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Coefficient Estimates

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An approach to the Bieberbach conjecture, due to Milin [1], aims at the estimation of logarithmic coefficients. A new estimation theory of these coefficients is obtained.

Substitution groups of formal power series lie in the background of the estimation theory. Inequalities arise from a subordination concept due to Littlewood [2]. They are used to construct Hilbert spaces of analytic functions which generalize those of the theory of square summable power series [3]. The construction of extension spaces [4] in that theory suggests a new interpretation of the Grunsky transformation [5]. Estimates result from an expansion derived from the Löwner equation, which is a continuous analogue of the subordination concept.

If $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ are formal power series with constant coefficient zero, the composition $f(g(z))$ is meaningful as a formal power series. The set of all power series with constant coefficient zero and with coefficient of z nonzero forms a group under substitution. A normal subgroup is given by the series with coefficient of z equal to one and coefficient of z^n zero for $1 < n < r$, r a given positive integer. The commutative subgroups of the group are computable.

THEOREM 1. *Let $f(z) = \sum a_n z^n$ be a given series with constant coefficient zero and coefficient of z equal to one, which is not equal to z, and let r be the least integer greater than one such that a_r is not zero. If c is a given complex number, a unique power series $g(z) = \sum b_n z^n$ exists, with constant coefficient zero and coefficient of z equal to one, such that*

$$f(g(z)) = g(f(z))$$

and such that $b_r = c$ and $b_n = 0$ when $1 < n < r$.

These are all the power series which commute with $f(z)$ under substitution.

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THEOREM 2. *Let $f(z)=\sum a_n z^n$ and $g(z)=\sum b_n z^n$ be power series with constant coefficient zero and with coefficient of z equal to one such that*

$$f(g(z))=g(f(z)).$$

If $f(z)$ is not equal to z and if r is the least positive integer greater than one such that a_r is not zero, then $b_n=0$ for $1<n<r$.

A differential equation is satisfied.

THEOREM 3. *Let $f(z)$ be a power series with constant coefficient zero and coefficient of z equal to one, which is not equal to z, and let r be the least integer greater than one such that the coefficient of z^r in $f(z)$ is not zero. For each complex number w, a unique power series $f(w,z)$ exists, with constant coefficient zero, with coefficient of z equal to one, and with coefficient of z^r equal to w, such that*

$$f(f(w,z))=f(w,f(z)).$$

The identity

$$f(a+b,z)=f(a,f(b,z))$$

holds for all complex numbers a and b. The coefficient of z^n in $f(w,z)$ is a polynomial of degree less than n in w for every n. A unique power series $\varphi(z)$ exists such that the differential equation

$$\partial/\partial w f(w,z)=\varphi(z)\,\partial/\partial z f(w,z)$$

is satisfied. The coefficient of z^r in $\varphi(z)$ is one and the coefficient of z^n in $\varphi(z)$ is zero when $n<r$.

The differential equation can be taken as the starting point for the construction of the group.

THEOREM 4. *Let r be an integer greater than one and let $\varphi(z)$ be a power series with coefficient of z^r equal to one and coefficient of z^n equal to zero when $n<r$. Then a unique power series $f(w,z)$ in z exists for each complex number w such that the coefficient of z^n is a polynomial of degree less than n in w and such that the differential equation*

$$\partial/\partial w f(w,z)=\varphi(z)\,\partial/\partial z f(w,z)$$

is satisfied with the initial condition $f(0, z) = z$. The coefficient of z^n in $f(w, z)$ is zero when $1 < n < r$ and the coefficient of z^r is w. The identity

$$f(a + b, z) = f(a, f(b, z))$$

holds for all complex numbers a and b.

A nonlinear equation is a consequence of the group property.

THEOREM 5. *In Theorem 4, a power series $f(z)$ with constant coefficient zero and coefficient of z equal to one is equal to $f(w, z)$ for some complex number w if, and only if,*

$$\varphi(z)\, \partial/\partial z f(z) = \varphi(f(z)).$$

The structure of commuting subgroups is more complicated for power series whose coefficient of z is not equal to one, but the special case serves as a useful guide, especially in matters of notation.

THEOREM 6. *If $\varphi(z)$ is a power series with constant coefficient one, then a unique power series $f(w, z)$ in z with constant coefficient zero exists such that the coefficient of z^n is a polynomial of degree at most n in w and such that the differential equation*

$$w\, \partial/\partial w f(w, z) = z\, \varphi(z)\, \partial/\partial z f(w, z)$$

is satisfied with the initial condition $f(1, z) = z$. The identity

$$f(ab, z) = f(a, f(b, z))$$

holds for all complex numbers a and b.

A nonlinear differential equation again enters as a result of the group property.

THEOREM 7. *In Theorem 6, a power series $f(z)$ with constant coefficient zero is equal to $f(w, z)$ for some complex number w if, and only if,*

$$z\, \varphi(z)\, \partial/\partial z f(z) = f(z)\, \varphi(f(z)).$$

The notation $\mathscr{C}(z)$ is used for the space of square summable power series $f(z) = \sum a_n z^n$,

$$\|f(z)\|^2_{\mathscr{C}(z)} = \sum |a_n|^2.$$

If $B(z)$ is a power series which converges in the unit disk and represents a function which is bounded by one in the disk, then the inequality

$$\|B(z)f(z)\|_{\mathscr{C}(z)} \leqslant \|f(z)\|_{\mathscr{C}(z)}$$

holds for every element $f(z)$ of $\mathscr{C}(z)$. If $f(z)$ is an element of $\mathscr{C}(z)$, define

$$\|f(z)\|^2_{\mathscr{H}(B)} = \sup[\|f(z) + B(z)\,g(z)\|^2_{\mathscr{C}(z)} - \|\,g(z)\|^2_{\mathscr{C}(z)}]$$

where the least upper bound is taken over all elements $g(z)$ of $\mathscr{C}(z)$. Then the set of elements of $\mathscr{C}(z)$ of finite $\mathscr{H}(B)$-norm is a Hilbert space $\mathscr{H}(B)$ in the $\mathscr{H}(B)$-norm. The space is contained in $\mathscr{C}(z)$ and the inclusion does not increase norms. The series

$$[1 - B(z)\,\bar{B}(w)]/(1 - z\bar{w})$$

belongs to the space for every point w in the unit disk and the identity

$$f(w) = \langle f(z), [1 - B(z)\,\bar{B}(w)]/(1 - z\bar{w})\rangle_{\mathscr{H}(B)}$$

holds for every element $f(z)$ of the space. If $f(z)$ is in $\mathscr{H}(B)$ and if $g(z)$ is in $\mathscr{C}(z)$, then $h(z) = f(z) + B(z)\,g(z)$ belongs to $\mathscr{C}(z)$ and the inequality

$$\|\,h(z)\|^2_{\mathscr{C}(z)} \leqslant \|f(z)\|^2_{\mathscr{H}(B)} + \|\,g(z)\|^2_{\mathscr{C}(z)}$$

is satisfied. Every element $h(z)$ of $\mathscr{C}(z)$ has a unique minimal decomposition for which equality holds.

The extension space $\mathscr{D}(B)$ of the space $\mathscr{H}(B)$ is a Hilbert space whose elements are pairs of power series with complex coefficients. By definition the pair $(f(z), g(z))$ belongs to $\mathscr{D}(B)$ if $f(z)$ belongs to $\mathscr{H}(B)$ and if $g(z) = \sum a_n z^n$ where

$$z^{n+1}f(z) - B(z)\,(a_0 z^n + \cdots + a_n)$$

belongs to $\mathscr{H}(B)$ for every nonnegative integer n and if the sequence of norms

$$\|z^{n+1}f(z) - B(z)(a_0 z^n + \cdots + a_n)\|^2_{\mathscr{H}(B)} + |a_0|^2 + \cdots + |a_n|^2$$

is bounded. The sequence is nondecreasing. Its limit is taken as the definition of $\|(f(z), g(z))\|^2_{\mathscr{D}(B)}$.

The transformation which takes $(f(z), g(z))$ into $f(z)$ is a partial isometry of $\mathscr{D}(B)$ onto $\mathscr{H}(B)$. The pair

$$([1 - B(z)\,\bar{B}(w)]/(1 - z\bar{w}), [B^*(z) - \bar{B}(w)]/(z - \bar{w}))$$

belongs to the space when w is in the unit disk. The identity

$$f(w) = \langle (f(z), g(z)), \\ ([1 - B(z)\bar{B}(w)]/(1 - z\bar{w}), [B^*(z) - \bar{B}(w)]/(z - \bar{w}))\rangle_{\mathscr{D}(B)}$$

holds for every element $(f(z), g(z))$ of the space. The transformation which takes $(f(z), g(z))$ into $(g(z), f(z))$ is an isometry of $\mathscr{D}(B)$ onto $\mathscr{D}(B^*)$, where $B^*(z) = \sum \bar{B}_n z^n$ if $B(z) = \sum B_n z^n$.

If r is a given nonnegative integer, let $\mathscr{C}_r(z)$ be the space of polynomials of degree at most r in the metric of $\mathscr{C}(z)$. Two power series $f(z)$ and $g(z)$ are said to be r-equivalent if the coefficient of z^n in $f(z)$ is equal to the coefficient of z^n in $g(z)$ for $n = 0,\ldots, r$. Every power series is r-equivalent to a unique polynomial of degree at most r. The $\mathscr{C}_r(z)$-norm of an arbitrary power series is defined as the $\mathscr{C}_r(z)$-norm of the r-equivalent polynomial of degree at most r. Assume that $B(z)$ is a power series which converges in the unit disk and represents a function which is bounded by one in the disk. Then the inequality

$$\|B(z)f(z)\|_{\mathscr{C}_r(z)} \leqslant \|f(z)\|_{\mathscr{C}_r(z)}$$

holds for every power series $f(z)$. The inequality allows a generalization of the space $\mathscr{H}(B)$.

If $f(z)$ is a power series, define

$$\|f(z)\|^2_{\mathscr{H}_r(B)} = \sup[\|f(z) + B(z)g(z)\|^2_{\mathscr{C}_r(z)} - \|g(z)\|^2_{\mathscr{C}_r(z)}]$$

where the least upper bound is taken over all power series $g(z)$. Then the $\mathscr{H}_r(B)$-norms of any two r-equivalent power series are equal. The set of polynomials of degree at most r which have finite $\mathscr{H}_r(B)$-norm is a Hilbert space $\mathscr{H}_r(B)$ in the $\mathscr{H}_r(B)$-norm. The space is contained in $\mathscr{C}_r(z)$ and the inclusion does not increase norms. If $f(z)$ and $g(z)$ are power series and if

$$h(z) = f(z) + B(z)g(z),$$

then the inequality

$$\|h(z)\|^2_{\mathscr{C}_r(z)} \leqslant \|f(z)\|^2_{\mathscr{H}_r(B)} + \|g(z)\|^2_{\mathscr{C}_r(z)}$$

is satisfied. Every power series $h(z)$ admits a minimal decomposition for which equality holds. The component series $f(z)$ and $g(z)$ in the minimal decomposition are unique within r-equivalence.

The space $\mathscr{H}_r(B)$ is properly related to the space $\mathscr{H}(B)$.

THEOREM 8. *If $B(z)$ is a power series which converges in the unit disk and represents a function which is bounded by one in the disk and if r is a* *nonnegative integer, then the transformation which takes every element of $\mathcal{H}(B)$ into the r-equivalent polynomial of degree at most r is a partial isometry of $\mathcal{H}(B)$ onto $\mathcal{H}_r(B)$.*

A fundamental concept in the estimation theory of coefficients is subordination. A power series $f(z)$ is said to be subordinate to a power series $g(z)$ if $f(z) = g(B(z))$ for a power series $B(z)$ with constant coefficient zero which converges in the unit disk and represents a function which is bounded by one in the disk. If $f(z)$ and $g(z)$ are power series with constant coefficient zero which converge in the unit disk and represent functions which have distinct values at distinct points of the disk, then $f(z)$ is subordinate to $g(z)$ if, and only if, the region onto which $f(z)$ maps the unit disk is contained in the region onto which $g(z)$ maps the unit disk.

Another basic Hilbert space is the space $\mathcal{C}(0)$ of power series $f(z) = \sum a_n z^n$ with constant coefficient zero such that

$$\|f(z)\|^2_{\mathcal{C}(0)} = \sum n\,|a_n|^2 < \infty.$$

The elements of the space are convergent power series in the unit disk. The series

$$\log \frac{1}{1 - z\bar{w}} = z\bar{w} + \frac{1}{2} z^2\bar{w}^2 + \frac{1}{3} z^3\bar{w}^3 + \cdots$$

belongs to the space for every point w in the unit disk and the identity

$$f(w) = \left\langle f(z), \log \frac{1}{1 - z\bar{w}} \right\rangle_{\mathcal{C}(0)}$$

holds for every element $f(z)$ of the space. The identity

$$\pi\,\|f(z)\|^2_{\mathcal{C}(0)} = \iint |f'(z)|^2\,dx\,dy$$

holds for every element $f(z)$ of the space, where integration is over the unit disk. The space contains every convergent power series with constant coefficient zero for which the integral on the right converges.

Let $B(z)$ be a power series with constant coefficient zero which converges in the unit disk and represents a function which is bounded by one and has distinct values at distinct points of the disk. Then the identity

$$\pi\,\|f(B(z))\|^2_{\mathcal{C}(0)} = \iint |f'(z)|^2\,dx\,dy$$

holds for every element $f(z)$ of $\mathcal{C}(0)$, where integration is over the region onto which $B(z)$ maps the unit disk. The transformation of $B(z)$-substitution, which takes $f(z)$ into $f(B(z))$, is bounded by one in $\mathscr{S}(0)$. It is isometric if, and only if, the complement in the unit disk of the region onto which $f(z)$ maps the unit disk has zero plane measure. Another generalization of the space $\mathscr{H}(B)$ results.

Define the $\mathscr{S}(B)$-norm of an element $f(z)$ of $\mathscr{S}(0)$ by

$$\|f(z)\|^2_{\mathscr{S}(B)} = \sup[\|f(z) + g(B(z))\|^2_{\mathscr{S}(0)} - \|g(z)\|^2_{\mathscr{S}(0)}]$$

where the least upper bound is taken over all elements $g(z)$ of $\mathscr{S}(0)$. The set $\mathscr{S}(B)$ of all elements of $\mathscr{S}(0)$ of finite $\mathscr{S}(B)$-norm is a Hilbert space in the $\mathscr{S}(B)$-norm. The inclusion of the space in $\mathscr{S}(0)$ does not increase norms. If $f(z)$ is in $\mathscr{S}(B)$ and if $g(z)$ is in $\mathscr{S}(0)$, then

$$h(z) = f(z) + g(B(z))$$

is an element of $\mathscr{S}(0)$ which satisfies the inequality

$$\|h(z)\|^2_{\mathscr{S}(0)} \leqslant \|f(z)\|^2_{\mathscr{S}(B)} + \|g(z)\|^2_{\mathscr{S}(0)}.$$

Every element $h(z)$ of $\mathscr{S}(0)$ has a unique minimal decomposition for which equality holds. The element $f(z)$ of $\mathscr{S}(B)$ in the minimal decomposition of an element $h(z)$ of $\mathscr{S}(0)$ is obtained from $h(z)$ under the adjoint of the inclusion of $\mathscr{S}(B)$ in $\mathscr{S}(0)$. If w is a point in the unit disk, the minimal decomposition of the element $\log 1/(1 - z\bar{w})$ of $\mathscr{S}(0)$ is obtained with

$$\log \frac{1 - B(z)\bar{B}(w)}{1 - z\bar{w}}$$

as the element of $\mathscr{S}(B)$. The identity

$$f(w) = \left\langle f(z), \log \frac{1 - B(z)\bar{B}(w)}{1 - z\bar{w}} \right\rangle_{\mathscr{S}(B)}$$

holds for every element $f(z)$ of $\mathscr{S}(B)$. Logarithms of power series with constant coefficient zero are determined by substitution in the formal expansion

$$\log \frac{1}{1-z} = z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \cdots.$$

An inequality is an immediate consequence of the kernel function computation.

THEOREM 9. *If $B(z)$ is a power series with constant coefficient zero* *which converges in the unit disk and represents a function which is bounded by one and has distinct values at distinct points of the disk, then the inequality*

$$\|\exp f(z)\|^2_{\mathcal{F}(B)} \leqslant \exp \|f(z)\|^2_{\mathcal{G}(B)}$$

holds for every element $f(z)$ of $\mathcal{G}(B)$.

If r is a positive integer, define $\mathcal{G}_r(0)$ to be the space of polynomials of degree at most r with constant coefficient zero in the norm of $\mathcal{G}(0)$. The $\mathcal{G}_r(0)$-norm is also defined for any power series with constant coefficient zero as the $\mathcal{G}_r(0)$-norm of the r-equivalent polynomial of degree at most r. Let $B(z)$ be a power series with constant coefficient zero which converges in the unit disk and represents a function which is bounded by one and has distinct values at distinct points of the disk. Then the inequality

$$\|f(B(z))\|_{\mathcal{G}_r(0)} \leqslant \|f(z)\|_{\mathcal{G}_r(0)}$$

holds for every power series $f(z)$ with constant coefficient zero. The inequality allows another generalization of the space $\mathcal{H}(B)$.

If $f(z)$ is a power series with constant coefficient zero, define

$$\|f(z)\|^2_{\mathcal{G}_r(B)} = \sup[\|f(z) + g(B(z))\|^2_{\mathcal{G}_r(0)} - \| g(z)\|^2_{\mathcal{G}_r(0)}]$$

where the least upper bound is taken over all power series $g(z)$ with constant coefficient zero. The $\mathcal{G}_r(B)$-norms of any two r-equivalent power series are equal. The set of polynomials of degree at most r, with constant coefficient zero, which have finite $\mathcal{G}_r(B)$-norm, is a Hilbert space $\mathcal{G}_r(B)$ in the $\mathcal{G}_r(B)$-norm. The space is contained in $\mathcal{G}_r(0)$ and the inclusion does not increase norms. If $f(z)$ and $g(z)$ are power series with constant coefficient zero and if

$$h(z) = f(z) + g(B(z)),$$

then the inequality

$$\| h(z)\|^2_{\mathcal{G}_r(0)} \leqslant \|f(z)\|^2_{\mathcal{G}_r(B)} + \| g(z)\|^2_{\mathcal{G}_r(0)}$$

is satisfied. Every power series $h(z)$ with constant coefficient zero admits a unique minimal decomposition for which equality holds. The component series $f(z)$ and $g(z)$ in the minimal decomposition are unique within r-equivalence.

The space $\mathcal{G}_r(B)$ is properly related to the space $\mathcal{G}(B)$.

THEOREM 10. *If $B(z)$ is a power series with constant coefficient zero which converges in the unit disk and represents a function which is bounded* *by one and has distinct values at distinct points of the disk and if r is a positive integer, then the transformation which takes every element of $\mathscr{S}(B)$ into the r-equivalent polynomial of degree at most r is a partial isometry of $\mathscr{S}(B)$ onto $\mathscr{S}_r(B)$.*

The adjoint of $B(z)$-substitution in $\mathscr{S}(0)$ is computable on polynomials.

THEOREM 11. *If $B(z)$ is a power series with constant coefficient zero which converges in the unit disk and represents a function which is bounded by one and has distinct values at distinct points of the disk, then the adjoint of $B(z)$-substitution in $\mathscr{S}(0)$ takes polynomials of degree at most r into polynomials of no larger degree for every positive integer r. If $u(z)$ and $v(z)$ are polynomials with constant coefficient zero, then the adjoint of $B(z)$-substitution in $\mathscr{S}(0)$ takes $u(z)$ into $v(z)$ if, and only if, $u(1/z) - v(1/B^*(z))$ is a power series. Every element of $\mathscr{S}(B)$ which has the same norm in $\mathscr{S}_r(B)$ as in $\mathscr{S}(B)$ is of the form $u(z) - v(B(z))$ for such polynomials $u(z)$ and $v(z)$.*

The computation of adjoints is a fundamental ingredient in the construction of the Grunsky transformation.

THEOREM 12. *Let $B(z)$ be a power series with constant coefficient zero which converges in the unit disk and represents a function which is bounded by one and has distinct values at distinct points of the disk. A unique transformation, called the Grunsky transformation of $\mathscr{S}(B)$ into $\mathscr{S}(B^*)$, exists which is bounded by one and which takes $f(z)$ into $g(z)$ whenever*

$$f(z) = u(z) - v(B(z))$$

and

$$g(z) = u(1/z) - v(1/B^*(z)) + \text{constant}$$

for polynomials $u(z)$ and $v(z)$ with constant coefficient zero such that the adjoint of $B(z)$-substitution takes $u(z)$ into $v(z)$. The adjoint of the Grunsky transformation of $\mathscr{S}(B)$ into $\mathscr{S}(B^*)$ is the Grunsky transformation of $\mathscr{S}(B^*)$ into $\mathscr{S}(B)$. The Grunsky transformations of $\mathscr{S}(B)$ into $\mathscr{S}(B^*)$ and of $\mathscr{S}(B^*)$ into $\mathscr{S}(B)$ are isometric if $B(z)$-substitution is isometric in $\mathscr{S}(0)$.

A continuous application of the Riemann mapping theorem is due to Löwner [6]. A Löwner family is a family of power series $f(t, z)$, indexed by a positive parameter t, with these properties: The series $f(t, z)$ has constant coefficient zero and coefficient of z equal to t for every index t. It converges in the unit disk and represents a function which has distinct values at distinct points of the disk. When a is less than b, $f(a, z)$ is subordinate to $f(b, z)$.

If $\psi_+(z)$ and $\psi_-(z)$ are power series with constant coefficient zero and coefficient of z positive, which converge in the unit disk and represent functions with distinct values at distinct points of the disk, and if $\psi_-(z)$ is subordinate to $\psi_+(z)$, then a Löwner family $f(t, z)$ exists such that $\psi_-(z) = f(a, z)$ and $\psi_+(z) = f(b, z)$ for positive numbers a and b, $a \leqslant b$.

A differential equation is satisfied.

THEOREM 13. *If a Löwner family of power series $f(t, z)$ is given, then the coefficients of $f(t, z)$ are absolutely continuous functions of t which satisfy the equation*

$$t\,\partial/\partial t\, f(t, z) = z\varphi(t, z)\,\partial/\partial z\, f(t, z)$$

where $\varphi(t, z)$ is a power series with constant coefficient one which converges in the unit disk and represents a function with positive real part in the disk for every index t, and the coefficients of $\varphi(t, z)$ are measurable functions of t.

A converse result is also due to Löwner.

THEOREM 14. *For each positive number t, let $\varphi(t, z)$ be a power series with constant coefficient one which converges in the unit disk and represents a function with positive real part in the disk. If the coefficients of $\varphi(t, z)$ are measurable functions of t, then a unique Löwner family of power series $f(t, z)$ exists such that*

$$t\,\partial/\partial t\, f(t, z) = z\varphi(t, z)\,\partial/\partial z\, f(t, z).$$

An expansion theorem holds for Löwner families.

THEOREM 15. *Assume that a Löwner family of power series $f(t, z)$ is given. When $0 < a < b < \infty$, let $B(b, a, z)$ be the unique power series with constant coefficient zero such that*

$$f(a, z) = f(b, B(b, a, z)).$$

Then a space $\mathscr{F}(B(b, a))$ exists and the coefficient of z in $B(b, a, z)$ is a/b. Let $\varphi(t, z)$ be the power series with constant coefficient one, defined for positive t, which appears in the Löwner equation

$$t\,\partial/\partial t\, f(t, z) = z\varphi(t, z)\,\partial/\partial z\, f(t, z).$$

If $h(t)$ is a square integrable function of t in the interval (a, b), then

$$F(z) = \int_a^b h(t)\, d \log \frac{B(t, a, z)}{az/t}$$

belongs to $\mathcal{G}(B(b, a))$ *and*

$$G(z) = \int_a^b h(t)\, d \log \frac{B^*(t, a, z)}{az/t}$$

belongs to $\mathcal{G}(B^*(b, a))$. *The inequalities*

$$\|F(z)\|^2_{\mathcal{G}(B(b,a))} \leqslant 2 \int_a^b |h(t)|^2/t\, dt$$

and

$$\|G(z)\|^2_{\mathcal{G}(B^*(b,a))} \leqslant 2 \int_a^b |h(t)|^2/t\, dt$$

are satisfied. Equality holds if

$$\varphi(t, z) = [\lambda(t) + z]/[\lambda(t) - z]$$

for a measurable function $\lambda(t)$ *of absolute value one in* (a, b). *In this case every element of* $\mathcal{G}(B(b, a))$ *is of the form* $F(z)$ *for some such choice of* $h(t)$, *every element of* $\mathcal{G}(B^*(b, a))$ *is of the form* $G(z)$ *for some such choice of* $h(t)$, *and* $B(b, a, z)$*-substitution is isometric in* $\mathcal{G}(0)$. *The Grunsky transformation of* $\mathcal{G}(B(b, a))$ *into* $\mathcal{G}(B^*(b, a))$ *takes* $F(z)$ *into* $-G(z)$ *and the Grunsky transformation of* $\mathcal{G}(B^*(b, a))$ *into* $\mathcal{G}(B(b, a))$ *takes* $G(z)$ *into* $-F(z)$.

An estimate is a corollary of the expansion.

THEOREM 16. *If* $B(z)$ *is a power series with constant coefficient zero which converges in the unit disk and represents a function which is bounded by one and has distinct values at distinct points of the disk, then*

$$\log \frac{B(z)}{zB'(0)}$$

belongs to $\mathcal{G}(B)$ *and the inequality*

$$\left\| \log \frac{B(z)}{zB'(0)} \right\|^2_{\mathcal{G}(B)} \leqslant 2 \log \frac{1}{|B'(0)|}$$

is satisfied.

Other estimates of this important series are obtained from the Löwner expansion using the theory of Legendre polynomials. The polynomials

$$S_n(z) = F(-n, n+1; 1; z)$$

form an orthonormal basis for $L^2(0, 1)$ and satisfy the recurrence relations

$$(2n+1)(1-2z)\,S_n(z) = nS_{n-1}(z) + (n+1)\,S_{n+1}(z).$$

An inequality is derived from the theory of the polynomials.

THEOREM 17. *If r is a positive integer and if $0 < a < 1$, then the inequality*

$$\left| \int_a^1 f(t)\,dt \right|^2 \leqslant 2(1-a)\left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right) \int_a^1 |f(t)|^2\, t\, dt$$

holds for every polynomial $f(z)$ of degree less than r.

An estimate of logarithmic coefficients is a corollary.

THEOREM 18. *If the coefficient $\varphi(t, z)$ in the Löwner equation is independent of t, then the inequality*

$$\left\| \log \frac{B(b, a, z)}{bz/a} \right\|^2_{\mathcal{C}_r(B(b,a))} \leqslant 4(1 - a/b)\left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right)$$

holds for every positive integer r.

A construction of Milin [1] estimates the coefficients of a power series from a knowledge of logarithmic coefficients. It is not known whether the estimate

$$\begin{aligned} &\|\exp f(z)\|^2_{\mathcal{H}_r(B)} \\ &\qquad \leqslant (r+1)\exp\left(\frac{1}{r+1}\sum_{n=1}^{r} \|f(z)\|^2_{\mathcal{C}_n(B)} - \sum_{n=1}^{r}\frac{1}{n+1}\right) \end{aligned}$$

holds for every element $f(z)$ of an arbitrary space $\mathcal{G}_r(B)$. Milin shows that the estimate is valid in the case $B(z) = 0$. The estimate is conjectured to be valid generally for the case $B(z) = B(b, a, z)$ when the coefficient $\varphi(t, z)$ in the Löwner equation is independent of t.

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Proof of Theorem 1. The choice of $g(z)$ is made so that

$$\begin{aligned} &-b_r z^r + b_r(z + a_r z^r + a_{r+1} z^{r+1} + \cdots)^r \\ &\qquad - b_{r+1} z^{r+1} + b_{r+1}(z + a_r z^r + a_{r+1} z^{r+1} + \cdots) + \cdots \\ &\quad = -a_r z^r + a_r(z + b_r z^r + b_{r+1} z^{r+1} + \cdots)^r \\ &\qquad - a_{r+1} z^{r+1} + a_{r+1}(z + b_r z^r + b_{r+1} z^{r+1} + \cdots)^{r+1} + \cdots. \end{aligned}$$

The coefficient of z^n on each side is zero when $n < 2r - 1$. When $r \leqslant n$, the coefficient of z^{n+r-1} on each side is a function of $b_1, \ldots, b_n$. The coefficients are equal when $n = r$ no matter what the choice of b_r. An inductive choice of b_n is made when $n > r$ so that the two coefficients are equal. A unique choice of b_n with this property exists because b_n appears on the left in the added term $na_r b_n$ and on the right in the added term $ra_r b_n$ where a_r is nonzero by hypothesis.

Proof of Theorem 2. Argue by contradiction, assuming that an integer s exists, $1 < s < r$, such that $b_n = 0$ when $1 < n < s$ and b_s is nonzero. Apply Theorem 1 with $f(z)$ and $g(z)$ interchanged. By the uniqueness part of the theorem, $f(z) = z$, which is contrary to hypothesis.

Proof of Theorem 3. Since $f(a, z)$ and $f(b, z)$ are power series which commute with $f(z)$, so is $f(a, f(b, z))$. Since $f(a, z) = z + az^r + \cdots$ and $g(z) = z + bz^r + \cdots$, it follows that $f(a, f(b, z)) = z + (a + b) z^r + \cdots$. By the uniqueness part of Theorem 1,

$$f(a + b, z) = f(a, f(b, z)).$$

If

$$f(w, z) = \sum f_n(w) z^n,$$

then $f_0(w) = 0$, $f_1(w) = 1$, $f_n(w) = 0$ when $1 < n < r$, and $f_r(w) = w$. By the proof of Theorem 1, $f_n(w)$ is a polynomial in the preceding coefficients when $n > r$. It follows that $f_n(w)$ is a polynomial in w for every n. So it is meaningful to take a partial derivative with respect to w. Since $f(a + w) = f(a, f(w, z))$, it follows that

$$\partial/\partial w f(a + w, z) = f'(a, f(w, z)) \, \partial/\partial w f(w, z).$$

If

$$\varphi(z) = \partial/\partial w f(w, z)$$

is evaluated at the point $w = 0$, then the coefficient of z^n in $\varphi(z)$ is zero when $n < r$ and the coefficient of z^r in $\varphi(z)$ is one. The desired differential equation is satisfied. If $\varphi(z) = \sum \varphi_n z^n$, then it implies that

$$\partial/\partial w f_n(w) = \sum_{0 < k < n} \varphi_{n+1-k} k f_k(w).$$

An inductive argument shows that the degree of $f_n(w)$ is less than n for every w.

Proof of Theorem 4. If $\varphi(z) = \sum \varphi_n z^n$ and

$$f(w, z) = \sum f_n(w)\, z^n,$$

the given differential equation is equivalent to the sequence of differential equations in the proof of Theorem 3. Since the initial conditions $f_1(0) = 1$ and $f_n(0) = 0$ when n is not one are given, an inductive argument shows that $f_n(w)$ is a uniquely determined polynomial of degree less than n in w for every nonnegative integer n. Note that $f_0(w) = 0$ and $f_1(w) = 1$.

It remains to show that the identity

$$f(a+b, z) = f(a, f(b, z))$$

is satisfied. Let a be held fixed and let $g(z)$ be the unique power series with constant coefficient zero and coefficient of z equal to one such that

$$g(f(a, z)) = z.$$

Then the coefficients of $g(f(a+b, z))$ are polynomials in b and

$$\begin{aligned} \partial/\partial b\, g(f(a+b, z)) &= g'(f(a+b, z))\, \partial/\partial b\, f(a+b, z) \\ &= g'(f(a+b, z))\, \varphi(z)\, \partial/\partial z\, f(a+b, z) \\ &= \varphi(z)\, \partial/\partial z\, g(f(a+b, z)). \end{aligned}$$

Since

$$g(f(a, z)) = z,$$

it follows that

$$g(f(a+b, z)) = f(b, z).$$

The desired identity now follows because $f(a, g(z)) = z$.

Proof of Theorem 5. Since

$$f(a+b, z) = f(a, f(b, z)),$$

differentiation with respect to a gives the identity

$$\varphi(z)\, \partial/\partial z\, f(a+b, z) = \varphi(f(b, z)) f'(a, f(b, z)).$$

The identity reads

$$\varphi(z)\, \partial/\partial z\, f(b, z) = \varphi(f(b, z))$$

when a is zero.

If $f(z)$ and $g(z)$ are power series with constant coefficient zero and coefficient of z equal to one such that

$$\varphi(z)\,\partial/\partial z\, f(z) = \varphi(f(z))$$

and

$$\varphi(z)\,\partial/\partial z\, g(z) = \varphi(g(z)),$$

then

$$\begin{aligned}\varphi(z)\,\partial/\partial z\, f(g(z)) &= f'(g(z))\,\varphi(z)\,\partial g/\partial z\\ &= \varphi(g(z)) f'(g(z))\\ &= \varphi(f(g(z))).\end{aligned}$$

It remains to show that a power series $f(z)$ is equal to z if $f(z) = \sum a_n z^n$ is a solution of the equation

$$\varphi(z)\,\partial/\partial z\, f(z) = \varphi(f(z))$$

with $a_0 = 0$, $a_1 = 1$, and $a_r = 0$. It is sufficient to show that $a_s = 0$ if $s > 1$, if $s \neq r$, and if $a_n = 0$ for $1 < n < s$. If $\varphi(z) = \sum \varphi_n z^n$, then

$$\begin{aligned}&(\varphi_r z^r + \varphi_{r+1} z^{r+1} + \cdots)(s a_s z^{s-1} + (s+1)\, a_{s+1} z^s + \cdots)\\ &\quad = -\varphi_r z^r + \varphi_r (z + a_s z^s + a_{s+1} z^{s+1} + \cdots)^r\\ &\qquad - \varphi_{r+1} z^{r+1} + \varphi_{r+1}(z + a_s z^s + a_{s+1} z^{s+1} + \cdots)^{r+1} + \cdots.\end{aligned}$$

The identity

$$s\varphi_r a_s = r\varphi_r a_s$$

is obtained on comparing the coefficient of z^{r+s-1} on each side. Since $\varphi_r = 1$ and since $s \neq r$, $a_s = 0$.

Proof of Theorem 6. If $\varphi(z) = \sum \varphi_n z^n$ and if

$$f(w, z) = \sum f_n(w)\, z^n,$$

then the desired differential equation is equivalent to the sequence of equations

$$w\,\partial/\partial w\, f_n(w) = \sum_{k=0}^{n} \varphi_{n-k}\, k f_k(w)$$

with the initial conditions $f_1(1) = 1$ and $f_n(1) = 0$ when $n > 1$. Since

$$w\,\partial/\partial w\, f_1(w) = f_1(w),$$

the solution is $f_1(w) = w$. Since the differential equation can be written

$$\partial/\partial w[f_n(w)/w^n] = \sum_{k=0}^{n-1} \varphi_{n-k} k f_k(w)/w^{n+1},$$

an inductive argument shows that $f_n(w)$ is a uniquely determined polynomial of degree at most n for every w. The remainder of the proof if similar to the proof of Theorem 4.

Proof of Theorem 7. The proof that

$$z\varphi(z)\,\partial/\partial z\, f(w, z) = f(w, z)\,\varphi(f(w, z))$$

is similar to the proof of Theorem 5. If $f(z)$ and $g(z)$ are power series with constant coefficient zero such that

$$z\varphi(z)\,\partial/\partial z\, f(z) = f(z)\,\varphi(f(z))$$

and

$$z\varphi(z)\,\partial/\partial z\, g(z) = g(z)\,\varphi(g(z)),$$

then

$$z\varphi(z)\,\partial/\partial z\, f(g(z)) = f(g(z))\varphi(f(g(z)).$$

It remains to show that a power series $f(z) = \sum a_n z^n$ is equal to z if its constant coefficient is zero, if its coefficient of z is one, and if it is a solution of the differential equation

$$z\varphi(z)\,\partial/\partial z\, f(z) = f(z)\,\varphi(f(z)).$$

It is sufficient to show that $a_r = 0$ if $r > 1$ and if $a_n = 0$ for $1 < n < r$. Then

$$\begin{aligned}(1 + \varphi_1 z + \varphi_2 z^2 + \cdots)&(ra_r z^r + (r+1)\,a_{r+1} z^{r+1} + \cdots)\\ &= (a_r z^r + a_{r+1} z^{r+1} + \cdots)\\ &\quad - \varphi_1 z^2 + \varphi_1 (z + a_r z^r + a_{r+1} z^{r+1} + \cdots)^2 + \cdots.\end{aligned}$$

The identity $r\,a_r = a_r$ is obtained on comparing the coefficient of z^r on each side. Since $r > 1$, $a_r = 0$.

Proof of Theorem 8. If $f(z)$ belongs to $\mathscr{H}(B)$, then

$$\|f(z)\|^2_{\mathscr{H}_r(B)} = \sup[\|f(z) + B(z)\,g(z)\|^2_{\mathscr{C}_r(z)} - \|g(z)\|^2_{\mathscr{C}_r(z)}]$$

where the least upper bound is taken over all polynomials $g(z)$ of degree at most r. Since the norm of $\mathscr{C}_r(z)$ is dominated by the norm of $\mathscr{C}(z)$,

$$\|f(z)\|^2_{\mathscr{H}_r(B)} \leqslant \sup[\|f(z) + B(z)\,g(z)\|^2_{\mathscr{C}(z)} - \|g(z)\|^2_{\mathscr{C}(z)}]$$

where the least upper bound is taken over all polynomials $g(z)$ of degree at most r. Since the same inequality remains valid when the least upper bound is taken over all elements $g(z)$ of $\mathscr{C}(z)$,

$$\|f(z)\|_{\mathscr{H}_r(B)} \leqslant \|f(z)\|_{\mathscr{H}(B)}.$$

If $h(z)$ is a polynomial of degree at most r, then the minimal decomposition of $h(z)$ as an element of $\mathscr{C}(z)$ is of the form

$$h(z) = f(z) + B(z)\,g(z)$$

where $f(z)$ is an element of $\mathscr{H}(B)$ and $g(z)$ is an element of $\mathscr{C}(z)$ such that

$$\|h(z)\|^2_{\mathscr{C}(z)} = \|f(z)\|^2_{\mathscr{H}(B)} + \|g(z)\|^2_{\mathscr{C}(z)}.$$

It follows that

$$\|h(z)\|^2_{\mathscr{C}(z)} \geqslant \|f(z)\|^2_{\mathscr{H}(B)} + \|g(z)\|^2_{\mathscr{C}_r(z)}.$$

Since $h(z)$ is a polynomial of degree at most r, the reverse inequality also holds. Since equality holds,

$$\|f(z)\|_{\mathscr{H}_r(B)} = \|f(z)\|_{\mathscr{H}(B)}$$

and $g(z)$ is a polynomial of degree at most r. The series $f(z)$ for which the identity has been obtained are those obtained from a polynomial of degree at most r under the adjoint of the inclusion of $\mathscr{H}(B)$ in $\mathscr{C}(z)$. These are the elements of $\mathscr{H}(B)$ which are orthogonal to elements of $\mathscr{H}(B)$ whose coefficient of z^n is zero for $n = 0,\ldots, r$. The construction also shows that the range of the transformation which takes every element of $\mathscr{H}(B)$ into the r-equivalent polynomial of degree at most r contains the range of the adjoint of the inclusion of $\mathscr{H}_r(B)$ in $\mathscr{C}_r(z)$. Since this range contains every element of $\mathscr{H}_r(B)$, every element of $\mathscr{H}_r(B)$ is r-equivalent to an element of $\mathscr{H}(B)$.

Proof of Theorem 9. The stated result is a special case of a more general theorem. Assume that $\mathscr{D}$ is a given Hilbert space of power series which converge in the unit disk and that the linear functional which takes $f(z)$ into

$f(w)$ is bounded in the space for every point w in the disk. Let $K(w, z)$ be the unique element of the space such that the kernel function identity

$$f(w) = \langle f(z), K(w, z)\rangle_{\mathcal{D}}$$

holds for every element $f(z)$ of the space. Then a Hilbert space $\mathcal{E}$ exists whose elements are convergent power series in the disk such that $\exp K(w, z)$ belongs to the space for every point w in the disk and such that the kernel function identity

$$f(w) = \langle f(z), \exp K(w, z)\rangle_{\mathcal{E}}$$

holds for every element $f(z)$ of the space. The inequality

$$\|\exp f(z)\|_{\mathcal{E}}^2 \leqslant \exp \|f(z)\|_{\mathcal{D}}^2$$

holds for every element $f(z)$ of the space.

For the proof observe that for every nonnegative integer r, the product

$$K(w_1, z_1) \cdots K(w_r, z_r)$$

is the kernel function of a tensor product Hilbert space of analytic functions of r complex variables $z_1,..., z_r$. The space contains $f_1(z_1) \cdots f_r(z_r)$ whenever $f_1(z) \cdots f_r(z)$ are elements of $\mathcal{D}$ and the square of the norm of the product is

$$\|f_1(z)\|_{\mathcal{D}}^2 \cdots \|f_r(z)\|_{\mathcal{D}}^2.$$

A Hilbert space $\mathcal{D}^r$ of power series in one variable exists such that the transformation which takes $f(z_1,..., z_r)$ into $f(z,..., z)$ is a partial isometry of the product space onto $\mathcal{D}^r$. The kernel function of $\mathcal{D}^r$ is $K^r(w, z)$.

When $r = 0$, the space $\mathcal{D}^r$ is the space of constants with the square of the norm equal to the square of the absolute value divided by $\exp K(0, 0)$. When r is a positive integer and $f_1(z),..., f_r(z)$ are elements of $\mathcal{D}$, the product $f(z) = f_1(z) \cdots f_r(z)$ belongs to $\mathcal{D}^r$ and the inequality

$$\|f(z)\|_{\mathcal{D}^r} \leqslant \|f_1(z)\|_{\mathcal{D}} \cdots \|f_r(z)\|_{\mathcal{D}}$$

is satisfied. The identity

$$\exp K(w, z) = \sum_{r=0}^{\infty} \frac{1}{r!} K^r(w, z)$$

shows that the desired space $\mathcal{E}$ exists and that a generalization of the theory of minimal decompositions applies. If an element $f_r(z)$ of $\mathcal{D}_r$ is chosen for every nonnegative integer r so that

$$\sum_{r=0}^{\infty} \frac{1}{r!} \|f_r(z)\|_{\mathcal{D}^r}^2 < \infty,$$

then

$$f(z) = \sum_{r=0}^{\infty} \frac{1}{r!} f_r(z)$$

belongs to $\mathcal{E}$ and the inequality

$$\|f(z)\|_{\mathcal{E}}^2 \leqslant \sum_{r=0}^{\infty} \frac{1}{r!} \|f_r(z)\|_{\mathcal{D}^r}^2$$

is satisfied. Every element $f(z)$ of $\mathcal{E}$ has a unique minimal decomposition for which equality holds. It follows that the inequality

$$\begin{aligned} \|\exp f(z)\|_{\mathcal{E}}^2 &\leqslant \sum_{r=0}^{\infty} \frac{1}{r!} \|f^r(z)\|_{\mathcal{D}^r}^2 \\ &\leqslant \sum_{r=0}^{\infty} \frac{1}{r!} \|f(z)\|_{\mathcal{D}}^{2r} \\ &\leqslant \exp \|f(z)\|_{\mathcal{D}}^2 \end{aligned}$$

holds for every element $f(z)$ of $\mathcal{D}$.

Proof of Theorem 10. The proof is similar to the proof of Theorem 8.

Proof of Theorem 11. If r is a given positive integer, then the set of elements of $\mathcal{S}(0)$ whose coefficient of z^n is zero for $n = 1,..., r$ is invariant under $B(z)$-substitution. It follows that its orthogonal complement in $\mathcal{S}(0)$, which is the set of polynomials of degree at most r with constant coefficient zero, is invariant under the adjoint of $B(z)$-substitution in $\mathcal{S}(0)$.

It follows from Theorem 10 that an element $f(z)$ of $\mathcal{S}(B)$ which has the same norm in $\mathcal{S}_r(B)$ as in $\mathcal{S}(B)$ is orthogonal to every element of $\mathcal{S}(B)$ whose coefficient of z^n is zero for $n = 1,..., r$. So at least one polynomial $u(z)$ is zero for $n = 1,..., r$. So at least one polynomial $u(z)$ of degree at most r with constant coefficient zero exists such that the identity

$$\langle h(z), u(z)\rangle_{\mathcal{S}(0)} = \langle h(z), f(z)\rangle_{\mathcal{S}(B)}$$

holds for every element $h(z)$ of $\mathcal{S}(B)$. If $v(z)$ is the polynomial of degree at most r obtained from $u(z)$ under the adjoint of $B(z)$-substitution in $\mathscr{C}(0)$, then by the theory of minimal decompositions,

$$f(z) = u(z) - v(B(z))$$

and

$$\|f(z)\|^2_{\mathscr{C}(B)} = \|u(z)\|^2_{\mathscr{C}(0)} - \|v(z)\|^2_{\mathscr{C}(0)}.$$

Assume that $u(z)$ and $v(z)$ are polynomials with constant coefficient zero. It remains to show that $v(z)$ is obtained from $u(z)$ under the adjoint of $B(z)$-substitution if, and only if, $u(1/z) - v(1/B^*(z))$ is a power series. It is easily verified that a polynomial $v(z)$ with constant coefficient zero is identically zero if $v(1/B^*(z))$ is a power series. It is therefore sufficient to show that $u(1/z) - v(1/B^*(z))$ is a power series whenever $v(z)$ is obtained from $u(z)$ under the adjoint of $B(z)$-substitution.

By linearity it is sufficient to make the computation in the case that $u(z)$ is a power of z. The verification is easily made if $u(z)$ is replaced by the series

$$u(z) = \log \frac{1}{1 - z\bar{w}}$$

for a point w in the unit disk. By the kernel function identity for $\mathscr{C}(0)$, the element of $\mathscr{C}(0)$ obtained from $u(z)$ under the adjoint of $B(z)$-substitution is

$$v(z) = \log \frac{1}{1 - z\bar{B}(w)}$$

where

$$u(1/z) - v(1/B^*(z)) = \log \frac{1 - \bar{B}(w)/B^*(z)}{1 - \bar{w}/z}$$

can be expanded as a power series in z because it is analytic in the unit disk. The desired conclusion in the case that $u(z)$ is a power of z is obtained on expanding these expressions in powers of $\bar{w}$ and comparing coefficients of $\bar{w}^n$ for every positive integer n. This formal procedure is easily justified since $B(z)$-substitution is bounded by one in $\mathscr{C}(0)$.

Proof of Theorem 12. The proof is an application of the Carathéodory mapping theorem [7]. Let $\gamma(t)$ be a continuous complex-valued function of t in the reals modulo one which has distinct values at distinct points, such that the origin belongs to the bounded component of the complement of the set of values of the function. Then a unique power series $\psi(z)$ exists, with constant coefficient zero and coefficient of z positive, which converges in the unit disk, which represents a function which has a continuous extension to the closed disk with distinct values at distinct points of the closed disk, and which maps the unit circle onto the set of values of γ.

A special case of Green's theorem is a corollary of the Carathéodory mapping theorem. If $\gamma(t)$ is a continuous complex valued function of t in the reals modulo one which has distinct values at distinct points and winds counterclockwise around the bounded component Ω of the set of values of the function, and if $f(z)$ is an analytic function of z in Ω which is continuous in the closure of Ω and of bounded variation on the boundary, then

$$\frac{1}{\pi}\iint_{\Omega} |f'(z)|^2\, dx\, dy = \frac{1}{2\pi i}\int_{\gamma} \bar{f}(z)\, df(z).$$

A norm estimate is now made when $f(z)$ and $g(z)$ are power series related by polynomials $u(z)$ and $v(z)$ as in the statement of the theorem. If $h(z)$ is in $\mathscr{C}(0)$, then by Green's theorem

$$\begin{aligned}
&\| g(z) + h(B^*(z))\|^2_{\mathscr{C}(0)} - \| u(z)\|^2_{\mathscr{C}(0)} \\
&\quad = \frac{1}{2\pi i}\int [\bar{g}(z) + \bar{h}(B^*(z)) - \bar{u}(1/z)]\, d[g(z) + h(B^*(z)) - u(1/z)] \\
&\quad = \frac{1}{2\pi i}\int [-\bar{v}(1/B^*(z)) + \bar{h}(B^*(z))]\, d[-v(1/B^*(z)) + h(B^*(z))]
\end{aligned}$$

and

$$\| h(z)\|^2_{\mathscr{C}(0)} - \| v(z)\|^2_{\mathscr{C}(0)} = \frac{1}{2\pi i}\int [\bar{h}(z) - \bar{v}(1/z)]\, d[h(z) - v(1/z)]$$

where each integral is a limit as r increases to one of a counterclockwise integration over a circle of radius r about the origin. Another application of Green's theorem gives the inequality

$$\begin{aligned}
&\frac{1}{2\pi i}\int [-\bar{v}(1/B^*(z)) + \bar{h}(B^*(z))]\, d[-v(1/B^*(z)) + h(B^*(z))] \\
&\qquad \leqslant \frac{1}{2\pi i}\int [\bar{h}(z) - \bar{v}(1/z)]\, d[h(z) - v(1/z)]
\end{aligned}$$

with equality if the complement in the unit disk of the region onto which $B^*(z)$ maps the unit disk has zero plane measure. It follows that the inequality

$$\| g(z) + h(B^*(z))\|^2_{\mathscr{C}(0)} - \| h(z)\|^2_{\mathscr{C}(0)} \leqslant \| u(z)\|^2_{\mathscr{C}(0)} - \| v(z)\|^2_{\mathscr{C}(0)}$$

is satisfied. By the arbitrariness of $h(z)$,

$$\|g(z)\|_{\mathscr{G}(B^*)} \leqslant \|f(z)\|_{\mathscr{G}(B)}.$$

Equality holds if $B(z)$-substitution is isometric in $\mathscr{G}(0)$ because equality holds in the inequality for boundary integrals and because $\mathscr{G}(B^*)$ is contained isometrically in $\mathscr{G}(0)$. This completes the construction of the Grunsky transformation of $\mathscr{G}(B)$ into $\mathscr{G}(B^*)$ since the functions $f(z)$ considered are dense in $\mathscr{G}(0)$.

As in the proof of Theorem 11, the polynomials $u(z)$ and $v(z)$ in the statement of the theorem can be replaced by norm limits of such polynomials. It follows that the Grunsky transformation of $\mathscr{G}(B)$ into $\mathscr{G}(B^*)$ takes

$$\log \frac{1 - B(z)\,\bar{B}(w)}{1 - z\bar{w}}$$

into

$$\log \frac{\bar{B}'(0)/B^*(z) - \bar{B}'(0)/\bar{B}(w)}{1/z - 1/\bar{w}}$$

for every point w in the unit disk. A similar construction holds for the Grunsky transformation of $\mathscr{G}(B^*)$ into $\mathscr{G}(B)$. Adjointness of the Grunsky transformations of $\mathscr{G}(B)$ into $\mathscr{G}(B^*)$ and of $\mathscr{G}(B^*)$ into $\mathscr{G}(B)$ is now obtained by a straightforward kernel function argument.

Proof of Theorem 13. Since $f(a, z)$ is subordinate to $f(b, z)$ when $a \leqslant b$,

$$f(a, z) = f(b, B(b, a, z))$$

for a power series $B(b, a, z)$ with constant coefficient zero which converges in the disk and represents a function which is bounded by one and has distinct values at distinct points of the disk. If some choice of index b is made, then a new Löwner family of series $B(t, z)$ is defined by $B(t, z) = B(b, bt, z)$ when $t \leqslant 1$ and by $B(t, z) = tz$ otherwise. For these reasons, it is sufficient to give a proof of the theorem in the case that $f(t, z) = tz$ when $t > 1$. Then $f(t, z) = B(t, z)$ represents a function which is bounded by one in the unit disk when $t \leqslant 1$. A space $\mathscr{G}(B(t))$ exists for each index t. When $a \leqslant b$, the space $\mathscr{G}(B(a))$ is contained in the space $\mathscr{G}(B(b))$ and the inclusion does not increase norms. The identity

$$B(a, z) = B(b, B(b, a, z))$$

holds for a power series $B(b, a, z)$ with constant coefficient zero which converges in the unit disk and represents a function which is bounded by one and has distinct values at distinct points of the disk. By Schwarz's lemma, the series $B(b, a, z)/z$ represents a function which is bounded by one in the unit disk. Since the coefficient of z in $B(b, a, z)$ is equal to a/b, another use of Schwarz's lemma shows that the inequality

$$\left| \frac{B(b, a, w)/w - a/b}{1 - a/b\, B(b, a, w)/w} \right| \leqslant |w|$$

holds at all points w of the unit disk. It follows that the inequality

$$\left| \frac{w - B(b, a, w)}{1 - \bar{w}\, B(b, a, w)} \right| \leqslant |w| \frac{1 - a/b}{(1 - |w|)^2}$$

holds at all points w of the disk. Since the inequality

$$\left| \frac{B(b, z) - B(b, w)}{1 - B(b, z)\, \bar{B}(b, w)} \right| \leqslant \left| \frac{z - w}{1 - z\bar{w}} \right|$$

holds by Schwarz's lemma, the inequality

$$\left| \frac{B(a, w) - B(b, w)}{1 - B(a, w)\, \bar{B}(b, w)} \right| \leqslant |w| \frac{1 - a/b}{(1 - |w|)^2}$$

holds at all points w of the disk. The inequality implies that $B(t, w)$ is an absolutely continuous function of t for every point w of the disk.

Consider the identity

$$b \frac{B(b, w) - B(a, w)}{b - a} = w \frac{1 + a/b}{1 - a/b} \frac{1 - B(b, a, w)/w}{1 + B(b, a, w)/w} \times \frac{1 + B(b, a, w)/w}{1 + a/b} \frac{B(b, w) - B(b, B(b, a, w))}{w - B(b, a, w)}$$

in the limit as a increases to b for a point b at which $\partial/\partial b\, B(b, w)$ exists. Since

$$\frac{B(b, w) - B(b, B(b, a, w))}{w - B(b, a, w)}$$

has the nonzero limit $\partial/\partial w\, B(b, w)$, it follows that a limit $\varphi(b, w)$ exists for

$$\frac{1 + a/b}{1 - a/b} \frac{1 - B(b, a, w)/w}{1 + B(b, a, w)/w}$$

and that

$$b\, \partial/\partial b\, B(b, w) = w\varphi(b, w)\, \partial/\partial w\, B(b, w).$$

But when $a < b$,

$$\frac{1 + a/b}{1 - a/b} \frac{1 - B(b, a, z)/z}{1 + B(b, a, z)/z}$$

is a power series with constant coefficient one which converges in the unit disk and represents a function with positive real part in the disk. It follows that a limiting power series $\varphi(b, z)$ exists for almost all b. It has constant coefficient one, converges in the unit disk, and represents a function with positive real part in the disk. The coefficients of $\varphi(t, z)$ are measurable functions of t and the desired differential equation is valid in the formal power series sense.

Proof of Theorem 14. For uniqueness, consider two Löwner families of power series $f_1(t, z)$ and $f_2(t, z)$ which satisfy the same differential equation. Since $f_\pm(a, z)$ is subordinate to $f_\pm(b, z)$ when $a \leqslant b$,

$$f_\pm(a, z) = f_\pm(b, B_\pm(b, a, z))$$

where $B_\pm(b, a, z)$ is a power series which converges in the unit disk and represents a function which is bounded by one in the disk. For each positive number b, the coefficients of $B_\pm(b, t, z)$ are absolutely continuous functions of t in the interval $(0, b]$ which satisfy the differential equation

$$t\, \partial/\partial t\, B_\pm(b, t, z) = z\varphi(t, z)\, \partial/\partial z\, B_\pm(b, t, z)$$

with the initial condition $B_\pm(b, t, z) = z$ when $t = b$. By the uniqueness of solutions of linear first-order differential equations with given initial conditions,

$$B_+(b, t, z) = B_-(b, t, z)$$

when $0 < t < b$. It follows that a power series $\psi(z)$ exists with constant coefficient zero and coefficient of z equal to one such that the identity

$$f_+(t, z) = \psi(f_-(t, z))$$

holds for all positive t. By the arbitrariness of t, the series converges in the complex plane and represents a function which has distinct values at distinct points of the plane. Since any such function is linear, $\psi(z) = z$.

For existence, consider first the case in which the series $\varphi(t, z) = \varphi(z)$ does not depend on t. For each complex number w, let $f(w, z)$ be the solution of the differential equation of Theorem 6. Let $\lambda(z)$ be the unique power series with constant coefficient zero and coefficient of z equal to one such that

$$z\varphi(z)\,\partial/\partial z\,\lambda(z) = \lambda(z).$$

Since

$$\lambda(z)\,\partial/\partial z\,\lambda(f(w,z)) = \lambda(f(w,z))\,\partial/\partial z\,\lambda(z)$$

by Theorem 7 and since the coefficient of z in $f(w, z)$ is w,

$$\lambda(f(w,z)) = w\lambda(z).$$

Since $\varphi(z)$ converges in the unit disk, $\lambda(z)$ converges in the unit disk. Since $\varphi(z)$ represents a function with positive real part in the disk,

$$-\partial/\partial\theta\, \mathrm{Re}\, i \log \lambda(re^{i\theta}) = \mathrm{Re}\, 1/\varphi(re^{i\theta}) \geqslant 0$$

when $0 < r < 1$. Since $\lambda(z)$ represents a function which has distinct values at distinct points of every circle of radius r about the origin, $0 < r < 1$, it represents a function which has distinct values at distinct points of the unit disk. The same inequality implies that the function maps the unit disk onto a star-like region. It follows that $f(t, z)$ represents a function which has distinct values at distinct points of the unit disk when $0 < t < 1$ and that the series $t\lambda(z)$ form a Löwner family when considered for positive t.

Existence will now be proved in the general case. By the compactness property resulting from Theorem 13, it is sufficient to consider the case in which a finite number of points $0 = t_0 < t_1 < \cdots < t_r$ exist such that $\varphi(t, z)$ is independent of t in each interval (t_{k-1}, t_k) and in the half-line (t_r, ∞).

To define $f(t, z)$ in (t_r, ∞), let $\varphi_\infty(z)$ be the value of $\varphi(t, z)$ when $t > t_r$. By Theorem 6, a unique solution $f_\infty(w, z)$ of the differential equation

$$w\,\partial/\partial w\, f_\infty(w,z) = z\varphi_\infty(z)\,\partial/\partial z\, f_\infty(w,z)$$

exists with the initial conditions $f_\infty(1, z) = z$, the nth coefficient of $f_\infty(w, z)$ being a polynomial of degree at most n in w which is zero when $n = 0$ and which is w when $n = 1$. Define

$$f(t,z) = t \lim_{w=0} f_\infty(w,z)/w$$

when $t_r \leqslant t$.

To define $f(t, z)$ in the interval $[t_{k-1}, t_k)$ when $f(t_k, z)$ is defined, let $\varphi_k(z)$ be the value of $\varphi(t, z)$ in the interval (t_{k-1}, t_k). By Theorem 6, a unique solution $f_k(w, z)$ of the differential equation

$$w\,\partial/\partial w\, f_k(w,z) = z\varphi_k(z)\,\partial/\partial z\, f_k(w,z)$$

exists with the initial condition $f_k(1, z) = z$, the nth coefficient of $f_k(t, z)$ being a polynomial of degree at most n in w which is zero when $n = 0$ and which is w when $n = 1$. Define

$$f(t, z) = f(t_k, f_k(t/t_k, z))$$

when $t_{k-1} \leqslant t \leqslant t_k$. The required properties of the series $f(t, z)$ are easily verified.

Proof of Theorem 15. The Löwner equation implies that the coefficients of $B(t, a, z)$ are absolutely continuous functions of t in (a, b) such that

$$t\, \partial/\partial t\, B(t, a, z) = -B(t, a, z)\, \varphi(t, B(t, a, z)).$$

The differential equation is used in the equivalent form

$$\partial/\partial t \log \frac{B(t, a, z)}{az/t} = -[\varphi(t, B(t, a, z)) - \varphi(t, B(t, a, 0))]/t.$$

It follows that the differential equation

$$t\, \partial/\partial t \log \frac{1 - B(t, a, z)\, \bar{B}(t, a, w)}{1 - z\bar{w}}$$
$$= B(t, a, z)\, \bar{B}(t, a, w) \frac{\varphi(t, B(t, a, z)) + \bar{\varphi}(t, B(t, a, w))}{1 - B(t, a, z)\, \bar{B}(t, a, w)}$$

and

$$t\, \partial/\partial t \log \frac{a/B(t, a, z) - a/B(t, a, \bar{w})}{t/z - t/\bar{w}}$$
$$= \frac{t\varphi(t, B(t, a, z))/B(t, a, z) - t\varphi(t, B(t, a, w))/B(t, a, \bar{w})}{t/B(t, a, z) - t/B(t, a, \bar{w})} - 1$$

hold at every point w of the unit disk. It will be shown that the inequality

$$\left| \sum c_k t\, \partial/\partial t \log \frac{B(t, a, w_k)}{a\, w_k/t} \right|^2$$
$$\leqslant 2 \sum c_i \bar{c}_k t\, \partial/\partial t \log \frac{1 - B(t, a, w_i)\, \bar{B}(t, a, w_k)}{1 - w_i \bar{w}_k}$$

holds at all points $w_1, \ldots, w_r$ of the unit disk and for all corresponding complex numbers $c_1, \ldots, c_r$.

By the Poisson representation of a function which is positive and harmonic in the unit disk, it is sufficient to make the verification when the special choice of $\varphi(t, z)$ is made. Equality holds in that case because

$$\frac{\varphi(t, z) - \varphi(t, 0)}{z} = \frac{2}{\lambda(t) - z}$$

and

$$\frac{\varphi(t, z) + \bar{\varphi}(t, w)}{1 - z\bar{w}} = \frac{2}{(\lambda(t) - z)(\bar{\lambda}(t) - \bar{w})}.$$

By the Schwarz inequality, it follows that

$$\left| \sum c_k \int_a^b h(t)\, d \log \frac{B(t, a, w_k)}{a\, w_k/t} \right|^2$$
$$\leqslant 2 \int_a^b |h(t)|^2/t\, dt \sum c_i \bar{c}_k \log \frac{1 - B(b, a, w_i)\, \bar{B}(b, a, w_k)}{1 - w_i \bar{w}_k}.$$

It follows that $F(z)$ belong to $\mathscr{S}(B(b, a))$ and that the desired norm inequality holds. Equality holds when the special choice of $\varphi(t, z)$ is made and

$$h(t) = t\, \partial/\partial t \log \frac{\bar{B}(t, a, w)}{a\, \bar{w}/t}$$

for a point w in the unit disk since then

$$F(z) = 2 \log \frac{1 - B(b, a, z)\, \bar{B}(b, a, w)}{1 - z\bar{w}}.$$

It follows that equality holds whenever $h(t)$ is a finite linear combination of such special functions of t or is a limit of such finite linear combinations in the mean square sense. To prove equality in general, it remains to show that no nontrivial choice of $h(t)$ exists such that $F(z)$ vanishes identically. By the theory of minimal decompositions, the restriction of $h(t)$ to any subinterval (a, c) of (a, b) is then a function with the same property in the subinterval. So

$$\int_a^c h(t)\, d \log \frac{B(t, a, z)}{a\, z/t} = 0$$

identically for $a < c < b$. Since the coefficient of z in the series is zero

$$\int_a^c h(t)\,dt = 0.$$

By the arbitrariness of c, $h(t)$ vanishes almost everywhere in (a, b).

A similar argument shows that $G(z)$ belongs to $\mathscr{S}(B^*(b, a))$, that the desired norm inequality holds, and that equality holds when the special choice of $\varphi(t, z)$ is made. For the computation of Grunsky transforms, it is sufficient to consider the case

$$h(t) = t\,\partial/\partial t \log \frac{\bar{B}(t, a, w)}{a\,\bar{w}/t}$$

for a complex number w of absolute value one, in which case $F(z)$ has already been computed. Since

$$\frac{\varphi(t, z)/z - \varphi(t, w)/w}{1/z - 1/w} - 1 = \frac{-2zw}{(\lambda(t) - z)(\lambda(t) - w)}$$

when the special choice of $\varphi(t, z)$ is made, a straightforward calculation shows that

$$-G(z) = \log \frac{a/B^*(b, a, z) - a/\bar{B}(b, a, w)}{b/z - b/\bar{w}}$$

which is the Grunsky transform of $F(z)$ by the proof of Theorem 12. A similar argument shows that $-F(z)$ is the Grunsky transform of $G(z)$. It is easily seen that every element of $\mathscr{S}(B(b, a))$ is of the form $F(z)$ for some choice of $h(t)$ because the set of such functions is a closed vector subspace containing the kernel functions. For the same reason, every element of $\mathscr{S}(B^*(b, a))$ is of the form $G(z)$ for some such choice of $h(t)$.

It remains to show that $B(b, a, z)$-substitution is isometric in $\mathscr{S}(0)$ when the special choice of $\varphi(t, z)$ is made. By the proof of Theorem 14, it is sufficient to give a proof in the case that $\lambda(t) = \lambda$ is independent of t. It will be shown that $B(b, t, z)$-substitution is isometric in $\mathscr{S}(0)$ for all t in (a, b). Since $B(b, t, z)$-substitution is bounded by one in $\mathscr{S}(0)$, it is sufficient to show that the identity

$$\|f(B(b, t, z))\|_{\mathscr{S}(0)} = \|f(z)\|_{\mathscr{S}(0)}$$

holds for a dense set of elements $f(z)$ of $\mathscr{S}(0)$. A convenient choice of test function consists of the polynomials $f(z)$ such that $f'(z)$ is divisible by $z - \lambda$.

Then the left side is a differentiable function of t in (a, b). By the Löwner differential equation, it is sufficient to verify the identity

$$\langle z(\lambda + z)/(\lambda - z)\, \partial/\partial z f(z), f(z)\rangle_{\mathscr{G}(0)} + \langle f(z), z(\lambda + z)/(\lambda - z)\, \partial/\partial z f(z)\rangle_{\mathscr{G}(0)} = 0$$

for every such polynomial $f(z)$. The identity is verified by a straightforward calculation since it can be written

$$\langle (\lambda + z)/(\lambda - z)\, \partial/\partial z f(z), \partial/\partial z f(z)\rangle_{\mathscr{H}(z)} + \langle \partial/\partial z f(z), (\lambda + z)/(\lambda - z)\, \partial/\partial z f(z)\rangle_{\mathscr{H}(z)} = 0.$$

Proof of Theorem 16. The desired estimate is obtained by choosing $h(t) = 1$ in Theorem 15. If

$$u(z) = \log \frac{1}{1 - z\bar{w}} \quad \text{and} \quad v(z) = \log \frac{1}{1 - z\bar{B}(w)}$$

for a point w in the unit disk, then the adjoint of $B(z)$-substitution takes $u(z)$ into $v(z)$ by the proof of Theorem 11. By the kernel function identity in $\mathscr{G}(0)$,

$$\left\langle u(z) - v(B(z)), \log \frac{B(z)}{zB'(0)} \right\rangle_{\mathscr{G}(B)} = \log \frac{\bar{B}(w)}{\bar{w}\bar{B}'(0)}$$

which is the constant coefficient in

$$u(1/z) - v(1/B^*(z)) = \log \frac{1 - \bar{B}(w)/B^*(z)}{1 - \bar{w}/z}.$$

The desired conclusion in the case that $u(z)$ and $v(z)$ are polynomials is obtained by a power series expansion in $\bar{w}$ as in the proof of Theorem 11.

Proof of Theorem 17. It is sufficient to give a proof of the inequality in the case $a = 0$, for then it follows by a change of variable that

$$\left| \int_a^1 f(t)\, dt \right|^2 \leqslant 2\left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right) \int_a^1 |f(t)|^2 (t - a)\, dt$$

for every polynomial $f(z)$ of degree less than r, where

$$t - a \leqslant (1 - a)\, t$$

for $a < t < 1$. Since $S_r(z)$ is orthogonal to every polynomial of degree less than r, the identity

$$\int_0^1 f(t)\,dt = \int_0^1 f(t)\,[1 - S_r(t)]/t\;t\,dt$$

holds for every such polynomial. By the Schwarz inequality,

$$\left|\int_0^1 f(t)\,dt\right|^2 \leqslant \int_0^1 |f(t)|^2\,t\,dt \times \int_0^1 [1 - S_r(t)]^2/t\,dt$$

where

$$\begin{aligned}\int_0^1 [1 - S_r(t)]^2/t\,dt &= \int_0^1 [1 - S_r(t)]/t\,dt\\ &= 2\left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right)\end{aligned}$$

because

$$\int_0^1 [S_{n-1}(t) - S_n(t)]/t\,dt = 2/n$$

for every $n = 1, \ldots, r$ by the orthogonality and recurrence relations for Legendre polynomials.

Proof of Theorem 18. Since the coefficient $\varphi(t, z)$ in the Löwner equation is independent of t, the series $B(b, a, z)$ is a function $B(a/b, z)$ of a/b. By Theorem 6, the coefficient of z^n in $B(t, z)$ is a polynomial of degree at most n in t with constant coefficient zero. It follows that the coefficient of z^n in

$$\log \frac{B(t, z)}{tz}$$

is a polynomial of degree at most n in t with constant coefficient zero. The desired inequality is now obtained from Theorems 15 and 17 by a straightforward calculation.

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