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Note

Distance Degree Regular Graphs

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In this note, inequalities between the distance degrees of distance degree regular graphs and to characterize the graphs when one of the equalities holds are proved.

1. INTRODUCTION

In this note, we shall consider only finite undirected graphs without loops and multi-edges.

A graph G with the vertex set V and the edge set E is denoted by $G=(V,E)$. Then $d(u,v)$ denotes the distance between two vertices u and v. The diameter of G is denoted by $d(G)$. For every vertex u and nonnegative integer i, we define

$$\Gamma_i(u)=\{v\in V; d(u,v)=i\}$$
$$\tilde{\Gamma}_i(u)=\{v\in V; d(u,v)\leqslant i\}.$$

For any finite set A, $|A|$ denotes the number of the elements of A.

We call a graph G *distance degree regular*, if the relation

$$|\Gamma_i(u)|=|\Gamma_i(v)|$$

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holds for any vertices u, v and nonnegative integer i. In this case, the ith distance degree of a distance degree regular graph G is the number $|\Gamma_i(u)|$, which will be denoted by $k_i(G)$ or simply by k_i. We remark that if G is distance degree regular, then

$$|\tilde{\Gamma}_i(u)| = \sum_{j=0}^{i} k_j. \tag{1}$$

holds.

We shall show

THEOREM 1. *Let G be a connected and distance degree regular graph and $d(G) \geqslant 2$. Then, for every integer r, $1 \leqslant r < d(G)$, we have*

$$3k_r(G) \geqslant 2(k_1(G) + 1).$$

THEOREM 2. *Let G be a connected and distance degree regular graph and $d(G) \geqslant 2$. If*

$$3k_r(G) = 2(k_1(G) + 1) \tag{2}$$

holds for some integer r, $1 \leqslant r < d(G)$, we have

$$G \cong C_n[K_m],$$

where $n = 2d(G) + 1$ or $2d(G)$ and $m = k_r(G)/2$.

For the definition of the composition $G_1[G_2]$ of two graphs G_1 and G_2, see Harary [1 p. 22]. To prove the above theorems, we shall use the following simple lemma.

LEMMA 3. *Let a, b, and c be vertices of G such that $d(a, b) = n$, $d(b, c) = m$ and $d(a, c) = n + m$, then we have*

$$\tilde{\Gamma}_m(a) \cup \tilde{\Gamma}_n(c) \subset \tilde{\Gamma}_{m+n}(b).$$

In particular

$$|\tilde{\Gamma}_{m+n}(b)| \geqslant |\tilde{\Gamma}_m(a)| + |\tilde{\Gamma}_n(c)| - |\tilde{\Gamma}_m(a) \cap \tilde{\Gamma}_n(c)|. \tag{3}$$

The theorems are obvious for $r = 1$, so we assume that $G = (V, E)$ is connected and distance degree regular and $d(G) > 2$ and $1 < r < d(G)$ in the rest of this note.

2. Proof of Theorem 1

For every $(u, v) \in E$ and positive integers i and j, the following hold:

$$\text{If } |i-j| \geqslant 2 \text{ then } \Gamma_i(u) \cap \Gamma_j(v) = \varnothing. \tag{4}$$

$$|\Gamma_{i+1}(u) \cap \Gamma_i(v)| + |\Gamma_i(u) \cap \Gamma_i(v)| + |\Gamma_{i-1}(u) \cap \Gamma_i(v)| = k_i. \tag{5}$$

Since

$$|\Gamma_1(u) \cap \Gamma_0(v)| = |\Gamma_0(u) \cap \Gamma_1(v)| = 1,$$

it follows from (5) by induction on i that

$$|\Gamma_{i+1}(u) \cap \Gamma_i(v)| = |\Gamma_i(u) \cap \Gamma_{i+1}(v)|. \tag{6}$$

By (1) and (3), we also get, if $d(u, v) = r$, then

$$|\tilde{\Gamma}_{r-1}(u) \cap \tilde{\Gamma}_1(v)| = |\Gamma_{r-1}(u) \cap \Gamma_1(v)| \geqslant 1 + k_1 - k_r. \tag{7}$$

Now we choose two vertices u and z such that $d(u, z) = r + 1$. Let $(u, v, w, \ldots, z)$ be one of the shortest paths from u to z. By (7),

$$|\Gamma_{r-1}(v) \cap \Gamma_1(z)| \geqslant 1 + k_1 - k_r.$$

Since

$$\Gamma_{r-1}(v) \cap \Gamma_1(z) \subset \Gamma_r(u) \cap \Gamma_{r-1}(v), \tag{8}$$

we have

$$|\Gamma_r(u) \cap \Gamma_{r-1}(v)| \geqslant 1 + k_1 - k_r. \tag{9}$$

We also have

$$|\Gamma_{r+1}(u) \cap \Gamma_r(v)| + |\Gamma_r(u) \cap \Gamma_r(v)| \geqslant 1 + k_1 - k_r. \tag{10}$$

To prove this inequality, we consider three cases.

Case i. There exists a vertex $x \in \Gamma_1(z) \cap \Gamma_{r+2}(u) \cap \Gamma_{r+1}(v)$. Since $d(w, x) = r$, (7), and

$$\Gamma_{r-1}(w) \cap \Gamma_1(x) \subset \Gamma_{r+1}(u) \cap \Gamma_r(v),$$

we have

$$|\Gamma_{r+1}(u) \cap \Gamma_r(v)| \geqslant 1 + k_1 - k_r.$$

Case ii. There exists a vertex $x \in \Gamma_1(z) \cap \Gamma_{r+1}(u) \cap \Gamma_{r+1}(v)$. Since $d(w, x) = r$, we have

$$|\Gamma_{r-1}(w) \cap \Gamma_1(x)| \geqslant 1 + k_1 - k_r,$$

by (7). We also get

$$\Gamma_{r-1}(w) \cap \Gamma_1(x) \subset (\Gamma_{r+1}(u) \cap \Gamma_r(v)) \cup (\Gamma_r(u) \cap \Gamma_r(v))$$

for this case. Hence (10) holds again.

Case iii. There exists no vertex in $\Gamma_1(z)$ belonging to $(\Gamma_{r+2}(u) \cup \Gamma_{r+1}(u)) \cap \Gamma_{r+1}(v)$. Thus

$$\begin{aligned} \tilde{\Gamma}_1(z) &= \left(\bigcup_{i=r}^{r+2} \Gamma_i(u)\right) \cap \left(\bigcup_{j=r-1}^{r+1} \Gamma_j(v)\right) \cap \tilde{\Gamma}_1(z) \\ &\subset (\Gamma_{r+1}(u) \cap \Gamma_r(v)) \cup \Gamma_r(u). \end{aligned}$$

Hence

$$1 + k_1 \leqslant |\Gamma_{r+1}(u) \cap \Gamma_r(v)| + k_r.$$

Theorem 1 follows from (5), (9), and (10).

3. Proof of Theorem 2

Let u and x be any two vertices such that $d(u, x) = r + 1$ and $(u, v, w, \ldots, x)$ be one of the shortest paths from u to x. Condition (2) implies that $1 + k_1 - k_r = k_r/2$, and forces equality in (8) and (9), so we have

$$\tilde{\Gamma}_{r-1}(v) \cap \tilde{\Gamma}_1(x) = \Gamma_r(u) \cap \Gamma_{r-1}(v) \tag{11}$$

$$|\Gamma_r(u) \cap \Gamma_{r-1}(v)| = k_r/2. \tag{12}$$

Condition (2) also forces equality in (7). If we write Eq. (7) in the form of (3) (from which it was derived) for the triple v, w, and x, we get

$$\tilde{\Gamma}_{r-1}(v) \cup \tilde{\Gamma}_1(x) = \tilde{\Gamma}_r(w), \tag{13}$$

For any $x, y \in V$, define $x \sim y$ if and only if

$$\tilde{\Gamma}_1(x) = \tilde{\Gamma}_1(y).$$

This is an equivalence relation. We show that each equivalence class spans a complete graph $K_{k_r/2}$, and that the quotient $G/\sim$ is isomorphic to a cycle C_n with $n = 2d(G)$ or $2d(G) + 1$.

Suppose $u \in V$, $v \in \tilde{\Gamma}_1(u)$, $v \not\sim u$, and $x \in \Gamma_r(v) \cap \Gamma_{r+1}(u)$. If $y \sim x$, then $y \in \Gamma_r(v) \cap \Gamma_{r+1}(u)$. Conversely, if $y \in \Gamma_r(v) \cap \Gamma_{r+1}(u)$, then (11) and (13) imply that $y \sim x$. Thus $\Gamma_r(v) \cap \Gamma_{r+1}(u)$ is a single equivalence class. Moreover any equivalence class is of this form for some u and v (given $x \in V$ we can choose a vertex $u \in \Gamma_{r+1}(x)$ and a path $(u, v,..., x)$ of length $r+1$ from u to x; then $u \not\sim v$ and the equivalence class containing x is precisely $\Gamma_r(v) \cap \Gamma_{r+1}(u)$.)

We now show that $|\Gamma_r(v) \cap \Gamma_{r+1}(u)| = k_r/2$ or, equivalently, $\Gamma_r(v) \cap \Gamma_r(u) = \varnothing$ (by (5) and (12)). Let $(u, v, w,..., z, x)$ be one of the shortest path from u to x. For any $a \in \Gamma_{r-1}(u) \cap \Gamma_r(v)$ ($\neq \phi$ by (6)), we have

$$d(w, a) = r + 1. \tag{14}$$

by (13) and $d(w, a) \leqslant r + 1$. Hence there exists a path of length $r+1$ of form $(w, v, u,..., a)$ from w to a. By (13) we have

$$\tilde{\Gamma}_{r-1}(v) \cup \tilde{\Gamma}_1(a) = \tilde{\Gamma}_r(u).$$

This implies $\Gamma_r(u) \cap \Gamma_r(v) \subset \tilde{\Gamma}_1(\mathrm{a})$. Interchanging the role of u and v, we have $d(b, z) = 1$ for any $b \in \Gamma_r(u) \cap \Gamma_r(v)$. Thus

$$d(w, a) \leqslant d(a, b) + d(b, z) + d(z, w) \leqslant r.$$

This contradicts (14). Hence $\Gamma_r(u) \cap \Gamma_r(v) = \varnothing$, so each equivalence class has size $k_r/2$ as claimed. Finally, since $k_1 = 2(k_r/2) + (k_r/2 - 1)$ the quotient graph has degree 2.

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Reference

1. F. Harary, "Graph Theory," Addison–Wesley, Reading, Mass., 1969.