An Iterative Method for the Extrapolation of Band-Limited Functions

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We derive an iterative algorithm to extrapolate band-limited functions using the operators of Fourier transform and projection on convex subsets. The algorithm allows the use of a priori knowledge in the extrapolation procedure.

1. INTRODUCTION

Entire functions of exponential growth such that the real part is slowly increasing are the Fourier transforms of distributions of compact support ([10], p. 311). Physicists call them band-limited functions ([2], p. 83). In practice, very often we know the functions only in a region $B$ of $\mathbb{R}^m$ and we want to know the functions on the whole of $\mathbb{R}^m$, or at least in a region as large as possible. Theoretically, the problem has a unique solution if $B \neq \emptyset$ ($\bar{B}$ is the interior of $B$) due to the fact that an entire function is identically zero if it is zero on an open ball. In the following we give an iterative method to extrapolate a band-limited function knowing the support of its Fourier transform. In addition to the iterative algorithms studied by Gerchberg [3], Papoulis [8] and Youla [13], the algorithm given in Section 2 below allows relaxation parameters and the use of other a priori knowledge. The improvement of the quality of the extrapolated function gained by the use of relaxation parameters and the incorporation of more a priori knowledge is

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illustrated in Section 3. Although we study the problem in connection with computed tomography from a limited range of views [11], the problem itself has theoretical value and applications in many other fields such as acoustics [12], positron tomography [9] and signal processing, which use the Fourier transform as a tool.

In the following we assume that the Hilbert space $\mathcal{H}$ is the space $L^2(\mathbb{R}^m)$ of square integrable complex valued functions defined on $\mathbb{R}^m$. Let $\mathcal{F}$ be a unitary operator on $\mathcal{H}$. For any $f \in \mathcal{H}, f = \Re f + i \Im f$, where both $\Re f$ and $\Im f$ are real valued and are called the real part and the imaginary part of $f$, respectively. If $f, g \in \mathcal{H}$, we write $f \preceq g$ if and only if $\Re f \preceq \Re g$ and $\Im f \preceq \Im g$ almost everywhere. We define

\[
\sup(f, g) = \sup(\Re f, \Re g) + i \sup(\Im f, \Im g),
\]
\[
\inf(f, g) = \inf(\Re f, \Re g) + i \inf(\Re f, \Re g).
\]

For any measurable set $S \subset \mathbb{R}^m$ the operator $\mathcal{S}$ defined by

\[
\mathcal{S}f = \chi_S f
\]

($\chi_s$ is the characteristic function of $S$) is an orthogonal projection operator. The support of a function $f$ (the complement of the maximal open set in which $f$ is a.e. zero) is denoted by $\text{supp} f$.

**Definition.** Let $A$ and $B$ be subsets of $E$. Let $b \in \mathcal{H}$ with the property that $\mathcal{B}b = b$. We say that a function $f$ is of type $(A, B)$ if

(i) $\text{supp} f \subset A$,

(ii) $\mathcal{B}f = b$.

The set of all functions of type $(A, B)$ is denoted by $\text{Cl}(A, B)$. In the following, we assume that $\text{Cl}(A, B) \neq \emptyset$ and the two sets $A$ and $B$ are closed.

Throughout this article, for any closed and convex set $C \subset \mathcal{H}$, $P_C$ designates the orthogonal projection operator on the set $C$ (by definition, $P_C(f) = g$ if $g$ is the unique point of $C$ such that for any $t \in C, \|f - g\| \leq \|f - t\|$).

2. Extrapolation of Functions

Suppose the value of a function $f$ is known in a measurable subset $B$. In the following we shall derive an iterative algorithm to extrapolate $f$ incorporating the following a priori knowledge,
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(a) \( \text{supp}\, \hat{f} \subseteq A \),
(b) \( l \leq \hat{f} \leq u \),
(c) \( \hat{f} \) is a real function,

where \( A \) is a known subset of \( \mathbb{R}^m \), \( l \) and \( u \) are two known functions of \( \mathcal{H} \). One way of extrapolating \( f \) incorporating the above a priori information is to find a function \( \hat{f} \in \text{Cl} (A, B) \) such that \( \hat{f} \) is in the intersection \( L \cap U \cap Z \), where

\[
L = \{ h \in \mathcal{H} \mid l \leq h \}, \\
U = \{ h \in \mathcal{H} \mid h \leq u \}, \\
Z = \{ h \in \mathcal{H} \mid \text{Im } h = 0 \}.
\]

These sets are closed and convex subsets of \( \mathcal{H} \). Starting from an arbitrary point, successive orthogonal projections on these subsets lead to a point of intersection. Precise mathematical formulation of the idea is given below.

**Lemma 0.** Let \( S \) be a closed subset of \( \mathbb{R}^m \) and \( f \in \mathcal{H} \). We have

\[
\text{supp } f \subseteq S \iff \mathcal{F}f = f. \tag{2}
\]

**Lemma 1.** Let \( L \) be a closed subspace of \( \mathcal{H} \). Let \( a \) and \( f \) be two points of \( \mathcal{H} \). Then

\[
P_{a+L}(f) = P_L(a) + P_{L^\perp}(f),
\]

where \( P_L(f) \) is the orthogonal projection of \( f \) on \( L \) and \( L^\perp \) is the orthogonal complement of \( L \).

**Proof.** \( g = P_L(f) + P_{L^\perp}(a) \) satisfies the following properties:

1. \( g = a - P_L(a) + P_L(f) \in a + L \).
2. For any \( t \in a + L \), \( \langle f - g, t - g \rangle = 0 \). Note that \( \langle f - g, t - g \rangle = \langle f - P_L(f) - P_{L^\perp}(a), t - g \rangle \) for some \( l \in L \). Hence \( \langle f - g, t - g \rangle = \langle P_{L^\perp}(f - a), l - P_L(f - a) \rangle = 0 \). Therefore \( g = P_{a+L}(f) \).

**Lemma 2.** If \( f \in \text{Cl}(A, B) \) and \( g = \hat{f} \) then

\[
g \in \ker(I - \mathcal{F}) \cap (g_0 + \ker(F \mathcal{B} F^{-1})), \tag{3}
\]

where \( g_0 = \mathcal{F} \hat{f} \).
Proof. From (2) it follows that $\mathcal{A} g = g$; hence $g \in \ker(I - \mathcal{A})$. We note that $g - g_0 \in \ker \mathcal{F} \mathcal{F}^{-1}$ because $\mathcal{F}^{-1} g = f$ and $\mathcal{F}^{-1} g_0 = \mathcal{F} f$; hence $\mathcal{B} \mathcal{F}^{-1} (g - g_0) = 0$ since $\mathcal{B}^2 = \mathcal{B}$. Therefore $g \in g_0 + \ker \mathcal{F} \mathcal{F}^{-1}$. The lemma is then proved.

We assume that $L$, $U$ and $Z$ are all nonempty.

**Lemma 3.** The orthogonal projections $P_L$, $P_U$ and $P_Z$ on the sets $L$, $U$ and $Z$ are given as follows:

$$
P_L(h) = \sup(l, h), \quad (4)$$

$$
P_U(h) = \inf(u, h), \quad (5)$$

$$
P_Z(h) = \Re(h). \quad (6)$$

Proof. By the definition of $L$, we conclude that $t = \sup(l, h) \in L$. Moreover for any $s \in L$, $\| h - t \| \leq \| h - s \|$. For,

$$
\| h - t \|^2 = \int_C (\Re l(x) - \Re h(x))^2 \, dx + \int_D (\Im l(x) - \Im h(x))^2 \, dx,
$$

where $C = \{ x \mid \Re h(x) < \Re l(x) \}$ and $D = \{ x \mid \Im h(x) < \Im l(x) \}$. If $\bar{C}$ and $\bar{D}$ designate the complements of $C$ and $D$, respectively, then we have

$$
\| h - s \|^2 = \int_C (\Re s(x) - \Re h(x))^2 \, dx + \int_{\bar{D}} (\Re s(x) - \Re h(x))^2 \, dx
$$

$$
+ \int_D (\Im s(x) - \Im h(x))^2 \, dx + \int_{\bar{C}} (\Im s(x) - \Im h(x))^2 \, dx.
$$

For all $s \in L$, we have $0 \leq \Re(l(x) - h(x)) \leq \Re(s(x) - h(x))$ for each $x \in C$ and $0 \leq \Im(l(x) - h(x)) \leq \Im(s(x) - h(x))$ for all $x \in D$. It follows that $P_L(h) = t$, i.e., $P_L$ is indeed given by (4). Similarly, we can prove relations (5) and (6).

**Theorem 4 (Bregman).** Let $\{ C_i \}_{i=0,1,...,r}$ be a finite family of closed and convex subsets of a separable Hilbert space and $P_i$ be the orthogonal projection on $C_i$, $i = 0, 1,..., r$. Let $f_0$ be an arbitrary point of the Hilbert space. Define a sequence $\{ f_n \}$ as follows:

$$
f_{n+1} = f_n + \rho_n (P_{i_n}(f_n) - f_n) \quad n \geq 0,
$$

where $i_n = n \mod(r + 1)$ and $i_0 = 0$. If $\bigcap_{i=0}^r C_i \neq \emptyset$ and $0 < \varepsilon_1 \leq \rho_n \leq \varepsilon_2 < 2$ for some real numbers $\varepsilon_1$ and $\varepsilon_2$ then the sequence $f_n \rightarrow f^*$ weakly, where $f^* \in \bigcap_{i=0}^r C_i$.

For the proof of the theorem, we refer the reader to [1]. Actually, the number of the convex sets $\{ C_i \}$ need not be finite. The strong convergence of
the above sequence in the case where the family \( \{C_i\} \) satisfies some additional conditions has been proved by Gubin et al. [4]. Halperin [5] shows the strong convergence of the sequence in the case of a finite family of subspaces \( \{C_i\} \).

**Algorithm.** Let \( f \in \text{Cl}(A, B) \) be a function such that \( l \leq f \leq u \) and \( \text{Im} f = 0 \). Let \( g_n \) be a sequence of functions defined as follows:

1. \( g_0 \) is as in Lemma 2.
2. \( g_1 = g_n + \lambda_n \) \( (\text{Re}(g_n) - g_n) \),
3. \( g_2 = g_n + \lambda_n^2 \) \( (g_n - g_n) \),
4. \( g_3 = g_n + \lambda_n^3 \) \( (\text{sup}(l, g_n^2) - g_n^2) \),
5. \( g_4 = g_n + \lambda_n^4 \) \( (\text{inf}(u, g_n^3) - g_n^3) \),
6. \( g_{n+1} = \mathcal{F}(\lambda_n^5 \mathcal{B}f + (1 - \lambda_n^5) \mathcal{B} \mathcal{F}^{-1}g_n + (I - \mathcal{B}) \mathcal{F}^{-1}g_n^4) \).

If for all \( i = 1, 2, 3, 4, 5 \), \( 0 < \varepsilon_1 \leq \lambda_n \leq \varepsilon_2 < 2 \) then as \( n \to \infty \), \( g_n \to g \) weakly, where \( g \) and \( \mathcal{F}^{-1}g \) satisfy all the above conditions on \( f \) and \( f \), respectively.

**Proof.** From Lemma 2 and the conditions on \( f \), we conclude that \( f \in \ker(I - \mathcal{A}) \cap (g_0 + \ker \mathcal{F} \mathcal{B} \mathcal{F}^{-1}) \cap L \cap U \cap Z \neq \emptyset \). The five sets to which \( f \) belongs are closed and convex subsets of \( \mathbb{R}^m \). The weak convergence of the above sequence to a point of intersection follows immediately from Theorem 4, Lemma 3 and Lemma 1.

**Remarks.**

(a) We can incorporate more a priori knowledge in the algorithm if we can interpret the knowledge in the form "\( f \) belongs to some closed and convex subset of \( \mathcal{H} \)" and if we know the orthogonal projection on the convex subset explicitly.

(b) In Lemma 2, we show that if \( f \in \text{Cl}(A, B) \) then \( f \) belongs to the intersection of two affine subspaces. In Chapter 6 of his Ph.D. dissertation [7], Lent gave different ways of representing the intersection of two affine subspaces which give rise to many different iterative algorithms to find a common point.

(c) In the case where the set \( A \) is a compact subset of \( \mathbb{R}^m \) and \( \mathcal{B} \neq \emptyset \) the above algorithm gives an iterative method to extend band-limited functions. In this case, by the Paley–Wiener theorem ([10], p. 307), \( \text{Cl}(A, B) \) consists of one element only. Furthermore if both \( A \) and \( B \) are compact intervals of \( \mathbb{R} \), the above algorithm with relaxation parameter \( \lambda = 1 \), and no other constraints on \( f \) beside compact support, coincides with the algorithm given in [8], where the decomposition of band-limited function in terms of prolate spheroidal functions [6] is used to prove the convergence of the algorithm.
3. Numerical Implementation and Illustration

The two basic steps in the Algorithm can be described in words as follows:

(i) To obtain $g_0$, form $\tilde{f}$, where $\tilde{f}(x) = f(x)$ if $x \in B$ and $\tilde{f}(x) = 0$ outside of $B$ and then compute $g_0 = \mathcal{F}\tilde{f}$.

(ii) The five intermediate iterative steps in part 2 of the algorithm can be implemented as follows:

(a) Keep the real part of $g_n$ as it is. Change the imaginary part of $g_n$ to $(1 - \lambda_n^1) \text{Im} g_n$. The function obtained is the function $g_n^1$.

(b) The function $g_n^2$ is given by the relations $g_n^2(x) = g_n^1(x)$ if $x \in A$ and $g_n^2(x) = (1 - \lambda_n^2) g_n^1(x)$ if $x \notin A$.

(c) The function $g_n^3$ is obtained from $g_n^2$ as follows: $g_n^3(x) = g_n^2(x)$ if $g_n^2(x) \geq l(x)$. If $g_n^3(x) < l(x)$ then $g_n^3(x) = g_n^2(x) + \lambda_n^3(l(x) - g_n^2(x))$.

(d) The procedure to get $g_n^4$ is similar to that given in (c).

(e) From a function $h$, where $h(x) = \mathcal{F}^{-1}g_n^4(x)$ if $x \notin B$ and $h(x) = \lambda_n^4 f(x) + (1 - \lambda_n^4) \mathcal{F}^{-1}g_n^4(x)$ if $x \in B$. Finally form $g_{n+1} = \mathcal{F} h$.

In [11], we have shown that the Fourier transform of the density function $g$ of a cross-section of an object is known in a well-defined region $B$ if the X-ray projection data $\mathcal{R}_\theta g$ are given only for $a \leq \theta \leq b$ with $b - a = 180^\circ - \epsilon$.

Fig. 1. Mathematical simulated phantom representing a human thorax.
Fig. 2. Reconstructed picture of the phantom from the simulated projection data taken from $-70^\circ$ to $70^\circ$ using the algorithm given in [11].

Fig. 3. The real part of the discrete Fourier transform of the phantom.
Fig. 4. The real part of the discrete Fourier transform of the picture given in Fig. 2.

Fig. 5. Reconstructed picture produced by the algorithm after the 11th iteration. A priori knowledge used: \( \text{Im}(g) = 0 \), support \( g \) is compact and \( g \geq 0 \).
Graphs of the density function of the picture given Figs. 1, 2, and 5 along the lines 47, 80, and 90. Figs. 1, 2, and 5.
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Figure 7
for some $\varepsilon < 0$. One way of improving the quality of the picture given in [11] is to extrapolate the function $f = \hat{g}$ incorporating available a priori knowledge. As an illustration of the performance of the algorithm, we have applied it to enhance the quality of the picture obtained by the algorithm given in [11].

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