that $p|_\lambda$ is a homeomorphism onto its image. As before, the isotopy can be done in such a way that λ' does not intersect any other loops of $X \cap A$, and the isotopy can be extended to a neighborhood of λ in X. Loops of $X \cap A'$ that are disjoint from A can be isotoped similarly.

The following argument shows that at this stage, for any component K of $X \backslash (A \cup A')$, $p|_{\partial K}$ is a local homeomorphism except along vertical lines of $K \cap (A \cap A')$. Every point of ∂K is either a point on the interior of an arc of $\partial X \cap (\partial F \times I)$, a point of $X \cap A \cap (\partial F \times I)$ or of $X \cap (A' \cap (\partial F \times I))$, a point of $A \cap A'$, or a point on the interior of an arc or loop of $X \cap (A \backslash A')$ or $X \cap (A' \backslash A)$. The map $p|_{\partial K}$ is a local homeomorphism near the first type of point by assumption. It is a local homeomorphism near the second type of point by Claim 2. It is a local homeomorphism near the third type of point (away from vertical lines) because X is pseudo-transverse to A and A'. Finally, $p|_{\partial K}$ is a local homeomorphism near the fourth type of point by the work done in Step 3.

Step 4: In this final step, X is isotoped so that p is locally injective everywhere except at vertical twist lines. The argument uses a fact about maps between surfaces.

Fact. *Let $f:(G, \partial G) \to (H, \partial H)$ be a map between surfaces such that $f|\partial G$ is a local homeomorphism and $f_*: \pi_1(G) \to \pi_1(H)$ is injective. Then there is a homotopy $f_\tau:(G, \partial G) \to (H, \partial H)$, with $\tau \in I, f_0 = f$, and $f_\tau|_{\partial G} = f_0|_{\partial G}$ for all τ, such that either* (1) *or* (2) *holds*:

1. *G is an annulus or Mobius band and $f_1(G) \subset \partial H$, or*
2. *$f_1: G \to H$ is a covering map.*

The case when G is a disk is easy to verify; all other cases are covered by [2, Theorem 13.1].

Pick a component K of $X\backslash(A\cup A')$, and let $L\times I$ be the corresponding component of $(F\times I)\backslash(A\cup A')$. Assume first that ∂K does not contain any vertical arcs of $A\cap A'$. By Claim 4, the map $\pi_1(K)\to\pi_1(L\times I)$ is injective. Since $p_*:\pi_1(L\times I)\to\pi_1(L)$ is an isomorphism, the composition $(p|_K)_*:\pi_1(K)\to\pi_1(L)$ is injective. Furthermore, by the discussion following Step 3, $p|\partial K:\partial K\to\partial L$ is a local homeomorphism on ∂K. Therefore, there is a homotopy $f_\tau:K\to L$ with $f_0=p|_K$ and $f_\tau|_{\partial K}=p|_{\partial K}$ such that either K is an annulus or Mobius band and $f_1(K)\subset\partial L$, or f_1 is a covering map.

If K is a Mobius band and $f_1(K)\subset\partial L$, then $f_0(\partial K)$ is a degree 2 loop in ∂L which is impossible since $\partial K\subset\partial L\times I$ is embedded. By Claim 6, it not possible for K to be an annulus and $f_1(K)$ to be a subset of ∂L. Therefore, f_1 must be a covering map. By Claims 5 and 6, the map $\pi_1(K)\to\pi_1(L\times I)$ is surjective. So the map $(p|_K)_*:\pi_1(K)\to\pi_1(L)$ is surjective. Therefore, f_1 must be a homeomorphism.

If ∂K contains vertical arcs of $A\cap A'$, then $p|_{\partial K}$ is still very close to a local homeomorphism – in fact, if $\hat{K}$ is the surface obtained by collapsing each vertical arc of $\partial K\cap(A\cap A')$ to a point, then $p|_K$ factors through a map $\hat{p}:\hat{K}\to L$ such that $\hat{p}|_{\partial K}$ is a local homeomorphism. So it is still possible to homotope $p|_K$ relative to ∂K to a map f_1 such that $f_1|_{\mathrm{int}(K)}$ is a homeomorphism.

The homotopy f_τ of K relative to ∂K in L induces a homotopy of K relative to ∂K in $L\times I$ which keeps the vertical coordinate of each point of K constant and changes its horizontal coordinate according to f_τ. At the end of the homotopy, the new surface K' itself will be embedded in $F\times I$, since $p|_{K'}$ is a homeomorphism.

Homotope X as described above for every component of $X\backslash(A\cup A')$. After doing this, each component of $X\backslash(A\cup A')$ is embedded in $F\times I$. But two components K'_1 and K'_2 in the same piece $L\times I$ of $(F\times I)\backslash(A\cup A')$ might intersect each other. If that happens, an additional homotopy of K can be tacked on to clear up the problem, as follows.

Let $K_1\subset L\times I$ and $K_2\subset L\times I$ be two components of $X\backslash(A\cup A')$ before the homotopy of Step 4 and let K'_1 and K'_2 be these components after the homotopy. Suppose that K'_1 intersects the component K'_2. Since K_1 and K_2 are disjoint, the component of $(L\times I)\backslash K_1$ that contains $L\times 0$ either contains all of K_2 or else contains no part of K_2. Therefore, either all loops of K_1 lie above the corresponding loops of K_2 or else they all lie below the corresponding loops. Since the homotopies of K_1 and K_2 did not move ∂K_1 and ∂K_2, the same statement holds for loops of $\partial K'_1$ and $\partial K'_2$. Therefore, it is possible to alter the vertical coordinates of K'_1 and K'_2 to make the two surfaces parallel, so that K'_1 lies entirely above K'_2 or entirely below K'_2. Therefore all components of $X\backslash(A\cup A')$ can be assumed disjoint after the homotopy of Step 4.

In its final position, X is embedded in $F\times I$. A theorem of Waldhausen [5, Corollary 5.5] states that if G and H are incompressible surfaces embedded in an irreducible 3-manifold, and there is a homotopy from G to H that fixes ∂G, then there is an isotopy from G to H that fixes ∂G. Therefore, the above sequence of homotopies can be replaced by an isotopy. $\square$

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References

[1] J. Hass, P. Scott, Intersections of curves on surfaces, Israel Journal of Mathematics 51 (1985) 90–120.
[2] J. Hempel, 3-Manifolds, Annals of Mathematics Studies, vol. 86, Princeton University Press, Princeton, NJ, 1976.
[3] W. Jaco, U. Oertel, An algorithm to decide if a 3-manifold is a haken manifold, Topology 23 (2) (1984) 195–209.
[4] W.S. Massey, Algebraic Topology: An Introduction, Springer, Berlin, 1967.
[5] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Annals of Mathematics 87 (1968) 56–88.