The chromatic difference sequence of the Cartesian product of graphs: Part II

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Abstract


This paper is a continuation of our earlier paper under the same title. We prove that the normalized chromatic difference sequences of the Cartesian powers of a Cayley graph on a finite Abelian group are stable; and that if the chromatic difference sequence of a Cayley graph on a finite Abelian group is achievable, then the chromatic difference sequences of its Cartesian powers are achievable too.

1. Introduction

We only consider simple undirected graphs in this paper, we omit motivations, most of the definitions and notations since this paper is a continuation of our earlier paper under the same title. See [3] for details. We concern with the chromatic difference sequence \( \text{cds}(G) = (r_1, r_2, \ldots, r_n) \) and the normalized chromatic difference sequence \( \text{ncds}(G) = \text{cds}(G)/|V(G)| \) introduced in [1]. We also concern with the Cartesian product \( G \boxtimes H \) of two graphs \( G \) and \( H \) whose chromatic numbers are equal, and the Cartesian power \( G^k \) of a graph \( G \). We are going to generalize the results about the cds and ncds of the Cartesian powers of circulant graphs to the cds and ncds of the Cartesian powers of Cayley graphs on finite Abelian groups.

The sequence \( \text{cds}(G) = (r_1, r_2, \ldots, r_n) \) is said to be achievable for a graph \( G \) if there exists an \( n \)-coloring of \( V(G) \) with color classes \( V_1, V_2, \ldots, V_n \) such that \( r_i = |V_i| \), \( i = 1, 2, \ldots, n \). Every achievable cds is nonincreasing. Every nonincreasing sequence
is the achievable cds of some complete n-partite graph. For arbitrary graph, its cds
is not necessarily achievable.

Let \( \Gamma \) be a group with additive structure. Let \( N \) be a subset of \( \Gamma \) closed under inverse, not containing 0. The Cayley graph on \( \Gamma \) with symbol \( N \), denoted by \( G(\Gamma; N) \) is the graph whose vertices are the elements of \( \Gamma \) and whose edges join two vertices \( u \) and \( v \) if and only if \( u - v \in N \). A Cayley graph is any graph that is isomorphic to the Cayley graph of some group with some symbol.

The set of vertices \( V(G^k) \) of the Cartesian power \( G^k \) for the Cayley graph \( G = G(\Gamma; N) \) is \( \{ (x_1, x_2, \ldots, x_k) : x_i \in \Gamma \text{ for } i = 1, 2, \ldots, k \} \). Two vertices \( (x_1, \ldots, x_k) \) and \( (y_1, \ldots, y_k) \) are adjacent in \( G^k \) if and only if there exists \( j \) such that \( x_j - y_j \in N \), \( 1 \leq j \leq k \); and \( x_i = y_i \) for \( i = 1, \ldots, k \), \( i \neq j \). Write \( (x_1, \ldots, x_k) + (y_1, \ldots, y_k) = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k) \). Let \( I, J \subseteq V(G^k) \), \( a \in V(G^k) \) and \( b \in V(G) \). Define \( I + a = \{ x + a : x \in I \}, (b, I) = \{ (b, x) : x \in I \} \) and \( I + J = \{ x + y : x \in I, y \in J \} \). Without loss of generality we regard \( (b, I) \) as a subset of \( V(G^{k+1}) \).

Our results rely on the following four lemmas.

**Lemma 1.1** (Nonhomomorphism Lemma). If there exists a homomorphism from \( G \) to \( H \) and \( H \) is vertex transitive, then \( \text{ncds}(G) \) dominates \( \text{ncds}(H) \).


**Lemma 1.2** (Nonincreasing Theorem). For any two graphs \( G \) and \( H \) with same chromatic number, \( \text{ncds}(G) \) dominates \( \text{ncds}(G \square H) \), and \( \text{ncds}(G^k) \) dominates \( \text{ncds}(G^{k+1}) \) for \( k = 1, 2, \ldots \).

**Lemma 1.3.** \( G \square H \) is vertex transitive if both \( G \) and \( H \) are vertex transitive.

For the proofs of Lemmas 1.2 and 1.3, see [3].

The following lemma is easy to prove.

**Lemma 1.4.** The Cayley graph is vertex transitive.

2. Main results

**Theorem 2.1.** There exists a homomorphism from \( G^{k+1} \) to \( G^k \) if \( G \) is a Cayley graph on an Abelian group.

**Proof.** Let \( G = G(\Gamma; N) \) where \( \Gamma \) is an Abelian group and \( N \) is the symbol. For any vertex \( x = (x_1, x_2, \ldots, x_{k+1}) \in V(G^{k+1}) \), define \( \tau(x) = (x_1, x_2, \ldots, x_{k-1}, x_k - x_{k+1}) \in G^k \). Now suppose \( x = (x_1, \ldots, x_{k+1}) \) and \( y = (y_1, \ldots, y_{k+1}) \) are two adjacent vertices in \( G^{k+1} \). Then there exists \( j \in \{ 1, 2, \ldots, k+1 \} \), such that \( x_j - y_j \in N \) and such that for all \( i = 1, 2, \ldots, k+1 \), \( i \neq j \), \( x_i = y_i \). We shall check that \( \{ \tau(x), \tau(y) \} \in E(G^k) \) according to the following three cases.
Theorem 2.2. Let $G$ be a Cayley graph on a finite Abelian group. Then $\text{ncds}(G^k) = \text{ncds}(G)$ for all positive integers $k$.

**Proof.** For any $k$, $\text{ncds}(G^k)$ dominates $\text{ncds}(G^{k+1})$, by the Nonincreasing Theorem; and $\text{ncds}(G^{k+1})$ dominates $\text{ncds}(G^k)$ by the Nonhomomorphism Lemma. A Cayley graph is vertex transitive, and so Lemma 1.3 and Theorem 2.1 assures that Lemma 1.1 applies. Therefore

$$\text{ncds}(G^k) = \text{ncds}(G^{k+1}).$$

Theorem 2.3. Let $G$ be a Cayley graph on a finite Abelian group. If $\text{cds}(G)$ is achievable, then $\text{cds}(G^k)$ is achievable for all $k$.

**Proof.** Let $G = G(\Gamma; N)$ where $\Gamma$ is a finite Abelian group and $N$ is the symbol. We proceed by induction on $k$. Let $\text{cds}(G) = (r_1, r_2, \ldots, r_n)$. Since $\text{cds}(G)$ is assumed achievable, there exist $n$ subsets $I^1, \ldots, I^n$ of $V(G)$ satisfying the following three conditions:

1. $|I^i| = r_i$ for $i = 1, 2, \ldots, n$;
2. $I^1, \ldots, I^n$ are pairwise disjoint;
3. each $I^1 \cup \cdots \cup I^t$ $(t = 1, 2, \ldots, n)$ induces a maximum $t$-colorable subgraph in $G$, and the partition $I^1, \ldots, I^n$ achieves the cds of this subgraph.

Obviously, for any $j \in \mathbb{Z}$, $I^i + j$, $I^i + j$, $\ldots$, $I^n + j$ are pairwise disjoint and each is an independent set since $G$ is a Cayley graph on a finite Abelian group. Hence they still satisfy the above three conditions. Thus the sequence $\text{cds}(G)$ can be achieved by each of the $|\Gamma|$ partitions:

$$(I^1 + j), (I^2 + j), \ldots, (I^n + j) \quad \text{for } j \in \Gamma.$$  

Now as the induction hypothesis supposes

$$\text{cds}(G^{k-1}) = (|\Gamma|^{k-2}; r_1, |\Gamma|^{k-2}; r_2, \ldots, |\Gamma|^{k-2}; r_n)$$

is achieved by each of the $|\Gamma|$ partitions:

$$I^1_{k-1} + (j, e^{k-2}), I^2_{k-1} + (j, e^{k-2}), \ldots, I^n_{k-1} + (j, e^{k-2}) \quad \text{for } j \in \Gamma$$

(here $e^{k-2} = (0, \ldots, 0)$ has $k-2$ coordinates, $(j, e^{k-2}) = (j, 0, \ldots, 0)$ has $k-1$ coordinates) and that, for each $j \in \Gamma$, we have

$$(4) \ |I^i_{k-1} + (j, e^{k-2})| = |\Gamma|^{k-2} r_i \quad \text{for } i = 1, 2, \ldots, n;$$
(5) $I_1^{k-1} + (j, e^{k-2}), I_2^{k-1} + (j, e^{k-2}), \ldots, I_n^{k-1} + (j, e^{k-2})$ are pairwise disjoint; 
(6) $(I_1^{k-1} + (j, e^{k-2})) \cup (I_2^{k-1} + (j, e^{k-2})) \cup \cdots \cup (I_n^{k-1} + (j, e^{k-2}))$ is a maximum $t$-colorable subgraph in $G^{k-1}$ and 
$$I_1^{k-1} + (j, e^{k-2}), I_2^{k-1} + (j, e^{k-2}), \ldots, I_n^{k-1} + (j, e^{k-2})$$
achieves the css of this subgraph for $t = 1, 2, \ldots, n$.

Now we first construct the following $n$ subsets of vertices of $G^k$: 
$$I_i^k = \bigcup_{j \in \Gamma} (j, I_i^{k-1} + (j, e^{k-2})) \quad \text{for } i = 1, 2, \ldots, n.$$ 

Then we construct, for each $j \in \Gamma$, $n$ subsets of $V(G^k)$ as follows: 
$$I_1^j + (j, e^{k-1}), I_2^j + (j, e^{k-1}), \ldots, I_n^j + (j, e^{k-1}).$$

We need to check that these subsets of vertices satisfy conditions (4)-(6) with $k - 1$ replaced by $k$ for each $j \in \Gamma$.

Check (4):
$$|I_i^k| = \left| \bigcup_{j \in \Gamma} (j, I_i^{k-1} + (j, e^{k-2})) \right|$$
$$= \sum_{j \in \Gamma} |(j, I_i^{k-1} + (j, e^{k-2}))|$$
$$= |\Gamma||\Gamma|^{k-2}r_i = |\Gamma|^{k-1}r_i \quad \text{for } i = 1, 2, \ldots, n.$$ 

For any $j \in \Gamma$, we have 
$$|I_i^k + (j, e^{k-1})| = |I_i^k| = |\Gamma|^{k-1}r_i \quad \text{for } i = 1, 2, \ldots, n.$$ 

Check (5): Suppose that for some $j \in \Gamma$, $(I_a^k + (j, e^{k-1})) \cap (I_b^k + (j, e^{k-1})) \neq \emptyset$ for $0 \leq a, b \leq n, a \neq b$, and the first coordinate of the common element is $c+j$. Since 
$$I_a^k + (j, e^{k-1}) = \bigcup_{s \in \Gamma} (s, I_a^{k-1} + (s, e^{k-2})) + (j, e^{k-1}),$$
$$I_b^k + (j, e^{k-1}) = \bigcup_{s \in \Gamma} (s, I_b^{k-1} + (s, e^{k-2})) + (j, e^{k-1}),$$
we have $c \in \Gamma$ such that 
$$(I_a^{k-1} + (c, e^{k-2})) \cap (I_b^{k-1} + (c, e^{k-2})) \neq \emptyset,$$
= \emptyset,$$
which contradicts the induction hypothesis. Hence (5) is true.

Check (6): Every maximum $t$-colorable subgraph of $G^k$ induces a $t$-colorable subgraph in $\{j\} \times G^{k-1}$ for $t = 1, 2, \ldots, n$ by hypothesis. The size of a maximum $t$-colorable subgraph in $G^{k-1}$ is $|\Gamma|^{k-2}(r_1 + \cdots + r_t)$. So the size of a maximum $t$-colorable subgraph of $G^k$ is at most 
$$|\Gamma||\Gamma|^{k-2}(r_1 + \cdots + r_t) = |\Gamma|^{k-1}(r_1 + \cdots + r_t).$$
The Cartesian product of graphs

Now for any \( j \in \Gamma \),
\[
|\left(I_t^k + (j, e^{k-1})\right) \cup \cdots \cup \left(I_t^k + (j, e^{k-1})\right)| = |\Gamma|^{k-1}r_1 + \cdots + |\Gamma|^{k-1}r_t \\
= |\Gamma|^{k-1}(r_1 + \cdots + r_t).
\]

In order that (6) be true we only need to prove that for each \( j \in \Gamma \),
\[
\left(I_t^k + (j, e^{k-1})\right) \cup \cdots \cup \left(I_t^k + (j, e^{k-1})\right) \quad (1 \leq t \leq n)
\]
is a \( t \)-colorable subgraph in \( G^k \).

If we can prove that for \( j \in \Gamma \), each of the following
\[
I_1^k + (j, e^{k-1}), I_2^k + (j, e^{k-1}), \ldots, I_t^k + (j, e^{k-1}), \ldots, I_n^k + (j, e^{k-1})
\]
is an independent set in \( G^k \), then the above statement will surely be true for any \( t \), \( 1 \leq t \leq n \).

Take any two vertices from
\[
\left(I_t^k + (j, e^{k-1})\right) = \bigcup_{i \in \Gamma} (t, I_i^k-1 + (t, e^{k-2})) + (j, e^{k-1})
\]
\((i = 1, 2, \ldots, n, j \in \Gamma)\), say \( x \) and \( y \). If \( x \) and \( y \) belong to the same set \( (a+j, I_i^k-1 + (a, e^{k-2})) \) then they are not adjacent since any two vertices in \( I_i^k-1 \) are not adjacent. If \( x \in (a+j, I_i^k-1 + (a, e^{k-2})) \) and \( y \in (b+j, I_i^k-1 + (b, e^{k-2})) \) with \( a \neq b \), \( \{a+j, b+j\} \in E(G) \), then \( \{a, b\} \in E(G^k) \). Then \( \{a+j, b+j\} \in E(G^k) \) and \( a \leq b \in N \). Suppose \( \{x, y\} \in E(G^k) \). Then \( \left(I_t^k-1 + (a, e^{k-2})\right) \cap \left(I_t^k-1 + (b, e^{k-2})\right) \neq \emptyset \) and hence there is a \((x_1, \ldots, x_{k-1}) \in I_t^k-1 \) and \((y_1, \ldots, y_{k-1}) \in I_t^k-1 \) such that \( x_1+a = y_1+b \) and \( x_i = y_i \) \((i = 2, \ldots, k-1) \). So \( x_1 - y_1 = b-a \in N, x_i = y_i \) \((i = 2, \ldots, k-1) \), which contradicts the fact that \( I_t^k-1 \) is an independent set. If \( x \in (a+j, I_i^k-1 + (a, e^{k-2})) \) and \( y \in (b+j, I_i^k-1 + (b, e^{k-2})) \) with \( a \neq b \), \( \{a+j, b+j\} \in E(G) \), then there is no edge between \( x \) and \( y \). □

The following seems reasonable:

**Conjecture.** If \( G \) is a Cayley graph on a finite Abelian group, then \( \text{cds}(G) \) is non-increasing and achievable.

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**References**